ABSTRACT. Perfect and proper equilibrium points have been refined very recently by García Jurado and Prada Sánchez who introduced the notions of strongly proper and perfectly proper equilibria. Here we further refine such concepts to strongly perfectly proper equilibrium points, we prove their existence and that they constitute a proper refinement.

I. INTRODUCTION

One of the most important solutions introduced in non-cooperative normal games is the concept of equilibrium point due to Nash [6]. Generally this point is not unique, for this reason it is convenient to establish criteria selecting the most convenient and intuitive ones.

In this way Selten [1975] introduced the concept of perfect equilibrium to eliminate solutions which are not stable against any arbitrarily slight perturbation of the game strategies. Only those equilibria which are self-enforcing under some arbitrarily slight player mistakes are acceptable.

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Myerson [1978] went further. He assumed that the players err "rationally" in the sense that their errors have probabilities that fall as the cost of the error rises. Myerson's concept of proper equilibrium eliminates those Nash equilibria that are not stable under any arbitrarily slight rational player mistake.

Recently García Jurado [1988] went further and he considered not only that players will tend to err with more probability towards that which cost less. But that besides it will tend to err more those players having a lower the cost. He introduced the concept of perfectly proper equilibrium point and proved its existence.

In another note García Jurado and Prada Sánchez [3] have introduced the concept of strongly proper equilibrium to rule out Nash equilibria that are not stable under any arbitrarily slight rational players error. Meaning rational is assuming that errors with equal cost have equal probabilities.

In this paper we introduce the concept of strongly perfectly proper equilibrium point, by short spp equilibrium point joining together the last two aspects.

2. CONCEPTS

In this section we introduce our notation and define the equilibrium concepts introduced by Selten, Myerson, García Jurado and García Jurado - Prada Sánchez.

Let \( \Gamma \) be a \( n \)-person game in normal form, i.e.

\[
\Gamma = (\phi_1, \ldots, \phi_n ; H_1, \ldots, H_n)
\]

where \( \phi_i \) is the finite set of pure strategies of player \( i \) and \( H_i \) defined from \( \phi_1 \times \ldots \times \phi_n \) to the reals \( \mathbb{R} \) is his payoff function. Let \( S_i \) be the set of mixed strategies of player \( i \), i.e.

\[
S_i = \left\{ s_i \in \mathbb{R}^{\phi_i} : \sum_{\phi_i \in \phi_i} s_i(\phi_i) = 1 \quad s_i(\phi_i) \geq 0 \quad \forall \phi_i \in \phi_i \right\}
\]
An $s_i \in S_i$ is completely mixed if $s_i(\phi_i) > 0 \ \forall \ \phi_i \in \phi_i$. Each $s = (s_1, \ldots, s_n) \in S = S_1 \times \cdots \times S_n$ is termed a combination of strategies, and for $\overline{s_i} \in S_i$, the combination of strategies $(s_1, \ldots, s_{i-1}, \overline{s_i}, s_{i+1}, \ldots, s_n)$ is denoted by $s \setminus \overline{s_i}$. An $s = (s_1, \ldots, s_n) \in S$ is completely mixed if all its components $s_i$ are. We recall that the set of pure strategies $\phi_i$ can be treated as a subset of $S_i$ by identifying each $\phi_i$ with that $s_i \in S_i$ for which $s_i(\phi_i) = 1$ and $s_i(\phi'_i) = 0$ for all $\phi'_i \in \phi_i \setminus \{\phi_i\}$.

The extension of $H_i$ to $S$ is defined by

$$H_i(s) = H_i(s_1, \ldots, s_n) = \sum_{(\phi_1, \ldots, \phi_n) \in \phi_1 \times \cdots \times \phi_n} H_i(\phi_1, \ldots, \phi_n) \prod_{j=1}^{n} s_j(\phi_j)$$

and $s_i, \overline{s_i} \in S_i$ are said to be payoff-equivalent for player $i$ if

$$H_i(s \setminus s_i) = H_i(s \setminus \overline{s_i}) \ \forall \ s \in S.$$ 

$\phi_i \in \phi_i$ is a best pure response to $s \in S$ for player $i \in N$ if

$$H_i(s \setminus \phi_i) = \max_{\phi_i \in \phi_i} H_i(s \setminus \phi_i).$$

The set of all player $i$'s best pure responses to $s$ will be denoted by $B_i(s)$. $\overline{s_i} \in S_i$ is a best response to $s \in S$ for player $i$ if for all $s_i \in S_i$

$$H_i(s \setminus s_i) \geq H_i(s \setminus \overline{s_i}).$$

$\overline{s} = (\overline{s_1}, \ldots, \overline{s_n}) \in S$ is a best response to $s \in S$ if for all $i$, $\overline{s_i}$ is a best response to $s$ for player $i$. $s \in S$ is a Nash equilibrium if it is a best response to itself.

An $s = (s_1, \ldots, s_n) \in S$ is an $\varepsilon$-perfect equilibrium if it is completely mixed and, $\forall \ \phi_i, \ \forall \overline{\phi_i}$ and $\forall i$...
An $s = (s_1, \ldots, s_n) \in S$ is a perfect equilibrium if there exists a pair of sequences $\{\epsilon_k\}_{k=1}^\infty$ and $\{s^k\}_{k=1}^\infty = \{(s_1^k, \ldots, s_n^k)\}_{k=1}^\infty$ such that:

a) $\epsilon_k > 0 \ \forall \ \kappa, \ \lim_{k \to \infty} \epsilon_k = 0$

b) $s_k$ is an $\epsilon_k$-perfect equilibrium $\forall \ \kappa$.

c) $\lim_{k \to \infty} s^k_i(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \Phi_i$ and $\forall \ i$.

An $s = (s_1, \ldots, s_n) \in S$ is an $\epsilon$-proper equilibrium if it is completely mixed and, $\forall \ \phi_i, \ \forall \ \overline{\phi}_i \in \Phi_i$ and $\forall \ i$

$$H_i(s \setminus \phi_i) < H_i(s \setminus \overline{\phi}_i) \Rightarrow s_i(\phi_i) < \epsilon s_i(\overline{\phi}_i).$$

An $s = (s_1, \ldots, s_n) \in S$ is a proper equilibrium if there exists a pair of sequences $\{\epsilon_k\}_{k=1}^\infty$ and $\{s^k\}_{k=1}^\infty = \{(s_1^k, \ldots, s_n^k)\}_{k=1}^\infty$ such that:

a) $\epsilon_k > 0 \ \forall \ \kappa, \ \lim_{k \to \infty} \epsilon_k = 0$

b) $s_k$ is an $\epsilon_k$-proper equilibrium $\forall \ \kappa$.

c) $\lim_{k \to \infty} s^k_i(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \Phi_i$ and $\forall \ i$.

A strongly $\epsilon$-proper equilibrium is a point $s = (s_1, \ldots, s_n) \in S$ if:

a) It is $\epsilon$-proper equilibrium

b) $\forall \ \phi_i, \ \overline{\phi}_i \in \Phi_i \setminus B_i(s)$ and $\forall \ i, \ s_i(\phi_i) = s_i(\overline{\phi}_i)$ if $\phi_i$ and $\overline{\phi}_i$ are payoff-equivalent for player $i$.

An $s = (s_1, \ldots, s_n) \in S$ is a strongly proper equilibrium point if there exists a pair of sequences $\{\epsilon_k\}_{k=1}^\infty$ and $\{s^k\}_{k=1}^\infty = \{(s_1^k, \ldots, s_n^k)\}_{k=1}^\infty$ such that:
a) \( \varepsilon_k > 0 \ \forall \ k, \lim_{k \to \infty} \varepsilon_k = 0 \)

b) \( s_k \) is a strongly \( \varepsilon_k \) - proper equilibrium \( \forall \ k \)

c) \( \lim_{k \to \infty} s_k(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \phi_i \) and \( \forall \ i. \)

García Jurado and Prada Sánchez in [3] have proved the existence of strongly proper equilibrium point which is a proper refinement of the concept of proper equilibrium point. They gave the following example

<table>
<thead>
<tr>
<th></th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 )</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_3 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( \alpha_4 )</td>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

The points \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_1)\) are proper equilibrium points but only the point \((\alpha_1, \beta_1)\) is strongly proper.

A point \( s = (s_1, \ldots, s_n) \) is an \( \varepsilon \) - perfectly proper equilibrium point if and only if:

a) \( s \) is completely mixed

b) If \( H_i(s \setminus \phi_i) - H_i(s \setminus \phi_j) < H_j(s \setminus \phi_i) - H_j(s \setminus \phi_j) \) with \( \phi_i \in B_i(s) \), \( \phi_j \in B_j(s) \) then \( s_j(\phi_j) < \varepsilon \ s_i(\phi_i) \) \( \forall \ \phi_i \in \phi_i \), \( \forall \ \phi_j \in \phi_j \) \( \forall \ i, j \in \{1, \ldots, n\} \).

It is said that a point \( s = (s_1, \ldots, s_n) \in S \) is a perfectly proper equilibrium point if and only if there exists a pair of sequences \( \{\varepsilon_k\}_{k=1}^{\infty} \) and \( \{s_k\}_{k=1}^{\infty} = \{(s_1^k, \ldots, s_n^k)\}_{k=1}^{\infty} \) such that

a) \( \varepsilon_k > 0 \ \forall \ k, \lim_{k \to \infty} \varepsilon_k = 0 \)

b) \( s_k \) is \( \varepsilon_k \) - perfectly proper \( \forall \ k \)
c) \( \lim_{k \to \infty} s_i^k(\phi_i) = s_i(\phi_i) \ \forall \ \phi_i \in \Phi_i \ \forall \ i \in \{1, \ldots, n\}. \)

García Jurado in [2] has introduced a concept properly refining the concept of proper equilibrium point.

In his doctoral thesis Van Damme [1983] shows the fact that enlarging dominated strategies the set of proper equilibrium might enlarge too. He used the following example

\[
\begin{array}{ccc}
\alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cc}
\alpha_2^1 & \alpha_2^2 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\]

It is clear that the unique proper equilibrium point in \( \Gamma_1 \) is \( (\alpha_1^1, \alpha_1^2) \). Consider the game \( \Gamma_2 \) which is obtained enlarging a strategy strictly dominated for the third player in the game \( \Gamma_1 \),

\[
\begin{array}{ccc}
\alpha_1^1 & \alpha_1^2 & \alpha_1^3 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cc}
\alpha_2^1 & \alpha_2^2 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\]

In \( \Gamma_2 \) the points \( (\alpha_1^1, \alpha_1^2, \alpha_1^3) \) and \( (\alpha_1^1, \alpha_2^2, \alpha_1^3) \) are proper equilibrium points. García Jurado in [6] proved that the point \( (\alpha_1^1, \alpha_2^2, \alpha_1^3) \) is not perfectly proper.

3. STRONGLY PERFECTLY PROPER EQUILIBRIUM POINT

As we mentioned in the Introduction we introduce a further concept which will result to be a proper refinement of the
concept of perfectly proper equilibrium point.

We say that a point \( s = (s_1, \ldots, s_n) \in S \) is \( \epsilon \) - strongly perfectly proper equilibrium point or briefly \( \epsilon \) - s.p.p. equilibrium point if:

a) \( s \) is completely mixed

b) if \( H_i(s\phi_i) - H_i(s\bar{\phi}_i) < H_j(s\phi_j) - H_j(s\bar{\phi}_j) \) with \( \phi_i \in B_i(s) \) and \( \phi_j \in B_j(s) \) then \( s_j(\bar{\phi}_j) \leq \epsilon s_i(\bar{\phi}_i) \) \( \forall \bar{\phi}_i \in \phi_i \),

\( \forall \bar{\phi}_j \in \phi_j \) \( \forall i, j \in \{1, \ldots, n\} \)

c) \( \forall \phi_i, \bar{\phi}_i \in \phi_i \cap B_i(s) \) and \( \forall i, s_i(\phi_i) = s_i(\bar{\phi}_i) \) if \( \phi_i \) and \( \bar{\phi}_i \) are payoff-equivalent for player \( i \).

A point \( s = (s_1, \ldots, s_n) \in S \) is called to be strongly perfectly proper equilibrium point or briefly s.p.p. equilibrium point if there exists a pair of sequences

\( \{\epsilon_k\}_{k=1}^{\infty}, \{(s^k_1, \ldots, s^k_n)\}_{k=1}^{\infty} = \{s_k\}_{k=1}^{\infty} \)

such that

a) \( \forall k \epsilon_k > 0; \lim_{k \to \infty} \epsilon_k = 0 \)

b) \( \forall k s_k \) is a \( \epsilon \) - s.p.p. equilibrium point

c) \( \lim_{k \to \infty} s^k_1(\phi_i) = s_1(\phi_i) \) \( \forall \phi_i \in \phi_i \), \( \forall i \in \{1, \ldots, n\} \).

It is clear that a s.p.p. equilibrium point is a perfectly proper equilibrium point. The inverse is not true in general. Consider the example

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 \\
\alpha_1 & 2 & 2 & 1 \\
\alpha_2 & 2 & 2 & 1 \\
\alpha_3 & 1 & 3 & 01 \\
\alpha_4 & 1 & 1 & 3 \\
\alpha_5 & 1 & 1 & 1 \\
\end{array}
\]

\[
B_1 \quad B_2 \quad B_3 
\]

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 \\
\alpha_1 & 2 & 1 & 1 \\
\alpha_2 & 2 & 1 & 1 \\
\alpha_3 & 1 & 3 & 01 \\
\alpha_4 & 1 & 1 & 3 \\
\alpha_5 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 \\
\alpha_1 & 2 & 1 & 1 \\
\alpha_2 & 2 & 1 & 1 \\
\alpha_3 & 1 & 3 & 01 \\
\alpha_4 & 1 & 1 & 3 \\
\alpha_5 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 \\
\alpha_1 & 2 & 1 & 1 \\
\alpha_2 & 2 & 1 & 1 \\
\alpha_3 & 1 & 3 & 01 \\
\alpha_4 & 1 & 1 & 3 \\
\alpha_5 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
\phi_1 & \phi_2 & \phi_3 \\
\alpha_1 & 2 & 1 & 1 \\
\alpha_2 & 2 & 1 & 1 \\
\alpha_3 & 1 & 3 & 01 \\
\alpha_4 & 1 & 1 & 3 \\
\alpha_5 & 1 & 1 & 1 \\
\end{array}
\]
It is easy to check that the points \((\mu a_1 + (1-\mu)a_5; \beta_1)\), \(\mu \in [0,1]\) and \((a_2, \beta_1)\) are perfectly proper equilibrium points. Consider first the point \((a_2, \beta_1)\). Taking the sequences

\[\varepsilon_k = \frac{1}{k+2}\]

\[s_1^k(a_1) = s_1^k(a_5) = \frac{1}{2(k+2)} \quad k = 4,5, \ldots\]

\[s_1^k(a_2) = 1 - \frac{300(k+2)^2+151}{300(k+2)^3}; \quad s_1^k(a_3) = \frac{1}{300(k+2)^3}; \quad s_1^k(a_4) = \frac{1}{2(k+2)^3}\]

\[s_2^k(\beta_1) = 1 - \frac{k+3}{2(k+2)^3}; \quad s_2^k(\beta_2) = \frac{1}{2(k+2)^3}; \quad s_2^k(\beta_3) = \frac{1}{2(k+2)^2}\]

then we have

\[H_1(s_1^k(a_2)) - H_1(s_1^k(a_3)) = 1 - \frac{1}{(k+2)^3}\]

\[H_1(s_1^k(a_2)) - H_1(s_1^k(a_1)) = \frac{k+1}{2(k+2)^3}\]

and

\[H_1(s_1^k(a_2)) - H_1(s_1^k(a_3)) = H_1(s_1^k(a_2)) - H_1(s_1^k(a_1)) < H_1(s_1^k(a_2)) - H_1(s_1^k(a_3))\]

\[s_1^k(a_3) \leq \varepsilon_k s_1^k(a_1) \quad \text{and} \quad s_1^k(a_3) \leq \varepsilon_k s_1^k(a_5)\]

hold true. The same for \(a_1\) and \(a_4\). For the second player

\[H_2(s_2^k(\beta_1)) - H_2(s_2^k(\beta_2)) = 1 - \frac{1}{(k+2)^3}\]

\[H_2(s_2^k(\beta_1)) - H_2(s_2^k(\beta_3)) = 1 - \frac{301}{300(k+2)^3}\]

and

\[H_2(s_2^k(\beta_1)) - H_2(s_2^k(\beta_3)) < H_2(s_2^k(\beta_1)) - H_2(s_2^k(\beta_2))\]

\[s_2^k(\beta_2) \leq \varepsilon_k s_2^k(\beta_3)\]

holds true.

Finally

\[H_1(s_1^k(a_2)) - H_1(s_1^k(a_3)) = H_1(s_1^k(a_2)) - H_1(s_1^k(a_1)) < H_2(s_2^k(\beta_1)) - H_2(s_2^k(\beta_2))\]
and

\[ H_2(s/\beta_1) - H_2(s/\beta_3) < H_1(s/\alpha_2) - H_1(s/\alpha_3) = H_1(s/\alpha_2) - H_1(s/\alpha_4) \]

\[ s_1^k(\alpha_3) < \varepsilon_k s_2^k(\alpha_3) \]

\[ s_1^k(\alpha_4) < \varepsilon_k s_2^k(\alpha_3) \]

and

\[ H_1(s/\alpha_2) - H_1(s/\alpha_3) = H_1(s/\alpha_2) - H_1(s/\alpha_1) < H_2(s/\beta_1) - H_2(s/\beta_3) \]

\[ s_2^k(\beta_3) < \varepsilon_k s_1^k(\alpha_1) \]

\[ s_2^k(\beta_3) < \varepsilon_k s_1^k(\alpha_3) \]

and in this way it is shown that the point \((\alpha_2, \beta_1)\) is a perfectly proper equilibrium point.

In an analogous way we will show that the point \((\mu \alpha + (1-\mu)\alpha_5, \beta_1)\) with \(\mu \in [0,1]\) is a perfectly proper equilibrium point. Indeed take the sequences \(\varepsilon_k = \frac{1}{k+2} \); \(s_1^k(\alpha_1) = s_1^k(\alpha_3) = \)

\[
= \frac{1}{2} \left[ 1 - \frac{300(k+2)+1}{300(k+2)-1} \right] ;
\]

\(s_2^k(\beta_3) = \frac{1}{(k+2)^2} \); \(s_2^k(\beta_3) = \frac{1}{(k+2)^2} \)

\[
= \frac{1}{300(k+2)^2} \left( 1 + \frac{1}{300(k+2)-1} \right) ;
\]

then we have

\[ H_1(s/\alpha_3) - H_1(s/\alpha_2) = H_1(s/\alpha_1) - H_1(s/\alpha_2) =
\]

\[
= \frac{1}{(k+2)^2} - \frac{1}{(k+2)^2} - \frac{1}{300(k+2)-1} \]

\[ H_1(s/\alpha_1) - H_1(s/\alpha_3) = H_1(s/\alpha_1) - H_1(s/\alpha_4) =
\]
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\[
1 = \frac{1}{(k+2)^2} - \frac{1}{(k+2)^2} - \frac{1}{300(k+2)-1}
\]

and

\[
H_1(s|\alpha_1) - H_1(s|\alpha_2) < H_1(s|\alpha_1) - H_1(s|\alpha_3) = H_1(s|\alpha_1) - H_1(s|\alpha_4)
\]

\[
s^k_1(\alpha_3) \leq \epsilon_k s^k_1(\alpha_2)
\]

\[
s^k_1(\alpha_4) \leq \epsilon_k s^k_1(\alpha_2).
\]

For the second player

\[
H_2(s|\beta_1) - H_2(s|\beta_2) = 1 - \frac{1}{(k+2)^2} \left(1 + \frac{1}{300(k+2)-1}\right)
\]

\[
H_2(s|\beta_1) - H_2(s|\beta_3) = 1 - \frac{3}{300(k+2)^2} \left(1 + \frac{1}{300(k+2)-1}\right)
\]

and

\[
H_2(s|\beta_1) - H_2(s|\beta_2) < H_2(s|\beta_1) - H_2(s|\beta_3)
\]

\[
s^k_2(\beta_3) \leq \epsilon_k s^k_2(\beta_2).
\]

Finally

\[
H_1(s|\beta_1) - H_1(s|\beta_2) < H_2(s|\beta_1) - H_2(s|\beta_2)
\]

\[
s^k_2(\beta_2) \leq \epsilon_k s^k_1(\alpha_2)
\]

\[
H_1(s|\alpha_1) - H_1(s|\alpha_2) < H_2(s|\beta_1) - H_2(s|\beta_3)
\]

\[
s^k_2(\beta_3) \leq \epsilon_k s^k_1(\alpha_2)
\]

and

\[
H_1(s|\alpha_1) - H_1(s|\alpha_3) < H_2(s|\beta_1) - H_2(s|\beta_3)
\]

\[
s^k_2(\beta_3) \leq \epsilon_k s^k_1(\alpha_3).
\]

Similarly for

\[
s^k_2(\beta_3) \leq \epsilon_k s^k_1(\alpha_4).
\]

The term \(H_1(s|\alpha_1) - H_1(s|\alpha_3)\) equals \(H_2(s|\beta_1) - H_2(s|\beta_2)\) and

\(H_1(s|\alpha_1) - H_1(s|\alpha_4)\) equals \(H_2(s|\beta_1) - H_2(s|\beta_2)\).
The strategy $\alpha_5$ does not have to appear in the inequalities since $s_1^k(\alpha_5) = s_1^k(\alpha_1)$.

Thus, the point $(\mu \alpha_1 + (1-\mu)\alpha_5, \beta_1)$ with $\mu \in [0,1]$ is a perfectly proper equilibrium point.

We have that perfectly proper equilibrium points are not always s.p.p. equilibrium points. For suppose otherwise, and let $(\epsilon_k)_{k=1}^{\infty}$ be a sequence satisfying the conditions of the definition and $(s_k)_{k=1}^{\infty} = \{(s_1^k,s_2^k)\}_{k=1}^{\infty}$ a sequence of strongly $\epsilon_k$ -perfectly proper equilibria converging to $(\alpha_2,\beta_1)$. Since

$$\lim_{k \to \infty} s_2^k(\beta_1) = 1 \text{ and } \lim_{k \to \infty} s_2^k(\beta_2) = \lim_{k \to \infty} s_2^k(\beta_3) = 0, \lim_{k \to \infty} s_1^k(\alpha_2) = 1;$$

$$\lim_{k \to \infty} s_1^k(\alpha_i) = 0 \text{ if } i \in \{1,3,4,5\} \text{ and } k \in \mathbb{N} \text{ such that } \forall \, k \geq K,$$

$$a_3, a_4 \notin B_1(s_k).$$

Hence $s_1^k(\alpha_3) = s_1^k(\alpha_4)$ if $k \geq K$, for the $s_k$ are strongly $\epsilon_k$ -perfectly proper and $a_3, a_4$ are payoff-equivalent for player 1. Accordingly, for $k \geq K$

$$H_2(s_k \backslash \beta_1) - H_2(s_k \backslash \beta_2) < H_2(s_k \backslash \beta_1) - H_2(s_k \backslash \beta_3)$$

which successively entails that $s_2^k(\beta_3) \leq \epsilon_k s_2^k(\beta_2)$.

This implies

$$H_1(s_k \backslash \alpha_2) < H_1(s_k \backslash \alpha_1)$$

which implies

$$H_1(s_k \backslash \alpha_1) - H_1(s_k \backslash \alpha_5) = 0 < H_1(s_k \backslash \alpha_1) - H_1(s_k \backslash \alpha_2)$$

and

$$s_1^k(\alpha_2) \leq \epsilon_k s_1^k(\alpha_5).$$

But this means that it is impossible that $\lim_{k \to \infty} s_1^k(\alpha_2) = 1$ which is a contradiction. Hence s.p.p. equilibrium points is a strict refinement of the concept of perfectly proper equilibrium.
4. EXISTENCE OF S.P.P. EQUILIBRIUM POINT

In this paragraph we will prove that any normal n-person game has an s.p.p. equilibrium point. The proof follows the ideas of García Jurado.

We first show that \( \forall \varepsilon_k \in (0,1) \exists \alpha_k \)-spp equilibrium \( s_k \).

Consider \( \varepsilon_k \in (0,1) \). Denoting \( |\phi_i| = m_i \), define,

\[
\forall \; i \in \{1,2,\ldots,n\}, \quad \gamma = \frac{\varepsilon_k}{\sum_{i=1}^{n} m_i}
\]

Consider the set \( S_i(\gamma) = \{s_i \in S_i \mid s_i(\phi_i) \geq \gamma \; \forall \; \phi_i \in \phi_i\} \)

\( \forall \; i \in \{1,2,\ldots,n\} \)

and let \( S(\gamma) = S_1(\gamma) \times \ldots \times S_n(\gamma) \).

Define now the multivalued function \( F: S(\gamma) \rightarrow P(S(\gamma)) \) given by

\[
F(s) = \{\sigma \in S(\gamma) \mid H_i(s \setminus \phi_i) - H_i(s \setminus \phi_i') < H_j(s \setminus \phi_j) - H_j(s \setminus \phi_j') \}
\]

with \( \phi_i, \phi_j \in B_i(s), \; \phi_j \in B_j(s) \Rightarrow \sigma_j(\phi_j) \leq \varepsilon_k \sigma_i(\phi_i') \; \forall \; \phi_i' \in \phi_i \)

\( \forall \; \phi_j' \in \phi_j, \; \forall \; i,j \sigma_i(\phi_i) = \sigma_i(\phi_i') \) payoff-equivalent for \( i, \phi_i', \phi_i \in \phi_i \backslash B_i(s) \} \).

\( S(\gamma) \) is compact convex and non-empty and \( F(s) \) is compact and convex \( \forall \; s \in S(\gamma) \). Moreover \( \forall \; s \in S(\gamma), \; F(s) \) is non-empty. Indeed, if we define

\[
A(s \setminus \phi_i') = \sum_{j=1}^{n} \{\phi_j' \in \phi_j \mid H_j(s \setminus \phi_j) - H_j(s \setminus \phi_j') < H_i(s \setminus \phi_i) - H_i(s \setminus \phi_i') \}
\]

with \( \phi_j \in B_j(s), \; \phi_i \in B_i(s) \} \; \forall \; \phi_i' \in \phi_i \; \forall \; i \in \{1,\ldots,n\} \)

and \( \sigma = (\sigma_1,\ldots,\sigma_n) \) such that for each player \( i \)
then \( \sigma \in F(s) \) and this for each \( s \in S(\gamma) \).

We observe that \( F \) is upper semicontinuous. By Kakutani's fixed point theorem (1941), \( F \) has a fixed point which is \( \varepsilon_k \) - s.p.p.

Now let \( \{ \varepsilon_k \}_{k=1}^{\infty} \) be a sequence such that it is possible to find a subsequence of \( \varepsilon_k \) - s.p.p. equilibrium points.

Because \( S \) is compact this subsequence converges to a point \( s \in S \) which is s.p.p. equilibrium point. (q.e.d.)

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