Boundedness of Fractional Integrals on Spaces of Homogeneous Type of Finite Measure

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I. Introduction

In this brief exposition we want to describe the main novelties in the extension of the theory of fractional integrals to spaces of homogeneous type with finite measure. The theory of fractional integrals was extended to $\mathbb{R}^n$ with Euclidean distance and a doubling weight in [GGW], and to homogeneous spaces of infinite measure in [GV1]. In [GV1] spaces of finite measure are considered, too, but the results apply only to function spaces with "homogeneous norms". This distinction between norms applies only to Lipschitz spaces and BMO, and due to duality, to $H^p$ spaces with $\frac{1}{1+\gamma} < p \leq 1$ where $\gamma$ is the order of the space. These spaces are, in fact, the ones where the novelties alluded to above occur, and we now list them.

A condition based on the smoothness of the function $1_{\alpha}$ turns out to be necessary and sufficient for boundedness of $I_\alpha$ on Lipschitz spaces and $H^p$ spaces, $p < 1$. The corresponding result for $H^1$ is related to the boundedness of $I_\alpha$ from BMO to Lip[\alpha]. In these two cases a sufficient condition is given which again is based on the smoothness of $I_\alpha$. Finally, for $H^p$ spaces the image under $I_\alpha$ of a general p-atom with or without vanishing integral may not have integral zero, but it can be shown that it is a molecule as defined in section II below. In section II we state the results in detail, and in section III we sketch the proofs. Complete proofs will appear in [GV2].

II. Statement of Results

In the sequel $(X, \delta, \mu)$ is a normal space of homogeneous type of finite measure and of order $\gamma$, $0 < \gamma \leq 1$. We may, and will assume that $\mu(X) = 1$. The diameter of the space is finite and will be denoted by $D$. $B_r(x)$ will denote the ball of center $x$ and radius $r$, i.e., $B_r(x) = \{y \in X : \delta(x, y) < r\}$. In order to define the kernel of the fractional integral without having to distinguish the case when the measure $\mu$ has atoms we shall adopt the following abuse of notation. If $0 < \alpha < 1$, we define
\[
\frac{1}{\delta(x,y)^{1-\alpha}} = \begin{cases} 
\delta(x,y)^{1-\alpha} & \text{if } x \neq y \\
0 & \text{if } x = y
\end{cases}
\]

The fractional integral of order \( \alpha, 0 < \alpha < 1 \), on measurable functions \( f \) is defined by

\[
I_\alpha f(x) = \int_X f(y) d\mu(y) \delta(x,y)^{1-\alpha}
\]

We shall now give the definition of molecules:

Let \( \epsilon > 0 \) and \((1 + \gamma)^{-1} < s < 1\). A measurable function \( M \) on \( X \) is an \( s \)-molecule if there exist a point \( x_0 \) in \( X \) and constants \( L > 0 \) and \( r > 0 \) such that

\[
|M(x)| \leq L \mu(B_r(x_0))^{-1/s} \quad \text{for all } x,
\]

(2.1)

\[
|M(x)| \delta(x_1,x_0)^{1+\epsilon} \leq L \mu(B_r(x_0))
\]

(2.2)

and

\[
|\int_X M d\mu| \leq L.
\]

(2.3)

For \( s = 1 \), (2.1) and (2.2) are the same but (2.3) is replaced by

\[
|\int_X M d\mu| \leq \frac{L}{1 + |\log \mu(B_r(x_0))|}
\]

(2.4)

The non-homogenous Lipschitz norms and BMO norm mentioned in the introduction are

\[
\|f\|_{Lip[\gamma]} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\delta(x,y)^{\gamma}}
\]

\[
\|f\|_1^* = \|f\|_1 + \sup_B m_B(|f - m_B(f)|)
\]

where \( m_B(h) = \frac{1}{\mu(B)} \int_B h(x) d\mu(x) \), and \( B \) ranges over all balls in \( X \). The letter \( c \) will denote a constant, not necessarily the same at each occurrence.

We now state the theorems:

Theorem 1. Let \((1 + \gamma)^{-1} < s \leq 1\) and let \( M \) be an \( s \)-molecule. Then \( M \) belongs to \( H^s \) and

\[
\|M\|_{H^s} \leq C.
\]

Where \( C \) is a constant that depends only on \( s, \epsilon \) and \( L \) for \( s < 1 \), and on \( \epsilon \) and \( L \) for \( s = 1 \).
Theorem 2. Let \( 0 < \alpha + \beta < \gamma \), and \( f \in \text{Lip}[\beta] \) then the following statements are equivalent:

i) \( I_\alpha f \in \text{Lip}[\alpha + \beta] \).

ii) \( I_\alpha f(x) \) converges absolutely for every \( x \) and there is a constant \( c \) independent of \( f \) such that

\[
\|I_\alpha f\|_{\text{Lip}[\alpha + \beta]} \leq c\|f\|_{\text{Lip}[\beta]}.
\]

Theorem 3. Let \( 0 < \alpha < \gamma \) and \( f \in \text{BMO} \). If \( I_\alpha f \) satisfies

\[
\sup_{x \neq y} \frac{|I_\alpha f(x) - I_\alpha f(y)|}{\delta(x, y)^\alpha} \leq c_{I_\alpha} < \infty
\]

then \( I_\alpha f(x) \) converges absolutely for every \( x \) in \( X \), and there exists a constant \( c \) independent of \( f \) such that

\[
\|I_\alpha f\|_{\text{Lip}[\alpha]} \leq c\|f\|^*.
\]

Theorem 4. Let \( (1 + \gamma)^{-1} < p < 1 \), \( 0 < \alpha < \gamma \) and \( 1 < \frac{1}{q} = \frac{1}{p} - \alpha \). Then the following statements are equivalent:

i) \( I_\alpha f \in \text{Lip}[\frac{1}{p} - 1] \)

ii) For any \( p \)-atom \( a \), \( I_\alpha a \) belongs to \( H^q \), and there exists a constant \( c \) independent of \( a \) such that

\[
\|I_\alpha a\|_{H^q} \leq c,
\]

and \( I_\alpha \) extends to a continuous linear map from \( H^p \) to \( H^q \).

Theorem 5. Let \( 0 < \alpha < \gamma \). If \( I_\alpha f \) satisfies (2.5) then for any \( (1 + \alpha)^{-1} \)-atom \( a \), \( I_\alpha a \) belongs to \( H^1 \), there is a constant \( c \) independent of \( a \) such that

\[
\|I_\alpha a\|_{H^1} \leq c,
\]

and \( I_\alpha \) extends to a continuous linear map from \( H^{1/(1+\alpha)} \) to \( H^1 \).

III. Sketch of the Proofs.

Theorem 1. We begin by decomposing the molecule \( M \) as the following sum

\[
M(x) = M_o(x) + J \frac{1}{\mu(B)} \chi_B(x),
\]

where \( J = \int M d\mu \) and \( \chi_B \) is the characteristic function of \( B = B_r(x_0) \). The function \( M_o \) is a molecule with vanishing integral and therefore by the results of Coifman and Weiss [GW] it is in \( H^* \), and \( \|M_o\|_{H^*} \leq C_1 \). The second term is a particular molecule with non-vanishing integral. Its \( H^* \) norm can be estimated using the \( \beta \)-maximal function of Macias and Segovia [MS2] or, as suggested by Professor Taibleson at this conference, by decomposing it into a
finite sum of atoms in the following way. Let $B_k = B_{2^k r}(x_0), k = 0, 1, ... K$, where $B_K = X$ but $B_{K-1} \neq X$. It is easy to see that $K \leq c_1 (1 + |\log \mu(B)|)$. For $k = 1, 2, ... K$, let

$$b_k = \frac{1}{\mu(B_{k-1})} \chi_{B_{k-1}} - \frac{1}{\mu(B_k)} \chi_{B_k}.$$ 

Then $b_k = \lambda_k a_k$, where $\lambda_k = \frac{\mu(B_k)^{1/s}}{\mu(B_{k-1})}$ and $a_k$ is an s-atom. One checks that

$$J \frac{\chi_B}{\mu(B)} = \sum_{k=1}^{K} J \lambda_k a_k + J.$$ 

From this decomposition it follows that

$$\| J \frac{1}{\mu(B)} \chi_B \|_{H^s} \leq C_2.$$ 

The constants $C_1$ and $C_2$ depend only on $L e_1$ and $s$.

Theorem 2. The implication $ii) \Rightarrow i)$ is immediate from the fact that $1 \in \text{Lip}_B[\beta]$. To prove that $i) \Rightarrow ii)$ we write for $x_1 \neq x_2$

$$I_\alpha f(x_1) - I_\alpha f(x_2) = \int X \frac{1}{\delta(x_1, y)^{1-\alpha}} - \frac{1}{\delta(x_2, y)^{1-\alpha}} \delta(y) - f(x_1) d\mu(y)$$

$$+ f(x_1) [I_\alpha 1(x_1) - I_\alpha 1(x_2)].$$

The first term is estimated by a standard argument and we obtain that its absolute value is less than or equal to $c \| f \|_{\text{Lip}[\beta]} \delta(x_1, x_2)^{\alpha+\beta}$. Using $i)$ we show that the second term in absolute value is less than or equal to $c \| f \|_{\infty} \delta(x_1, x_2)^{\alpha+\beta}$. On the other hand $\| I_\alpha f \|_{\infty} \leq c \| f \|_{\infty}$ because the kernel of $I_\alpha$ is in $L^1$.

Theorem 3. For $x_1 \neq x_2$, let $r = \delta(x_1, x_2), B = B_r(x_1)$, and write

$$I_\alpha f(x_1) - I_\alpha f(x_2) =$$

$$\int X \frac{1}{\delta(x_1, y)^{1-\alpha}} - \frac{1}{\delta(x_2, y)^{1-\alpha}} \delta(y) - m_B(f) d\mu(y)$$

$$+ m_B(f) [I_\alpha 1(x_1) - I_\alpha 1(x_2)].$$

The first term is estimated by a standard argument, and we obtain that its absolute value is less than or equal to $c \delta(x_1, x_2)^\alpha \| f \|^{*}$. The estimate of the second term depends on the inequality

$$| m_B(f) | \leq c \| f \|^{*} (1 + |\log \mu(B)|)$$
valid for f in BMO. This inequality together with (2.5) imply that the second term in absolute value is less than or equal to $c\|f\|^*\delta(x_1, x_2)^\alpha$.

Theorem 4. The implication ii) $\Rightarrow$ i) follows easily from Theorem 2 and duality. To prove that i) $\Rightarrow$ ii) we first consider a p-atom $a$ whose support is contained in ball centered at $x_0$, with $\int a\,d\mu = 0$. The proofs of (2.1) and (2.2) are done by standard arguments. To prove (2.3) we use $\int a\,d\mu = 0$ and a change of order of integration to write

$$\int I_\alpha a(x)d\mu(x) = \int \left[ \int \frac{a(y)}{\delta(x_1, y)^{1-\alpha}} d\mu(y) \right] d\mu(x) = \int a(y) I_\alpha a(y) d\mu(y) = \int a(y) [I_\alpha a(y) - I_\alpha a(x_0)] d\mu(y).$$

Since $I_\alpha a_1$ belongs to $Lip_1^{1,1}$, and $\alpha$ is a p-atom in supported $B_r(x_0)$ we have

$$|\int I_\alpha a\,d\mu| \leq \|I_\alpha a\|_{Lip_1^{1,1}} r^{\frac{1}{2}} \int |a|\,d\mu \leq D^{1/p-1}\|I_\alpha a\|_{Lip_1^{1,1}} = L.$$

It is easy to see that $I_\alpha a_1$ is an $H^q$ molecule, too. It is then shown by standard arguments involving the atomic decomposition that $I_\alpha$ extends to all $H^p$ as a bounded map into $H^q$.

Theorem 5. Again let $a$ be a $(1 + \alpha)^{-1}$ - atom supported in a ball $B$ of center $x_0$. The proof of (2.1) and (2.2) are done by standard arguments. To prove (2.4) we use $\int a\,d\mu = 0$ and a change of order of integration to write, as in the proof of Theorem 4

$$\int I_\alpha a(x)d\mu(x) = \int a(y) [I_\alpha a(y) - I_\alpha a(x_0)] d\mu(y).$$

Now we use (2.5) to obtain

$$|\int I_\alpha a\,d\mu| \leq c \frac{1}{1 + |\log(\mu(B))|}. $$

It is easy to see that $I_\alpha a_1$ is a 1 - molecule. Finally the extension of $I_\alpha$ to all of $H^{\frac{1}{2+\alpha}}$ is proved as in Theorem 4.
References


