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CONSTRUCTING BIRATIONAL GAMES WITH GIVEN EQUILIBRIUM POINTS

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ABSTRACT. This work is addressed to the problem of constructing birational games with predetermined equilibrium points. We develop techniques which generalize those introduced for bimatrix games.

A necessary and sufficient condition for a pair of strategies to be a unique equilibrium point of a birational game is given.

KEY WORDS. Two-person games. Birational games. Equilibrium points. Constructing birational games. Uniqueness.

I. INTRODUCTION

This work is concerned with presenting techniques for constructing birational games with predetermined equilibrium points. These techniques are similar to the method for constructing bimatrix games.

A birational game is defined by a quadruple (A,B;C,D) of real m×n matrices together with the Cartesian product X×Y of all

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m-dimensional probability vectors X and all n-dimensional probability vectors Y. When $B=D=J_{m,n}$, where $J_{m,n}$ is the mxn matrix with every element equal to 1, a birational game reduces to an ordinary bimatrix game.

If $(x,y) \in X \times Y$, the payoffs of the game (A,B;C,D) for player i, i = 1,2, is defined by $E(x,y) = xAy/xBy$ and $F(x,y) = xCy/xDy$, respectively. $E(x,y)$ ($F(x,y)$) is defined (though possibly equal to $+\infty$) provided its numerator and denominator are not simultaneously equal to 0.

A point (x,y) in $X \times Y$ is an equilibrium point of the game (A,B;C,D) if $xAy/xBy \geqslant \xi Ay/\xi By$ for all $\xi \in X$ and $xCy/xDy \geqslant$ $\geqslant xC\eta/xD\eta$ for all $\eta \in Y$. Marchi [1976] proved that every rational game (A,B;C,D) with $B > 0$ and $D > 0$ (i.e., every element b_{ij}, d_{ij} of B,D are positive real) has an equilibrium point.

Von Neumann [1937] in the course of analyzing a model of economic growth, has been the first in considering a two-person zero-sum game with nonlinear payoff function. Subsequent development of the model has been synthesized by Morgenstern and Thompson [1976]. The same payoff function appears in a special case of a stochastic game proposed by Shapley [1953]. Marchi [1976] extended and generalized the equilibrium points of a rational game to a n-person game with a rational payoff function. Marchi [1979] and Marchi, Tarazaga, Elorza [1983-4] applied such results to obtain a new approach to expanding economies.

II. CONSTRUCTING A GAME WITH GIVEN EQUILIBRIUM POINTS

Let (x,y) be an equilibrium point, with $\alpha = xAy/xBy$ and $\beta = xCy/xDy$ the corresponding payoffs. Denote by S(x), S(y), M(x) and M(y) the following sets:

$$S(x) = \{i: x_i > 0\} \qquad S(y) = \{j: y_j > 0\}$$

$$M(y) = \{i: A_{i.}y = \alpha B_{i.}y\} \quad \text{and} \quad M(x) = \{j: xC_{.j} = \beta xD_{.j}\}$$

where $A_{i.}$ ($B_{i.}$) is the ith row of A (B) and $C_{.j}$ ($D_{.j}$) the jth column of C (D).

THEOREM 2.1. *Let* (x,y) *represent a pair of probability vectors in* $X \times Y$, *let* α,β *be real numbers and let* $B,D > 0$ *be* $m \times n$ *real matrices. There exist two matrices* A *and* C *such that the birational game* $(A,B;C,D)$ *has* (x,y) *as an equilibrium point and* α,β *as the corresponding payoff to the players.*

Proof. Without loss of generality we assume that $\alpha > 0$ and $\beta > 0$ (because if $xAy/xBy < 0$ there exists $c > 0$ such that $x(A+cB)y/xBy > 0$; a similar result is valid for β). Regard x and y as linear transformations from E^m and E^n to E^1, respectively. Let $c_1,\dots,c_{m-1}$ and $a_1,\dots,a_{n-1}$ be bases respectively, for the nullspaces of x and y. For $i \in S(x)$ let $A_{i.} = \alpha B_{i.} + \sum_{k=1}^{n-1} \lambda_{ik} a_k$, where λ_{ik} are arbitrary real numbers and for $j \in S(y)$, let $C_{.j} = \beta D_{.j} + \sum_{k=1}^{m-1} \gamma_{kj} c_k$, where γ_{kj} are arbitrary real numbers. For $i \notin S(x)$, let $A_{i.} = (1/2)\alpha B_{i.}$. For $j \notin S(y)$, let $B_{.j} = (1/2)\beta B_{.j}$. Then the birational game $(A,B;C,D)$ has an equilibrium point (x,y).

A point (x,y) of $X \times Y$ is said to be a completely mixed point if $x_i > 0$, $i = 1,\dots,m$, and $y_j > 0$, $j = 1,\dots,n$.

THEOREM 2.2. *Let* (x,y) *be a completely mixed point and* $B,D > 0$. *A necessary and sufficient condition for the existence of a birational game* $(A,B;C,D)$ *that has* (x,y) *as its unique completely-mixed equilibrium point is that* $m = n$.

Proof. Assume $m = n$. Choose, as before, a basis $a_1,\dots,a_{n-1}$ for the nullspace of y. Let $c_1,\dots,c_{n-1}$ be a basis for the nullspace of x. Let A be the $n \times n$ matrix whose i-th row is $A_{i.} = \alpha B_{i.} + a_i$, $i = 1,\dots,n-1$. Let $A_{n.} = \alpha B_{n.}$. Let C be the $n \times n$ matrix whose j-th column is $C_{.j} = \beta D_{.j} + c_j$ $j = 1,\dots,n-1$. Let $C_{.n} = \beta C_{.n}$. Then $(A,B;C,D)$ has an equilibrium point (x,y). Let (x^*,y^*) be another completely-mixed equilibrium point with payoffs α^*, β^*. Then $Ay^* = \alpha^* By^*$ and $x^*C = \beta^* x^* D$, in particular $A_{n.}y^* = \alpha^* B_{n.}y^*$ and $x^*C_{.n} = \beta^* x^* D_{.n}$. It follows from the definitions of $A_{.n}$ and $B_{.n}$ that $\alpha^* = \alpha$ and $\beta^* = \beta$; but the systems $(A-\alpha B)z = 0$ and $(C-\beta D)w = 0$ have rank $n-1$, implying $x = x^*$ and $y = y^*$.

The proof of the necessity is similar to the proof given by Millham [1973, Theorem 2].

We generalize Theorem 2.2 for arbitrary probability vectors.

THEOREM 2.3. *Let* (x,y) *be a pair of probability vectors and* $B,D > 0$. *A necessary and sufficient condition for the existence of a birational game that has* (x,y) *as its unique equilibrium point is that* $|S(x)| = |S(y)|$.

We omit the proof since it is similar to that of Kreeps [1974]. In our case we have to replace systems (1) and (2) in Kreeps' paper by the following

$$\left.\begin{aligned} \sum_{i=1}^{m} x_i(c_{ij}-c_{i1}-\beta(d_{ij}-d_{i1})) &= 0 \qquad j = 2,\ldots,k_2. \\ \sum_{i=1}^{m} x_i(c_{ij}-c_{i1}-\beta(d_{ij}-d_{i1})) &\leq 0 \qquad j = k_2+1,\ldots,n. \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} \sum_{j=1}^{n} y_j(a_{ij}-a_{1j}-\beta(b_{ij}-b_{1j})) &= 0 \qquad i = 2,\ldots,k_1. \\ \sum_{j=1}^{n} y_j(a_{ij}-a_{1j}-\beta(b_{ij}-b_{1j})) &\leq 0 \qquad j = k_1+1,\ldots,m. \end{aligned}\right\} \quad (2)$$

Under the conditions given by Millhan [1973, Theorem 4] two equilibrium points in bimatrix games are interchangeable. This result is found to hold true also for birational games.

THEOREM 2.4. *Let* $(x^1,y^1,\alpha^1,\beta^1),(x^2,y^2,\alpha^2,\beta^2)$ *be two equilibrium points and payoffs for a birational game* $(A,B;C,D)$. *A necessary and sufficient condition for* $(x^1,y^1,\alpha^1,\beta^1)$, $(x^2,y^2,\alpha^2,\beta^2)$ *to be interchangeable is* $S(x^1) \subseteq M(y^2)$, $S(x^2) \subseteq M(y^1)$, $S(y^1) \subseteq M(x^2)$, $S(y^2) \subseteq M(x^1)$.

Proof. Suppose (x^1,y^2) and (x^2,y^1) are equilibrium points. Then $x_i^1 > 0$ implies $A_{i.}y^2 = \alpha^2 B_{i.}y^2$ or $S(x^1) \subseteq M(y^2)$, and similarly $x_i^2 > 0$ implies $A_{i.}y^1 = \alpha^1 B_{i.}y^2$ or $S(x^2) \subseteq M(y^1)$. The remainder of the necessity aspects of the proof are clear.

Suppose, on the other hand, that the given conditions holds. The condition $S(x^1) \subseteq M(y^2)$ implies that if $A_{i.}y^2 < \alpha^2 B_{i.}y^2$ then $x_i^1 = 0$, and the condition $S(y^2) \subseteq M(x^1)$ implies that if $x^1 C_{.j} = \beta^1 x^1 D_{.j}$ then $y_j^2 = 0$, from which if follows that (x^1,y^2) is an equilibrium point with payoffs α^2 and β^1. The rest of the proof is identical in nature to that stated.

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