A CHARACTERIZATION OF EXTRINSIC
k-SYMMETRIC SUBMANIFOLDS OF $\mathbb{R}^N$

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SECTION 1.

In [2] D. Ferus introduced the notion of extrinsic symmetric submanifold of $\mathbb{R}^N$. This is a submanifold $M$ of $\mathbb{R}^N$ that is locally symmetric in the usual sense and such that for each $p$ the local symmetry $T_p: M \to M$ extends to an isometry of the ambient which is the identity on the normal space. Ferus proved that such a submanifold has parallel second fundamental form and obtained a complete classification by applying his previous results about submanifolds with this type of second fundamental form.

On the other hand in [9] W. Strübing completed the remarkable results of Ferus by giving a direct proof of the fact that every submanifold of $\mathbb{R}^N$, with parallel second fundamental form is in fact extrinsic symmetric. This was known to be a fact by Ferus' classification. One has then

(1.1) THEOREM. Let $(M, g)$ be a Riemannian manifold and let $i: M^n \to \mathbb{R}^N$ be an isometric immersion.

Then $M$ is an extrinsic symmetric submanifold if and only if the second fundamental form of the immersion is parallel.

In [8], the notion of extrinsic $k$-symmetric submanifold of $\mathbb{R}^N$ was introduced by extending Ferus' definition to the case of $k$-symmetric manifolds in the sense of [6], [7], [10]. This definition is global in nature and [8] contains a complete description of these submanifolds if they are compact and $k$ is odd.

The methods in [8] are quite different from those in [2] be-
cause extrinsic k-symmetric submanifolds of $\mathbb{R}^N$ do not have parallel second fundamental form in the sense of [2] and [9]. However one can define a new type of covariant derivative for the second fundamental form in terms of the canonical connection $\nabla^c$ of the k-symmetric manifold $M$ [6, p.23]. For symmetric spaces, i.e. $k=2$, $\nabla^c$ coincides with the Levi-Civita connection but this is not the case in general k-symmetric spaces. The proof given by Ferus in [2, p.83] of the above mentioned property of the second fundamental form does not extend to the new definition but, with a different method, one can prove that extrinsic k-symmetric submanifolds in the sense of [8] have canonically parallel second fundamental form (see (2.9)). This is the motivation of the following theorem which is the main result of this paper (see Sec.2 for definitions).

(1.2) THEOREM. Let $(M, g, \{B_x: x \in M\})$ be a compact connected Riemannian regular $s$-manifold of order $k$ and let $S$ denote its symmetry tensor. Let $i: M^n \to \mathbb{R}^{n+q}$ be an isometric imbedding and denote by $a$ its second fundamental form. Then $M$ is an extrinsic k-symmetric submanifold of $\mathbb{R}^{n+q}$ if and only if

i) $(\nabla^c a)(v, a) = 0$ in $M$ and

ii) $a_a(SX, SX) = a_a(X, X)$ for some point $a \in M$ and every $X \in M_a$.

The paper is organized as follows: In Sec.2 we recall the definition of extrinsic k-symmetric submanifold from [8] and introduce $\nabla^c$ proving that, for these submanifolds, one has $\nabla^c a = 0$ (2.9). The results of this section yield a proof of the fact that the conditions are necessary. In Sec.3 we study the nature of the $\nabla^c$-geodesics as curves in $\mathbb{R}^{n+q}$ to prove that the conditions are sufficient.

SECTION 2

Let $M^n$ be a compact Riemannian manifold and let $i: M^n \to \mathbb{R}^{n+q}$
be an isometric imbedding with the following properties.

i) For each \( p \in M \), there is an isometry \( \sigma_p : \mathbb{R}^{n+q} \to \mathbb{R}^{n+q} \) such that \( \sigma_p^k = \text{id} \), \( \sigma_p(p) = p, \sigma_p(M_p^I) = \text{identity on } M_p^I \).

ii) \( \sigma_p(i(M)) = i(M) \).

iii) Let \( \theta_p = (\sigma_p|M) \). The collection \( \{\theta_p : p \in M\} \) defines on \( M \) a Riemannian regular \( s \)-structure of order \( k \) [6,p.4-6]. Notice that condition (iii) implies that \( p \) is an isolated fixed point of \( \theta_p \) in \( M \) for each \( p \in M \).

If these conditions are satisfied we say that \( M \) is an \textit{extrinsic} \( k \)-symmetric submanifold of \( \mathbb{R}^{n+q} \).

We denote by \( g \) the Riemannian metric on \( M \) and by \( \langle , \rangle \) the inner product on \( \mathbb{R}^{n+q} \). \( \nabla \) and \( \nabla^E \) shall denote the corresponding Levi-Civita connections on \( M \) and \( \mathbb{R}^{n+q} \) respectively.

Associated to our isometric imbedding we have the second fundamental form \( \alpha \), the shape operator and the normal connection \( \nabla^\perp \).

On our Riemannian regular \( s \)-manifold we may consider the canonical connection \( \nabla^C \) [6,p.24] and two important tensors namely \( D(X,Y) = \nabla_X Y - \nabla^C_X Y \) and \( S \) which is defined by \( S_X = \theta_{X*} |X \) \( \forall X \in M \). These two tensors and the metric are parallel with respect to \( \nabla^C \) i.e.

\[
(2.1) \quad \nabla^C g = 0 \quad , \quad \nabla^C D = 0 \quad , \quad \nabla^C S = 0 \quad [6, \text{p.25}] 
\]

For the second fundamental form of an isometric imbedding one defines its "covariant derivative" as

\[
(2.2) \quad (\nabla^C_X \alpha)(Y,Z) = \nabla^C_X \alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(X, \nabla^C_X Z). 
\]

This derivative is used by Ferus and Strübing in the characterization of extrinsic 2-symmetric submanifolds of \( \mathbb{R}^N \). It is
obtained from the connections $\nabla$ in $TM$ and $\nabla^\perp$ in $TM^\perp$.

Here we propose to use a different combination which, as we shall see, will be more convenient for our purposes. Namely we define

$$
(\nabla^c_X a)(Y,Z) = \nabla^\perp_X(a(Y,Z)) - a(\nabla^c_X Y,Z) - a(Y,\nabla^c_X Z)
$$

an call it the canonical covariant derivative of $a$.

If our submanifold $M \subset \mathbb{R}^{n+q}$ happens to be a symmetric space one knows that $\nabla^c = \nabla$ and then both derivatives coincide.

As usual we have

$$
\nabla^c_X (fa) = (Xf) + f(\nabla^c_X a)
$$

$$
\nabla^c_{fX + gY} a = f(\nabla^c_X a) + g(\nabla^c_Y a)
$$

In a coherent way, we can define the canonical covariant derivative for the shape operator

$$
(\nabla^c_X a)\xi Y = \nabla^c_X (A_\xi Y) - A_\xi \nabla^\perp_X \xi Y - A_\xi (\nabla^c_X Y)
$$

and they are obviously related by

$$
\text{LEMMA.} \quad g((\nabla^c_X a)\xi Y, Z) = <(\nabla^c_X a)\xi Y, Z, \xi> \quad X, Y, Z \text{ tangent fields on } M, \xi \text{ a normal field.}
$$

Let $a$ be a point in $M$ which we shall keep fixed. Let $N_a$ be a normal neighborhood of $a$ in $M$ and such that $\sigma_a(N_a) = N_a$. Let $X \in M_a$ and $\xi \in M_a^\perp$; we shall denote by $X^*$ the "adapted" vector field on $N_a$ constructed from $X$ i.e. $X^*$ is constructed by $\nabla^c$-parallel translation along the $\nabla^c$-geodesics through $a$. It is easy to see that it is a well defined $C^\infty$ vector field on $N_a$ and that $\nabla^c_{X^*} U|_a = 0 \forall U \in M_a$.

Now it is easy to see that we can extend $\xi$ to a normal field $\xi^*$ defined on $N_a$ with the following properties.
(2.6) $\xi^*$ is $V^1$-parallel along each $V^c$-geodesic through a

(2.7) $\xi^*$ is $\sigma_a$-invariant on $N_a$ i.e.

$$\sigma_a|_{X}(\xi^*) = \xi^*_{\sigma_a}(X) \ \forall \ X \in N_a.$$ 

Associated to $X^*$ we can consider other two vector fields on $N_a$ namely $S\xi^*$ and $\theta_a\xi^*$. These fields are also parallel along each $V^c$-geodesic starting at $a$ because $V^cS = 0$ and $\theta_a$ is $V^c$-affine. Clearly we have $S\xi^* = \theta_a\xi^*$ on $N_a$ because they coincide at $a$ and are parallel along each $V^c$-geodesic through a.

(2.8) Proposition. At each point $a$ of the extrinsic $k$-symmetric submanifold we have $A_{\xi^*}(S\xi^* X) = SA_{\xi^*}X \ \forall \ X \in M_\xi \ \xi \in M_a^1$.

Proof. Let $\gamma(t)$ be a $V^c$-geodesic starting at $a$ and put $\beta(t) = \sigma_a(\gamma(t))$.

We have

$$A_{\xi^*}(\beta(t))[(\sigma_a|_{\gamma(t)})X^*(\gamma(t))] =$$

$$= A(\sigma_a|_{\gamma(t)})[X^*(\gamma(t))] =$$

$$= \sigma_a|_{\gamma(t)}[A_{\xi^*}(\gamma(t))X^*(\gamma(t))] .$$

Making $t = 0$ now we have $A_{\xi^*}S\xi^* X = S_a(A_{\xi^*}X)$.

(2.9) Lemma. For each $U, X \in M_\xi \ \xi \in M_a^1$ we have at the point $a$

$$[\nabla^c_{SU}A]_{\xi^*}X = S_{U\xi}(\nabla^c_{SU}A)_{\xi^*}X .$$

Proof. Let $\gamma(t)$ be a $V^c$-geodesic starting at $a$ with $\gamma(0) = U$.

By definition $(\nabla^c_{SU}A)_{\xi^*}X = \nabla^c_{SU}(A_{\xi^*}X^*)$ since $\nabla^1_{SU}\xi^* = 0 =

$\nabla^c_{SU}\xi^*$. Now $\nabla^c_{SU}(A_{\xi^*}X^*) = \nabla^c_{SU}(A_{\xi^*}(\xi^*(\gamma))) = $

$$= \nabla^c_{SU}(\sigma_a|_{\xi^*}(\gamma)X^*(\gamma)) = \sigma_a|_{\xi^*}(\nabla^c_{SU}(A_{\xi^*}X^*)) = S_{U\xi}(\nabla^c_{SU}A)_{\xi^*}X ,$$
since \( \nabla^1_U \xi^* = 0 = \nabla^c_U X^* \).

With the aid of these lemmas we can prove

(2.10) THEOREM. If \( i: M^n \to R^{n+q} \) is a extrinsic k-symmetric submanifold then \( A^c_\xi \) is parallel with respect to the canonical connection. i.e. \( (\nabla^c_U A) = 0 \quad \forall \ U \in M_p, \quad \forall \ p \in M. \)

Proof. Let us take our point \( a \in M \) and its normal neighborhood \( N_a \) as above. By (2.8) we have

\[
A^c_\xi X^* = S^{-1} A^c_\xi SX^*
\]

and then

\[
\nabla^c_{SU} (A^c_\xi X^*) = \nabla^c_{SU} (S^{-1} A^c_\xi SX^*).
\]

Now

\[
\nabla^c_{SU} (A^c_\xi X^*) = (\nabla^c_{SU} A)\xi X,
\]

and since \( \nabla^c_{SU} S^{-1} = 0 \) we have

\[
\nabla^c_{SU} (S^{-1} A^c_\xi SX^*) = S^{-1} [\nabla^c_{SU} (A^c_\xi SX^*)] = S^{-1} [(\nabla^c_{SU} A)\xi SX] = (\nabla^c_{SU} A)\xi X
\]

by (2.9).

Then we have proved

\[(\nabla^c_{SU} A)\xi X = (\nabla^c_{SU} A)\xi X\]

and since \( \xi \) and \( X \) are arbitrary we get

\[\nabla^c_{(I-S)SU} A = 0\]

which, since \( I-S \) is non singular, implies \( \nabla^c_U A = 0. \)

(2.11) COROLLARY. If \( i: M^n \to R^{n+q} \) is a extrinsic k-symmetric submanifold then its second fundamental form is canonically parallel i.e. \( (\nabla^c_U \alpha) = 0 \quad \forall \ U \in M_p, \quad \forall \ p \in M. \)
SECTION 3

In this section we prove that the conditions of theorem (1.2) are sufficient

(3.1) LEMMA. Let \((M^n, g, \{\theta_x: x \in M}\)) be a Riemannian regular s-manifold and let \(i: M^n \rightarrow R^{n+q}\) be an isometric immersion with the following properties

i) \((\nabla^c_x a) = 0\) in \(M\).

ii) For some point \(a \in M\), \(\alpha_a(SX, SX) = \alpha_a(X, X)\) \(\forall X \in M_a\).

Then at each point \(p \in M\) and for every \(X, Y \in M_p\), \(\alpha_p(SX, SY) = \alpha_p(X, Y)\).

Proof. This is straightforward and left to the reader. \(\square\)

Let \(C: I \rightarrow R^{n+q}\) be a regular \(C^\infty\) curve. We say that \(C\) is a Frenet curve in \(R^{n+q}\) of osculating rank \(r \geq 1\) if \(C\) is parametrized with respect to arc length, defined in an open non empty interval \(I\) and for each \(t \in I\) the derivatives \(\dot{C}(t), \ldots, C^{(r)}(t)\) are linearly independent and \(\dot{C}(t), \ldots, C^{(r+1)}(t)\) are linearly dependent.

(3.2) PROPOSITION. Let \((M, g, \{\theta_x: x \in M}\)) and \(i: M^n \rightarrow R^{n+q}\) an isometric imbedding with the same hypothesis of (3.1) and let \(\gamma\) be a \(V^c\)-geodesic in \(M\).

Then, except by a linear change of parameter, \(C(t) = i(\gamma(t))\) is a Frenet curve in \(R^{n+q}\) of osculating rank \(r\) for some \(1 \leq r \leq n+q\) and its Frenet curvatures are constant.

Let \(\gamma(0) = a\) and consider, in the interval where it is defined, the Frenet curve \(C_1(t) = i(\theta_a(\gamma(t)))\). Then \(C_1\) has the same osculating rank as \(C(t)\) and the corresponding Frenet curvatures are equal.
Proof. Let $\gamma(t)$ be a $V^C$-geodesic in $N_a$ starting at $a \in M$. It is clear that $g(\dot{\gamma}, \dot{\gamma})$ is constant and then, by a linear change of parameter, (which does not change the fact that $\gamma$ is geodesic) we can assume that $g(\dot{\gamma}, \dot{\gamma}) = 1$. This means that $C$ is parametrized by arc length. Since $i$ is an imbedding we can identify $M$ and $i(M)$ and then $C(t) = \gamma(t)$.

Consider the first two derivatives of $C$,

$$\dot{C}(t) = \dot{\gamma}(t)$$
$$\ddot{C}(t) = \nabla^E_{\dot{\gamma}} \dot{\gamma} = D(\dot{\gamma}, \dot{\gamma}) + \alpha(\dot{\gamma}, \dot{\gamma}).$$

Then, we have $\ddot{C} = T_2 + N_2$ (tangent and normal components) and by (2.1) and (i)

$$\nabla^c_{\dot{\gamma}} T_2 = 0 = \nabla^c_{\dot{\gamma}} N_2.$$ 

Assume now that we have proved that, for each $j \leq i$,

$$C(j) = T_j + N_j \quad \text{(tangent and normal) with}$$

$$\nabla^c_{\dot{\gamma}} T_j = 0 = \nabla^c_{\dot{\gamma}} N_j.$$ 

We shall see that this is also the case for $i+1$.

$$C(i+1)(t) = \nabla^E_{\dot{\gamma}} T_i + \nabla^E_{\dot{\gamma}} N_i =$$

$$= \nabla^c_{\dot{\gamma}} T_i + D(\dot{\gamma}, T_i) + \alpha(\dot{\gamma}, T_i) - A_{N_i} \dot{\gamma} + \nabla^c_{\dot{\gamma}} N_i =$$

$$= [D(\dot{\gamma}, T_i) - A_{N_i} \dot{\gamma}] + \alpha(\dot{\gamma}, T_i) = T_{i+1} + N_{i+1}.$$ 

Now by (2.1), (i) and the inductive hypothesis we get

$$\nabla^c_{\dot{\gamma}} T_{i+1} = 0. \text{ Similarly, by (i) and the inductive hypothesis,}$$

$$\nabla^c_{\dot{\gamma}} N_{i+1} = 0.$$ 

Then, for each $k \geq 1$,

$$C(k)(t) = T_k(t) + N_k(t), \quad \nabla^c_{\dot{\gamma}} T_k = 0 = \nabla^c_{\dot{\gamma}} N_k.$$ 

Let $I$ be the open interval where $\gamma$ is defined. For each $t \in I$
let \( r(t) \) be the natural number \( (1 \leq r(t) \leq n+q) \) such that
\( \dot{C}(t), \ldots, C^{(r)}(t) \) are linearly independent and \( \dot{C}(t), \ldots, C^{(r+1)}(t) \)
are linearly dependent. Let \( t_0 \in I \) be a point such that
\( r(t_0) \leq r(t) \ \forall \ t \in I \).

There are some real numbers \( a_1, \ldots, a_{r(t_0)+1} \), not all zero,
such that \( \sum a_j C^{(j)}(t_0) = 0 \) (sum from \( j=1 \) to \( r(t_0)+1 \)).

With these real numbers we define a couple of real \( C^\infty \) functions on \( I \).

\[
\begin{align*}
  h(t) &= \| \sum a_j T_j(t) \|^2 \\
  f(t) &= \| \sum a_j N_j(t) \|^2
\end{align*}
\]

They satisfy \( h(t_0) = f(t_0) = 0 \) and by (2.1) and (3.4)
\[ h'(t) = 0 \quad \forall \ t \in I \]
and therefore \( h(t) = 0 \ \forall \ t \in I \).

Similarly by (3.4) \( f'(t) = 0 \) and again, \( f(t) = 0 \ \forall \ t \in I \).

We have then, \( r(t) = r(t_0) \ \forall \ t \in I \) and therefore, \( C(t) \) is a
Frenet curve on \( I \). Let \( r = r(t_0) \).

We have to prove now that the Frenet curvatures of \( C(t) \) are
constant on \( I \).

In fact, we shall prove that for each \( j, 1 \leq j \leq r \), we can write:
\[
V_j(t) = P_j(t) + Q_j(t) \quad \text{(tangent and normal)}
\]
(3.5)
\[
\nabla^C \gamma P_j = 0 = \nabla^C \gamma Q_j, \quad k_{j-1}(t) = \text{constant}
\]

Let us proceed by induction on \( j \). For \( j = 1 \)
\[
\begin{align*}
  V_1(t) &= \dot{C}(t) = P_1(t) + Q_1(t) \quad Q_1 = 0 \\
  \nabla^C \gamma P_1 &= \nabla^C \gamma \dot{Q}_1 = 0, \quad k_0(t) = \| \dot{C}(t) \| = 1
\end{align*}
\]
Assume that (3.5) is true for each \( j \leq i < r \). We have to show this for \( i+1 \). Now we have
We shall show first that $k_i = \text{constant}$ and then complete the other parts of (3.5). We know [9, p.39, (10)] that

\[(3.7) \quad k_i(t) = \|V'_i(t) + k_{i-1}V_{i-1}(t)\| \quad (> 0 \text{ for } 1 \leq i < r)\]

Then, replacing the values of $V_{i-1}$ and the derivative, one gets

\[(3.8) \quad [k_i(t)]^2 = \|D(\dot{\gamma}, P_i) - A_{Q_i} \dot{\gamma} + k_{i-1}P_{i-1}\|^2 + \]
\[+ \|\alpha(\dot{\gamma}, P_i) + k_{i-1}Q_{i-1}\|^2 = u(t) + v(t)\]

and, by induction, it is easy to see that $u$ and $v$ are constant.

Once that we know this we can compute $V_{i+1}$ (recall $k_i > 0$ for $1 \leq i < r$).

\[(3.9) \quad V_{i+1}(t) = \frac{1}{k_i} [V'_i(t) + k_{i-1}(t)V_{i-1}(t)] = \]
\[= \frac{1}{k_i} [(D(\dot{\gamma}, P_i) - A_{Q_i} \dot{\gamma} + k_{i-1}P_{i-1}) + (\alpha(\dot{\gamma}, P_i) + k_{i-1}Q_{i-1})] = \]
\[= P_{i+1}(t) + Q_{i+1}(t)\]

and, since $k_i = \text{constant}$, we have $V^c_{Y P_{i+1}} = 0$. Similarly one can easily get $V^c_{Y Q_{i+1}} = 0$. In this way we have proved (3.5).

Let us prove now the second part of (3.2). Let $\gamma_1(t) = \Theta_a(\gamma(t))$ and let $r_1$ be the rank of $\dot{\gamma}_1(t) = i(\gamma_1(t))$. We have its Frenet frame $V_{11}, V_{12}, \ldots, V_{1r_1}$ and we can write (3.5) for the curve $C_1$

\[(3.10) \quad V^c_{Y P_{1j}} = 0 = V^c_{Y Q_{1j}} \quad , \quad k_{1(j-1)} = \text{constant} .\]

Our next objective is to prove the following identities.

For each $j \quad 1 \leq j \leq r$
(3.11) \( P_{1j}(0) = SP_j(0) \), \( Q_{1j}(0) = Q_j(0) \), \( k_{ij-1} = k_{j-1} \).

Clearly, they are true for \( j = 1 \) so we assume that they hold for each \( j \leq i < r \) and prove them for \( i + 1 \).

Let us write (3.8) for \( \gamma_1 \):

\[
[k_{1i}]^2 = \|D(\theta_{a*}\gamma, P_{1i}) - A_{Q_{1i}} \theta_{a*}\gamma + k_{1i-1}P_{1i-1}\|^2 +
\|\alpha(\theta_{a*}\gamma, P_{1i}) + k_{1i-1}Q_{1i-1}\|^2.
\]

At \( t = 0 \), we have by induction and the remarked properties of \( D, A_{\gamma} \) and \( \alpha \) that

\[
[k_{1i}]^2 = \|S[D(\gamma, P_{1i}) - A_{Q_{1i}} \gamma + k_{1i-1}P_{1i-1}]\|^2 + \|\alpha(\gamma, P_{1i}) + k_{1i-1}Q_{1i-1}\|^2
\]

and therefore, since they are positive,

\[ k_{1i} = k_i. \]

In order to complete the proof of (3.11), we write, for \( \gamma_1 \), the formula (3.9).

\[
V_{1i+1}(t) = \frac{1}{k_i} [D(\theta_{a*}\gamma, P_{1i}) - A_{Q_{1i}} \theta_{a*}\gamma + k_{1i-1}P_{1i-1} +
\alpha(\theta_{a*}\gamma, P_{1i}) + k_{1i-1}Q_{1i-1}]
\]

and again, by taking \( t = 0 \), we get

\[
V_{1i+1}(0) = \frac{1}{k_i} [S[D(\gamma, P_{1i}) - A_{Q_{1i}} \gamma + k_{1i-1}P_{1i-1}] + \alpha(\gamma, P_{1i}) +
\alpha(\gamma, P_{1i}) + k_{1i-1}Q_{1i-1}]
\]

from which (3.11) follows.

It is easy to see now, that (3.11) implies \( r_1 \geq r \), because \( S \) is non singular.

Now we can define, for \( j = 2, \ldots, k \), new geodesics in \( M \) by

\[ \gamma_j(t) = \theta_{a}(\gamma_j(t)) \]
and if we call $r_j$ the rank of $C_j$ then
$$r = r_k \geq r_{k-1} \geq \ldots \geq r_1 \geq r$$
which shows $r_1 = r$. This finishes the proof of (3.2).

Let us complete now the proof of (1.2).

The conditions are sufficient:

Given the tensor $S$ on $M$ we can define, for each $p \in M$, an isometry $\sigma_p : \mathbb{R}^{n+q} \to \mathbb{R}^{n+q}$ by

$$\sigma_p(v) = \begin{cases} S_p(v) & \text{if } v \in M_p \\ v & \text{if } v \in M_p^\perp \end{cases}$$

As we mentioned before we identify $M$ and $i(M) \subset \mathbb{R}^{n+q}$. We have to prove that $\sigma_p(M) \subset M$ and that $\sigma_p|_M = \theta_p$ for each $p \in M$.

At this point we need to make the following observation due to O. Kowalski (private communication).

Let $M = G/K$ be a compact $k$-symmetric space where $G$ is the connected component of the identity of the group of symmetries. Let $g$ and $k$ be the Lie algebras of $G$ and $K$ respectively. Let $\theta$ be the automorphism of $G$ induced by the symmetry at the origin $0 = [K]$ of $M$. Then $(G,K,\theta)$ is a "regular homogeneous $s$-manifold" ([6] p.53). Let $g = k \oplus m$ be the decomposition of $g$ given by ([6] II. 24) which makes $G/K$ reductive with respect to that decomposition. Let $\langle X,Y \rangle = -B(X,Y)$, where $B$ is the Killing form on $g$. This is a scalar product invariant by every automorphism of $g$. Let $m'$ be the orthogonal complement of $k$ in $g$ with respect to this scalar product. This gives a new decomposition $g = k \oplus m'$.

(3.12) **Lemmas.** The two decompositions $g = k \oplus m$ and $g = k \oplus m'$ coincide.

**Proof.** Let $\theta_*$ be the automorphism of $g$ induced by $\theta$. Then, by definition, if $A = \text{Id}_g - \theta_*$, one has
$k = \ker(A)$ and $m = \text{Im}(A)$

(These are the Fitting 0-component and Fitting 1-component of $g$, relative to $A$, respectively).

Now $\theta_\ast$ leaves $m'$ invariant and then $A_i^\ast(g) \supset m'$ $\forall$ $i \geq 1$. But since the dimensions of $m$ and $m'$ coincide we have $m = m'$.

(3.13) COROLLARY. The canonical connection $\nabla^c$ of the regular homogeneous $s$-manifold $(G,K,\theta)$ and the canonical connection $\tilde{\nabla}$ of $G/K$ with respect to the decomposition $g = k \oplus m$ coincide.

Proof. This follows from the fact that the canonical connection of a homogeneous space $G/K$, reductive with respect to the decomposition $g = k \oplus m$, is uniquely determined by the choice of $m$ ([6] p. 29, I. 6).

(3.14) COROLLARY. Let $(M,g,(\theta_x: x \in M))$ be a compact connected Riemannian regular $s$-manifold of order $k$. Let $\nabla^c$ be its canonical connection and $p$ be a point in $M$. Then given any point $x \in M$ there exists a $\nabla^c$-geodesic in $M$ joining $p$ to $x$.

Proof. Let $g = k \oplus m$ be the orthogonal decomposition with respect to the Killing form $B$ on $g$. The restriction of $(-B)$ to $m$ induces on $M$ a new Riemannian metric $h(X,Y)$ which makes $M$ a naturally reductive homogeneous space [5, II, p. 203]. One knows ([1, p. 55]) that the canonical connection $\tilde{\nabla}$ on $M$, with respect to the decomposition $g = k \oplus m$, has the same geodesics that the Riemannian connection corresponding to the metric $h$. Then the corollary follows from (3.13) and the theorem of Hopf-Rinow [4, p. 56].

Let $\gamma$ be this $\nabla^c$-geodesic joining $p$ to $x$; we may assume $\gamma(0) = p$. Put $\gamma_1 = \theta_p(\gamma)$. By (3.2) $\gamma$ and $\gamma_1$ are Frenet curves in $\mathbb{R}^{n+q}$ of the same osculating rank $r$ ($1 \leq r \leq n+q$) and their corresponding curvatures are equal and constant.
By keeping the same notation as in the proof of (3.2) we call $V_j$ and $V_1j$ the Frenet frames of $\gamma$ and $\gamma_1$ respectively. By the nature of the curvatures in this case it is enough to show that
\[ \sigma_p(V_i(0)) = V_i(0) \quad i = 1, \ldots, r \]

To that end we have plenty of information in the proof of (3.2). Clearly this identity follows from (3.5), (3.10) and (3.11) and then $\sigma_p(M) \in M$. It is now clear that $\sigma_p|M = \theta_p$ and the proof of (1.2) is complete. \hfill \Box

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