CHARACTERIZATION OF LIPSCHITZ SPACES VIA THE COMMUTATOR OPERATOR OF COIFMAN, ROCHBERG, AND WEISS

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Let $f \in L^1_{\text{loc}} \cap S'$, $0 < \beta < k \leq n$, $k$ an integer (in particular, $k = \lfloor \beta \rfloor + 1$), and $n$ - dimension of the ambient space. Let $K(x) = \Omega(x)/|x|^n$ be a Calderón-Zygmund kernel, i.e., $\int_{\mathbb{S}^{n-1}} \Omega = 0$, $\Omega \in C^\infty(\mathbb{S}^{n-1})$, $\Omega$ homogeneous of degree 0.

Define:
\[
C_{f,k}g(x) = \int_{\mathbb{R}^n} \Delta^k_h f(x)K(h)g(x + h) \, dh,
\]
\[
\bar{C}_{f,k}g(x) = \int_{\mathbb{R}^n} \Delta^k_{-h} f(x)K(h)g(x + h) \, dh,
\]
where $\Delta^k_h$ is the $k$-th difference operator; i.e., $\Delta^{k+1}_h f(x) = \Delta^k f(x + h) - \Delta^k f(x)$, $\Delta^k f(x) = f(x + h) - f(x)$.

We have the following theorem, which generalizes a theorem by Janson ([j]), dealing with the case $k = 1$.

**Theorem 1** Let $1 < p < q < +\infty; 1/p - 1/q = \beta/n$. Under above assumptions, the following are equivalent:

i) $f = f_1 + P$, with $f_1 \in \Lambda_\beta$ and $P$ a polynomial of degree $< k$.

ii) $C_{f,k} : L^p \to L^p$ is a bounded operator.

iii) $\bar{C}_{f,k} : L^p \to L^p$ is a bounded operator.

If, in particular $k = \lfloor \beta \rfloor + 1$, then i) says that $f \in \Lambda_\beta$.

The key ingredient in the proof is the following characterization of Lipschitz spaces, which appears to be new.

**Theorem 2** Suppose $f \in L^1_{\text{loc}} \cap S'$, $0 < \beta < k$, $k$ - integer (in particular, $k = \lfloor \beta \rfloor + 1$). Then, the following conditions are equivalent:

i) $f = f_1 + P$ with $f_1 \in \Lambda_\beta$ and $P$ a polynomial of degree $< k$.

ii) There exists an open set $U \subset \mathbb{R}^n$ such, that for every $x_0 \in \mathbb{R}^n$ and $t > 0$,
\[
\left| \frac{1}{|Q|} \int_{Q} \int_{Q} [\Delta^k_{y-z} f(x)] \, dy \, dz \right| \leq ct^\beta,
\]
with $c$ independent of $z \in U$, $x_0$ and $t$. Here, $Q = Q(z_0,t)$ (cube centered at $z_0$, sidelength $t$, sides parallel to the axes), and $Q' = Q(z_0 + zt,t)$.

iii) The same condition as ii), with (1) replaced by:
If any of these conditions hold (thus all hold) then \( \|f_1\|_{A_\phi} \) is comparable with smallest \( c \) in ii) and iii).

Proof of Theorem 2: i)\( \Rightarrow \)ii) \& i)\( \Rightarrow \)ii): This direction follows immediately from the pointwise estimate on the differences of \( f \). ii)\( \Rightarrow \)i) \& iii)\( \Rightarrow \)i): The key idea is to write

\[
\frac{1}{|Q| |Q'|} \int_Q \int_{Q'} \left[ \Delta_{\mu}^k f(x) \right] dy \, dz \leq c t^{\theta}.
\]  

(2)

for appropriate functions \( \mu \) and \( \nu \). Here \( \mu_1(x) = 1/t^n \mu(x/t) \); similarly for \( \nu \). It turns out, that if \( z_1, \ldots, z_n \) are linearly independent, then the vector valued functions \( (\hat{\mu}^{z_1}, \ldots, \hat{\mu}^{z_n}) \) and \( (\hat{\nu}^{z_1}, \ldots, \hat{\nu}^{z_n}) \) satisfy a so-called Tauberian condition; i.e., their absolute value is not identically 0 along any ray (see [1-t]). Thus, \( \mu = (\mu^{z_1}, \ldots, \mu^{z_n}) \) and \( \nu = (\nu^{z_1}, \ldots, \nu^{z_n}) \) are compactly supported, finite, vector valued Borel measures, such, that \( \hat{\mu} \) and \( \hat{\nu} \) satisfy Tauberian condition. We invoke now a version of Calderón's reproducing formula, due to Janson and Taibleson [1-t]):

Theorem 3 (Janson and Taibleson) There exist vector valued functions \( \eta = (\eta_1, \ldots, \eta_n) \) and \( \omega = (\omega_1, \ldots, \omega_n) \) such that \( \eta_i, \omega_i \in \mathcal{S}; i = 1, \ldots, n \) associated with \( \mu \) and \( \nu \) respectively, and an integer \( N \) (\( N \) is the order of \( f \) as a tempered distribution), so that:

\[
f = \int_0^\infty f * \mu_t * \eta_t \, dt, \quad \mu_t * \eta_t = \sum_{i=1}^n \mu_t^{i} * \eta_i t,
\]

\[
f = \int_0^\infty f * \nu_t * \omega_t \, dt, \quad \nu_t * \omega_t = \sum_{i=1}^n \nu_t^{i} * \omega_i t,
\]

both integrals converging in \( \mathcal{S}_N' = \mathcal{S}'/P_{N-1} \) where \( P_{N-1} \) is the space of polynomials of degree at most \( N - 1 \).

From this, following the argumentation in [1-t], we deduce, that both (3) and (4) (separately) imply that \( f = f_0 + P \), with \( f_0 \in \Lambda_\beta \) and \( P \) a polynomial. We then show, that the polynomial satisfying either (1) or (2) (i.e., \( P \) in the place of \( f \)) has degree at most \( k - 1 \). To see this, observe, that the double integrals in (1) and (2) are polynomials in \( t \), of degree equal to the degree of \( P \) (provided we replace \( x_0 \) by \( tx_0 \)). Thus, coefficients (in \( t \)) of order \( k \) and higher vanish. Each of these coefficients is a polynomial in \( z \), and therefore is a zero polynomial (in \( z \).
The homogeneous part of each such polynomial of the highest order coincides with the corresponding homogeneous part of \( P \), up to a non-zero constant. Thus \( P \) has degree at most \( k - 1 \). This finishes the proof of Theorem 2.

We now prove Theorem 1.

Proof of Theorem 1: i) \( \Rightarrow \) ii) & i) \( \Rightarrow \) iii): This part follows immediately from the pointwise estimates on the differences of \( f \), and the boundedness properties of the Riesz potentials.

ii) \( \Rightarrow \) i) & iii) \( \Rightarrow \) i): Using the method in [j], we can show, that ii) implies ii) of Theorem 2 and iii) similarly implies iii) of Theorem 2. This finishes the proof of Theorem 1.

References


