A CONVERSE OF THE ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY

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SUMMARY. In this note we prove: Let \((x_v) (v = 1, 2, \ldots)\) be a sequence of positive real numbers. If \((x_v) (v = 0, 1, \ldots)\) with \(x_0 = 0\) is concave, then

\[
\frac{1}{n} \sum_{v=1}^{n} x_v < \frac{e}{2} \prod_{v=1}^{n} x_v^{1/n}
\]

where the constant \(e/2\) is best possible.

For the famous inequality between the (unweighted) arithmetic and geometric means of positive real numbers \(x_1, \ldots, x_n\), i.e.

\[
G_n = \prod_{v=1}^{n} x_v^{1/n} \leq \frac{1}{n} \sum_{v=1}^{n} x_v = A_n
\]

with equality holding if and only if \(x_1 = \ldots = x_n\), one can find in literature many proofs, extensions, sharpenings and variants [1], [3], [5], [6].

In several interesting papers different authors have published upper bounds for the difference \(A_n - G_n\) as well as for the ratio \(A_n/G_n\). The resulting inequalities are called complementary or converse inequalities; see [3].

In 1933 J. Favard [4] discovered a remarkable inequality which is complementary to the inequality between the arithmetic mean and the geometric mean of a positive, integrable function.

He proved:

If \(f: [a,b] \rightarrow \mathbb{R}\) is positive, continuous and concave, then
\[ \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{e}{2} \exp \left( \frac{1}{b-a} \int_a^b \log f(t)dt \right) \]

where the constant \( e/2 \) cannot be replaced by a smaller number.

Interesting extensions of this result were given by L. Berwald [2].

It is natural to ask: Does there exist a discrete analogue of Favard's inequality? The following proposition gives an affirmative answer to this question.

**Theorem.** Let \((x_v) (v = 1, 2, \ldots)\) be a sequence of positive real numbers. If \((x_v) (v = 0, 1, \ldots)\) with \(x_0 = 0\) is concave, then

\[ \frac{1}{n} \sum_{v=1}^{n} x_v < \frac{e}{2} \prod_{v=1}^{n} x_v^{1/n} \]

(1)

where the constant \( e/2 \) is best possible.

**Proof.** We establish (1) by induction on \( n \).

For \( n = 1 \) inequality (1) is obviously true.

Next we assume that (1) holds. Then we obtain

\[ \prod_{v=1}^{n+1} x_v = x_{n+1} \prod_{v=1}^{n} x_v > x_{n+1} \left( \frac{e}{2} \frac{1}{n} \sum_{v=1}^{n} x_v \right)^n \]

and we have to show that

\[ x_{n+1} \left( \frac{e}{2} \frac{1}{n} \sum_{v=1}^{n} x_v \right)^n \geq \left( \frac{2}{e} \frac{1}{n+1} \sum_{v=1}^{n+1} x_v \right)^{n+1}. \]

(2)

Because of

\[ e > (1 + \frac{1}{n+1})^{n+1} \]

inequality (2) is proved if we can establish

\[ \frac{x_{n+1}}{2} (1 + \frac{1}{n+1})^{n+1} \left( \frac{1}{n} \sum_{v=1}^{n} x_v \right)^n \geq \left( \frac{1}{n+1} \sum_{v=1}^{n+1} x_v \right)^{n+1} \]

which can be written as
Now we prove the inequalities
\[
\frac{1}{2n} \leq \frac{x_{n+1}}{\sum_{v=1}^{n} x_{v}} \leq \frac{2}{n} .
\] (4)

Since \((x_v)\) is concave we obtain
\[
\sum_{i=0}^{2k-2} (x_{n+1-k+i} + x_{n+3-k+i}) \leq 2 \sum_{i=0}^{2k-2} x_{n+2-k+i} \quad (k = 1, \ldots, n+1)
\]
which is equivalent to
\[
x_{n+1-k} \cdot x_{n+1+k} \leq x_{n+2-k} \cdot x_{n+k} .
\]

This means that
\[
k \mapsto x_{n+1-k} \cdot x_{n+1+k} \quad (k = 1, \ldots, n+1)
\]
is decreasing and we get
\[
x_{n+1-k} \leq x_{n+1-k} \cdot x_{n+1+k} \leq x_{n} \cdot x_{n+2} \leq 2x_{n+1}
\]
which implies
\[
\sum_{k=1}^{n} x_{n+1-k} \leq \sum_{k=1}^{n} 2x_{n+1} ,
\]
i.e.
\[
\frac{1}{2n} \leq \frac{x_{n+1}}{\sum_{i=1}^{n} x_{i}} .
\]

Next we note that the sequence \((x_v/v) \ (v = 1, 2, \ldots)\) is decreasing. Indeed, since
\[
\frac{x_{0} \cdot x_{2}}{2} = \frac{x_{2}}{2} \leq x_{1}
\]
and because of
\[
(v+1) \left[ \frac{x_{v+1}}{x_v} - \frac{x_v}{v+1} \right] \leq (v-1) \left[ \frac{x_v}{v} - \frac{x_{v-1}}{v-1} \right]
\]
the assertion follows immediately by induction.

We prove the second inequality of (4) by induction on \( n \). For \( n=1 \) the inequality is obviously valid. From the monotonicity of \( (x_{\nu}/\nu) \) and from the induction hypothesis we obtain

\[
(n+1)x_{n+2} - 2 \sum_{\nu=1}^{n+1} x_{\nu} \leq (n+2)x_{n+1} - 2 \sum_{\nu=1}^{n+1} x_{\nu} =
\]

\[
= nx_{n+1} - 2 \sum_{\nu=1}^{n} x_{\nu} \leq 0
\]

which completes the proof of double-inequality (4).

We denote by \( \delta \) the function

\[
\delta : \left[ \frac{1}{2n}, \frac{2}{n} \right] \to \mathbb{R},
\]

\[
\delta(x) = \frac{1}{2} \frac{(n+2)^{n+1}}{n^n} x - (1+x)^{n+1}.
\]

Because of

\[
\delta''(x) = -n(n+1)(1+x)^{n-1} < 0
\]

we conclude that \( \delta \) is concave which implies

\[
\delta(x) \geq \min \{ \delta\left(\frac{1}{2n}\right), \delta\left(\frac{2}{n}\right) \}.
\]

(5)

We have

\[
\delta\left(\frac{2}{n}\right) = 0
\]

(6)

and we will prove

\[
\delta\left(\frac{1}{2n}\right) = -\frac{1}{4} \frac{(2n+1)^{n+1}}{(2n)^{n+1}} \left( \frac{n+2}{n+\frac{2}{2}} \right)^{n+1} - 4 > 0.
\]

(7)

In order to establish (7) we show that the sequence

\[
L_n = \left( \frac{n+2}{n+\frac{2}{2}} \right)^{n+1} (n = 1, 2, \ldots)
\]

is increasing. We define

\[
g(x) = (x+1) \log \frac{x+\frac{2}{2}}{x+\frac{2}{2}}, x \geq 1.
\]
Differentiation yields
\[ g'(x) = \log \frac{x+2}{x+\frac{1}{2}} - \frac{3(x+1)}{2(x+\frac{1}{2})(x+2)}. \]

Replacing in
\[ \log \left(1 + \frac{1}{y^2}\right) \geq \frac{2}{2y+1}, \quad y > 0, \]
(see [6, p. 273]) by \( \frac{2x+1}{3} \) we obtain for \( x \geq 1 \):
\[ \log \frac{x+2}{x+\frac{1}{2}} > \frac{2}{\frac{3(2x+1)+1}{2}} \geq \frac{3(x+1)}{2(x+\frac{1}{2})(x+2)} \]
which yields
\[ g'(x) > 0 \quad \text{for} \quad x \geq 1. \]

Hence we get for \( n \geq 1 \):
\[ L_n \geq L_1 = 4 \]
and therefore
\[ f \left( \frac{1}{2n} \right) \geq 0. \]

From (5), (6) and (7) we conclude
\[ f(x) \geq 0 \quad \text{for all} \quad x \in \left[ \frac{1}{2n}, \frac{2}{n} \right]. \]

This proves inequality (3).

Finally we have to show that in (1) the constant \( e/2 \) cannot be replaced by a smaller number.

We assume that the inequality
\[ \frac{1}{n} \sum_{v=1}^{n} x_v < c \prod_{v=1}^{n} x_v^{1/n} \quad \text{(8)} \]
holds for every sequence \( (x_v) \) \((v = 1, 2, \ldots)\) of positive real numbers such that \( (x_v) \) \((v = 0, 1, \ldots)\) with \( x_0 = 0 \) is concave.

Let \( x_v = v/n \) \((v = 0, 1, \ldots; n \in \mathbb{N})\). Then we obtain from (8):
\[ \frac{n+1}{2n} < c \frac{(n!)^{1/n}}{n} \quad \text{or} \quad \frac{n+1}{2(n!)^{1/n}} < c. \]
If $n$ tends to $\infty$ we get:

$$e/2 \leq c.$$ 

Hence, the constant $e/2$ is best possible in (1).

REFERENCES


