BOUNDDEDNESS OF SINGULAR INTEGRAL OPERATORS ON $H_\omega$.

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Abstract: We study the boundedness of singular integral operators on Orlicz-Hardy spaces $H_\omega$, in the setting of spaces of homogeneous type. As an application of this result, we obtain a characterization of $H_\omega IR^n$ in terms of the Riesz Transforms.

§ 1. NOTATION AND DEFINITIONS

Let $X$ be a set. A function $d : X \times X \rightarrow IR^+ \cup \{0\}$ shall be called a quasi-distance on $X$ if there exists a finite constant $K$ such that

(1.1) $d(x, y) = 0$ if and only if $x = y$

(1.2) $d(x, y) = d(y, x)$

and

(1.3) $d(x, y) \leq K[d(x, z) + d(z, y)]$

for every $x, y$ and $z$ in $X$.

In a set $X$, endowed with a quasi-distance $d(x, y)$, the balls

$$B(x, r) = \{y : d(x, y) < r\}, \ r > 0,$$

form a basis for the neighbourhoods of $x$ in the topology induced by the uniform structure on $X$.

We shall say that a set $X$, with a quasi-distance $d(x, y)$ and a non-negative measure $\mu$ defined on a $\sigma$-algebra of subsets of $X$ containing the balls $B(x, r)$, is a normal
space of homogeneous type if there exist four positive finite constants \( A_1, A_2, K_1 \) and \( K_2, K_2 \leq 1 \leq K_1 \), such that

\[
(1.4) \quad A_1 r \leq \mu(B(x, r)) \quad \text{if } r \leq K_1 \mu(X)
\]

\[
(1.5) \quad B(x, r) = X \quad \text{if } r > K_1 \mu(X)
\]

\[
(1.6) \quad A_2 r \geq \mu(B(x, r)) \quad \text{if } r \geq K_2 \mu(\{x\})
\]

\[
(1.7) \quad B(x, r) = \{x\} \quad \text{if } r < K_2 \mu(\{x\}).
\]

We note that, under these conditions, there exists a finite constant \( A \), such that

\[
(1.8) \quad 0 < \mu(B(x, 2r)) \leq A \mu(B(x, r))
\]

holds for every \( x \in X \) and \( r > 0 \).

We shall say that a normal space of homogeneous type \((X, d, \mu)\) is of order \( \alpha, 0 < \alpha < \infty \), if there exists a finite constant \( K_3 \) satisfying

\[
(1.9) \quad |d(x, z) - d(y, z)| \leq K_3 r^{1-\alpha} d(x, y) \alpha
\]

for every \( x, y \) and \( z \) in \( X \), whenever \( d(x, z) < r \) and \( d(y, z) < r \) (See [MS]).

Throughout this paper \( X = (X, d, \mu) \) shall denote a normal space of homogeneous type of order \( \alpha, 0 < \alpha \leq 1 \).

Let \( \rho \) be a positive function defined on \( IR^+ \). We shall say that \( \rho \) is of upper type \( m \) (respectively, lower type \( m \)) if there exists a positive constant \( c \) such that

\[
(1.10) \quad \rho(st) \leq ct^m \rho(s),
\]
for every \( t \geq 1 \) (respectively, \( 0 < t \leq 1 \)). A non-decreasing function \( \rho \) of finite upper type such that \( \lim_{t \to -0} \rho(t) = 0 \) is called a growth function.

For \( \rho(t) \) a positive right-continuous non-decreasing function satisfying \( \lim_{t \to 0^+} \rho(t) = 0 \) and \( \lim_{t \to \infty} \rho(t) = \infty \), the function

\[
\Phi(t) = \int_0^t \rho(s)ds
\]

will be called a Young function.

Given \( \Phi(t) \) a Young function of finite upper type, we define the Orlicz space \( L_\Phi \) by

\[
L_\Phi = \{ f : \int \Phi(|f(x)|)dx < \infty \},
\]

and we denote by

\[
\| f \|_{L_\Phi} = \inf \{ \lambda : \int \Phi \left( \frac{|f(x)|}{\lambda} \right) \leq 1 \}
\]

the Luxemburg norm.

Given a Young function, we consider the complementary Young function of \( \Phi \) defined by

\[
\psi(t) = \int_0^t q(s)ds , \text{ with } q(s) = \sup_{\rho(t) \leq s} t.
\]

For \( \Phi(x) \) a Young function, the Hölder inequality

\[
(1.12) \quad \| f \|_{L_\Phi} \| g \|_{L_\Psi} \leq \int f(x)g(x)dx
\]

holds for every \( f \in L_\Phi \) and \( g \in L_\Psi \).

We shall understand that two positive functions are equivalent if their ratio is bounded above and below by two positive constants.

Let \( \rho \) be a growth function. We shall say that a function \( \psi(x) \) belongs to \( \text{Lip}(\rho) \), if

\[
\| \psi \|_{\text{Lip}(\rho)} = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{\rho(d(x, y))} < \infty
\]

holds. When \( \rho(t) \) is the function \( t^\beta \), \( 0 < \beta < \infty \), we shall say that \( \psi(t) \) is in \( \text{Lip}(\beta) \) and, in this case, \( \| \psi \|_\beta \) indicates its norm.
The space of distributions \((E^\alpha)'\), introduced by Maciá and Segovia in [MS], is the dual space of \(E^\alpha\) consisting of all functions with bounded support belonging to Lip \((\beta)\) for some \(0 < \beta < \alpha\).

For \(x \in X\) and \(0 < \gamma < \alpha\), we consider the class \(T_\gamma(x)\) of functions \(\psi\) belonging to \(E^\alpha\) satisfying the following condition: there exists \(r\) such that \(r \geq K_2\mu(\{x\})\), \(\text{supp}\psi \subseteq B(x, r)\) and

\[
(1.13) \quad r \| \psi \|_\infty \leq 1 \quad \text{and} \quad r^{1+\gamma} \| \psi \|_\gamma \leq 1.
\]

Given \(\gamma, 0 < \gamma < \alpha\), we define the \(\gamma\)-maximal function \(f_\gamma^*(x)\) of a distribution \(f\) on \(E^\alpha\) by

\[
(1.14) \quad f_\gamma^* = \sup\{|f(\psi)| : \psi \in T_\gamma(x)\}.
\]

(1.15) **Definition:** Let \(\rho\) be a growth function plus a nonnegative constant or \(\rho \equiv 1\). A \((p, q)\)-atom, \(1 < q \leq \infty\), is a function \(a(x)\) on \(X\) satisfying:

\[
(1.16) \quad \int_X a(x) d\mu(x) = 0,
\]

\[
(1.17) \quad \text{the support of } a(x) \text{ is contained in a ball } B \text{ and}
\]

\[
(1.18) \quad \left[\mu(B)^{-1} \int_B |a(x)|^q \right]^{1/q} \leq [\mu(B)\rho(\mu(B))]^{-1} \quad \text{if } q < \infty
\]

or

\[
\| a \|_\infty \leq [\mu(B)\rho(\mu(B))]^{-1}, \quad \text{if } q = \infty.
\]

Clearly, when \(\rho(t) = t^{1/p-1}, p \leq 1\), a \((p, q)\)-atom is a \((p, q)\)-atom in the sense of [M-S].

Let \(\omega\) be a growth function of positive lower type \(l\) such that \(l(1+\alpha) > 1\). For every \(\gamma\) with \(0 < \gamma < \alpha\) and \(l(1+\gamma) > 1\), we define

\[
(1.19) \quad H_\omega = H_\omega(X) = \left\{ f \in (E^\alpha)': \int \omega[f^*_\gamma(x)] d\mu(x) < \infty \right\}.
\]

and we denote

\[
(1.20) \quad \| f \|_{H_\omega} = \| f_\gamma^* \|_{H_\omega} = \inf \left\{ \lambda > 0 : \int \omega \left[ \frac{f^*_\gamma(x)}{\lambda^{1/p}} \right] d\mu(x) \leq 1 \right\}.
\]
Let \( \omega \) be a growth function of positive lower type \( l \). If \( \rho(t) = t^{-1}/\omega^{-1}(t^{-1}) \), we define the atomic Orlicz Space \( H^{\rho,q}(X) = H^{\rho,q}, 1 < q \leq \infty \), as the space of all distributions \( f \) on \( E^\alpha \) which can be represented by

\[
(1.21) \quad f(\psi) = \sum_i b_i(\psi),
\]

for every \( \psi \) in \( E^\alpha \), where \( \{b_i\}_i \) is a sequence of multiples of \((\rho,q)\)-atoms such that if \( \text{supp}(b_i) \subseteq B_i \), then

\[
(1.22) \quad \sum_i \mu(B_i)\omega\left(\|b_i\|_q\mu(B_i)^{-1/q}\right) < \infty.
\]

Given a sequence of multiples of \((\rho,q)\)-atoms, \( \{b_i\}_i \), we set

\[
(1.23) \quad \Lambda_q(\{b_i\}) = \inf \left\{ \lambda : \sum_i \mu(B_i)\omega\left(\frac{\|b_i\|_q\mu(B_i)^{-1/q}}{\lambda^{1/l}}\right) \leq 1 \right\}
\]

and we define

\[
(1.24) \quad \|f\|_{H^{\rho,q}} = \inf \Lambda_q(\{b_i\}),
\]

where the infimum is taken over all possible representations of \( f \) of the form (1.21).

It has been shown in [V] that the spaces \( H_\omega \) and \( H^{\rho,q} \) are equivalent. More precisely, in that paper the following Theorem is proved

**Theorem A:** Let \( \omega \) be a function of lower type \( l \) such that \( l(1+\alpha) > 1 \). Assume that \( \omega(s)/s \) is non-increasing. Let \( \rho(t) \) be the function defined by \( t\rho(t) = 1/\omega^{-1}(t^{-1}) \). Then \( H_\omega \equiv H^{\rho,q} \) for every \( 1 < q \leq \infty \).

We observe that the statement of the Theorem A implies in particular that the definition of \( H_\omega \) is independent of \( \gamma, 0 < \gamma < \alpha \) and \( l(1+\gamma) > 1 \). Furthermore, from proposition (3.1) in [V], we may assume without lost of generality, that \( \omega \) is, in addition, continuous, strictly increasing and a subaditive function.
§ 2. BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS ON HARDY-ORLICZ SPACES

In this section \((X, d, \mu)\) shall mean a normal space of homogeneous type of order \(\alpha\), \(0 < \alpha \leq 1\) and \(K\) shall denote the constant appearing in (1.3).

We assume that a singular kernel is a measurable function \(k : X \times X \to \mathbb{R}\) satisfying the following conditions:

\begin{align*}
(2.1) \ |k(x, y)| & \leq c d(x, y)^{-1} \text{ for } x \neq y \\
(2.2) \text{ There exist } \delta, \ 0 < \delta \leq \alpha, \text{ such that } & |k(x, y) - k(x', y)| + |k(y, x) - k(y, x')| \leq c d(x, x')^\delta d(x, y)^{-1-\delta}, \\
& \text{provided } d(x, y) > 2 d(x, x'). \\
(2.3) \text{ Let } 0 < r < R < \infty, \text{ then } & \\
\text{a) } & \int_{r \leq d(x, y) < R} k(x, y)d\mu(y) = 0, \text{ for every } x \in X. \\
\text{and } & \\
\text{b) } & \int_{r \leq d(x, y) < R} k(y, x)d\mu(y) = 0, \text{ for every } x \in X. \\
\end{align*}

Given \(\epsilon > 0\), we define

\[ T_\epsilon f(x) = \int_{\epsilon \leq d(x, y) < 1/\epsilon} k(x, y)f(y)d\mu(y). \]

For singular integrals, in the context of spaces of homogeneous type, conditions for their boundedness on \(L^2\) were given in [A], [D-J-S], [M-T] and [M-S-T].

In the sequel we shall assume that \(T\) is a bounded singular integral operator on \(L^2(X)\) associated to a kernel \(k(x, y)\) satisfying (2.1), (2.2) and (2.3). Under these assumptions we shall obtain, in Theorem 2.20, the boundedness of \(T\) on the spaces \(H_\omega\).
In order to prove the main theorem we shall need some previous results.

(2.4) LEMMA. Let \( k(x, y) \) be a kernel satisfying (2.1) and (2.3). Let \( \Phi(t) \) be a Lipschitz function defined on \([0, \infty)\) such that \( \Phi(t) = 0 \) for \( t \geq 2 \). Assume that \( \Phi(t) \) satisfies one of the following two conditions:

a) \( \Phi(t) = 1 \) for \( t \leq 1 \), or

b) \( \Phi(t) = 0 \) for \( t \leq 1 \).

Let \( 0 < r < R < \infty \), then

\[
\int_{r \leq d(x,y) < R} k(x,y)\Phi(d(x,y))d\mu(y) = 0, \text{ for every } x \in X.
\]

PROOF. We prove the lemma for \( \Phi \) satisfying (a). The other case follows the same lines. Given \( 0 < r < R \), we have three possibilities:

i) \( 2 \leq r \),

ii) \( 0 < r < 2 < R \)

iii) \( 0 < r < R \leq 2 \).

If \( r \geq 2 \) the lemma follows immediately. Suppose that (ii) holds. Since \( k(x, y) \) satisfies (2.3) and \( \Phi(t) = 1 \) for \( t \leq 1 \), it is enough to assume that \( r \geq 1 \) in this case. Given \( \varepsilon > 0 \), let \( P = \{t_0, t_1, \cdots t_N\} \) be a partition of the interval \([r, 2]\), with \( \Delta t_i = t_i - t_{i-1} < \delta \) and \( \delta \) a constant depending on \( \varepsilon \) to be determined later. Then we have

\[
\int_{r \leq d(x,y) < R} k(x,y)\Phi(d(x,y))d\mu(y) = \sum_{i=1}^{N} \int_{t_{i-1} \leq d(x,y) < t_i} k(x,y)[\Phi(d(x,y)) - \Phi(t_i)]d\mu(y)
\] 

\[
+ \sum_{i=1}^{N} \Phi(t_i) \int_{t_{i-1} \leq d(x,y) < t_i} k(x,y)d\mu(y).
\]
Using that $\Phi$ is a Lipschitz function and applying (2.1) and (2.3), we obtain

$$
| \int_{r \leq d(x,y) < R} k(x,y)\Phi(d(x,y))d\mu(y) | \leq c\delta \sum_{i=1}^{N} \int_{t_{i-1} \leq d(x,y) < t_i} |k(x,y)|d\mu(y)
$$

$$
\leq c\delta \int_{1 \leq d(x,y) < 2} |k(x,y)|d\mu(y)
$$

$$
\leq c\delta.
$$

Choosing $\delta$ such that $c\delta < \varepsilon$, we conclude the proof of (ii). The remaining case (iii) follows the same line.

(2.5) REMARK. Let $\Phi$ be as in Lemma (2.4). For $\varepsilon > 0$, the kernel $k(x,y)\Phi(\frac{d(x,y)}{\varepsilon})$ satisfies (2.1) and, from Lemma (2.4), also verifies (2.3). On the other hand, since $X$ is of order $\alpha$, (2.2) holds with constant independent of $\varepsilon$.

Let $\psi_1$ and $\psi_2$ in $C^\infty([0,\infty))$ satisfying the following conditions: supp$\psi_1 \subset [1/2, \infty)$ and $\psi_1(t) = 1$ if $t \geq 1$; supp$\psi_2 \subset [0, 2]$ and $\psi_2(t) = 1$ for $t \leq 1$. For $f \in L^p, 1 \leq p < \infty$, we define

$$
\tilde{T}_\varepsilon f(x) = \int k(x,y)\psi_1\psi_2(\varepsilon d(x,y))f(y)d\mu(y).
$$

(2.6) LEMMA. Let $k(x,y)$ be a singular kernel satisfying (2.1), (2.2) and (2.3). Then,

$$
\| \tilde{T}_\varepsilon f - T f \|_{L^1} \to 0, \text{ as } \varepsilon \to 0.
$$

PROOF. We have

$$
\tilde{T}_\varepsilon f(x) = \int_{\varepsilon/2 \leq d(x,y) \leq \varepsilon} k(x,y)\psi_1(\frac{d(x,y)}{\varepsilon})f(y)d\mu(y) + T_\varepsilon f(x)
$$

$$
+ \int_{1/\varepsilon \leq d(x,y) < 2/\varepsilon} k(x,y)\psi_2(\varepsilon d(x,y))f(y)d\mu(y) = T^1_\varepsilon f(x) + T_\varepsilon f(x) + T^2_\varepsilon f(x).
$$

Since $T_\varepsilon f(x)$ converges to $T f$ in $L^2$, we only need to prove that $T^i_\varepsilon f$ converges to zero in $L^2$ for $i = 1, 2$. Clearly from (2.1), we have

$$
T^i_\varepsilon f(x) \leq cM f(x), \text{ for } i = 1, 2.
$$
From (2.7) and by the density in $L^2$ of the Lipschitz $\gamma$ functions with bounded support, it is enough to prove the convergence of $T_i f$ for such functions. Let $f$ be a function with bounded support belonging to $\text{Lip}(\gamma)$. Then by Lemma (2.4), we get

\[(2.8) \quad |T_i^2 f(x)| = \int_{\epsilon^2 < d(x, y) < \epsilon} k(x, y) \psi_1 \left( \frac{d(x, y)}{\epsilon} \right) |f(y) - f(x)| d\mu(y) \leq c \| f \|_{\gamma} \epsilon^\gamma.\]

On the other hand from (2.1), we obtain

\[
|T_i^2 f(x)| \leq \epsilon \int_{1/\epsilon \leq d(x, y) < 2/\epsilon} |\psi_2(\epsilon d(x, y))||f(y)| d\mu(y) \\
\leq \epsilon \| f \|_{L^1} \left( \int_{1/\epsilon \leq d(x, y) < 2/\epsilon} |\psi_2(\epsilon d(x, y))|^2 d\mu(y) \right)^{1/2} \\
\leq c \| f \|_{L^1} \epsilon^{1/2}. \tag{2.9}
\]

By (2.7), (2.8), (2.9) and the Lebesgue dominated convergence Theorem, the desired conclusion follows, ending the proof of the Lemma.

(2.10) LEMMA. (Partition of unity). Let $x \in X$ and $r > 0$. Then, there exists a sequence $\{\Phi_j(x, y)\}_{j \geq 0}$ of non-negative functions satisfying:

(2.11) the support of $\Phi_j^r$ for $j \geq 1$ is contained in the ring $C(x, (2K)^j r, (2K)^{j+2} r)$,

(2.12) the support of $\Phi_0^r$ is contained in $B(x, 4Kr)$ and $\Phi_0^r(x) = 1$ on $B(x, 3Kr)$,

(2.13) there exists a constant $c$ shuch that for every $j \geq 0, \Phi_j^r \in \text{Lip}(\alpha)$ as functions of $y$ with $\| \Phi_j^r \|_{\alpha} \leq c(2K)^{-j\alpha r - \alpha}$,

(2.14) $\sum_{j \geq 0} \Phi_j^r(x, y) = 1$ for every $y \in X$.

PROOF. Let $\eta(t)$ and $\gamma(t)$ in $C^\infty([0, \infty))$ satisfying: $0 \leq \eta(t) \leq 1$, $\text{supp} \, \eta \subset [0, 4K]$, $\eta(t) = 1$ if $0 \leq t \leq 3K$, $0 \leq \gamma(t) \leq 1$, $\text{supp} \gamma \subset [2K, 8K^3]$ and $\gamma(t) = 1$ if $3K \leq t \leq 6K^2$.\]
Taking \( \psi_0(x, y) = \eta(d(x, y)/r) \) and \( \psi_j(x, y) = \gamma(d(x, y))/\gamma(\frac{d(x, y)}{r} + 1) \) for every \( j \geq 1 \), it follows easily that \( \Phi_j(x, y) = \psi_j(x, y) \sum_{k \geq 0} \psi_k(x, y) \) for \( j \geq 0 \), satisfy all the conditions in the lemma.

**Lemma (2.15).** Let \( k(x, y) \) be a kernel satisfying (2.1), (2.2) and (2.3). Let \( b(x) \) be a multiple of a \((\rho, \infty)\) atom with support contained in \( B(x_0, r) \). Assume that \( \{\Phi_j(x, y)\}_{j \geq 0} \) is as in Lemma (2.10) and \( T_j^* \) is the operator associated to the kernel \( k_j^* = k(x, y)\Phi_j(x, y) \), for \( j \geq 0 \). Then

(2.16) the support of \( T_j^* b \) is contained in \( B(x_0, (2K)^{j+3}r) \) for \( j \geq 0 \),

(2.17) \( \| T_j^* b \|_\infty \leq \frac{c \| b \|_\infty}{(2K)^{j+1}r} \) for \( j \geq 1 \), \( \| T_0^* b \|_L^1 \leq c \| b \|_\infty \mu(B(x_0, r)) \), and

(2.18) \( \int T_j^* b(x) d\mu(x) = 0 \) for every \( j \geq 0 \).

**Proof.** Let us first note that if \( x \not\in C(x_0, (2K)^{j+2}r) \), then \( T_j^* b(x) = 0 \) for every \( j \geq 1 \). For \( j = 0 \), it is clear that \( \text{supp} (T_0^* b) \subset B(x_0, 8K^2 r) \), and hence (2.16) follows. Next we shall prove (2.17). By remark (2.5), we get

(2.19) \( (2K)^{j+1}r \leq d(x, x_0) \leq (2K)^{j+3}r \).

Therefore if \( x \not\in C(x_0, (2K)^{j+2}r) \), then \( T_j^* b(x) = 0 \) for every \( j \geq 1 \). For \( j = 0 \), it is clear that \( \text{supp} (T_0^* b) \subset B(x_0, 8K^2 r) \), and hence (2.16) follows. Next we shall prove (2.17). By remark (2.5), we get

(2.17)

(2.19) \( (2K)^{j+1}r \leq d(x, x_0) \leq (2K)^{j+3}r \).

Therefore if \( x \not\in C(x_0, (2K)^{j+2}r) \), then \( T_j^* b(x) = 0 \) for every \( j \geq 1 \). For \( j = 0 \), it is clear that \( \text{supp} (T_0^* b) \subset B(x_0, 8K^2 r) \), and hence (2.16) follows. Next we shall prove (2.17). By remark (2.5), we get

\[
\| T_0^* b \|_2 \leq c \| b \|_2 \leq c \| b \|_\infty \mu(B(x_0, r))^{1/2}.
\]

On the other hand, since \( X \) is a normal space, from (2.5) and (2.19) we obtain, that for any \( j \geq 1 \),

\[
|T_j^* b(x)| = \left| \int [K(x, y)\Phi_j(x, y) - K(x, x_0)\Phi_j(x, x_0)]b(y)d\mu(y) \right|
\leq c \| b \|_\infty \int \frac{d(y, x_0)\delta}{d(x_0, x)^{1+\delta}} d\mu(y)
\leq \frac{c}{(2K)^{j(1+\delta)}} \| b \|_\infty.
\]

Finally, (2.18) is a consequence of Lemma (2.4).
Now we are in position to prove the main result.

**THEOREM 2.20** Let $T$ be a singular integral operator associated to a kernel $k(x,y)$ satisfying (2.1), (2.2) with $\delta > 1/l - 1$ and (2.3). Assume that $l(1 + \alpha) > 1$. Then, $T$ is a bounded operator from $H_\omega$ into $H_\omega$.

**PROOF:** By the density of $L^2(X)$ in $H_\omega$, it is enough to show the theorem for $f \in L^2(X) \cap H_\omega$. Given $\epsilon > 0$, from Theorem A and (1.24), there exists a sequence $\{b_k\}_k$ of multiples of $(\rho, \infty)$ atoms with $\text{supp}(b_k) \subset B_k = B(x_k, r_k)$, such that $f = \sum_k b_k$ in $(E^\alpha)'$ and

\[
(2.21) \quad \| f \|_{H_\omega} (1 + \epsilon) \geq \Lambda_\infty(\{b_k\}).
\]

If we are able to prove that

\[
(2.22) \quad T f = \sum_k T b_k \text{ in } (E^\alpha)' ,
\]

we will get $T f \in H_\omega$ and $\| T f \|_{H_\omega} \leq c \| f \|_{H_\omega}$. In fact, let $\{\Phi^j_k\}_j$ be a partition of the unity as in Lemma (2.10) associated to $B_k$, therefore

\[
(2.23) \quad T f = \sum_k \sum_j T_j^k b_k + \sum_k T_0^k b_k \text{ in } (E^\alpha)' .
\]

Furthermore, Lemma (2.15) implies that $\{T_j^k b_k\}_{j,k}$ are multiples of a $(\rho, \infty)$ atom. Hence, from (1.24) it follows that

\[
(2.24) \quad \| T f \|_{H_\omega} \leq \Lambda_2(\{T_j^k b_k\}_{j,k}) + \Lambda_2(\{T_0^k b_k\}_k).
\]

Let $\eta \geq 1$ be a constant to be determined later, $\lambda = \eta \Lambda_\infty(\{b_k\}_k)$ and $B^j_k \supset \text{supp}(T_j^k b_k), j \geq 0$. We now estimate

\[
(2.25) \quad \sum_k \sum_{j \geq 1} \mu(B_j^k) \omega \left( \frac{\| T_j^k b_k \|_2 M(B_j^k)^{-1/2}}{\lambda^{1/2}} \right).
\]

By (1.8), (2.16) and (2.17), the sum (2.25) is bounded by

\[
c \sum_k \sum_{j \geq 1} (2K)^j \mu(B_j) \omega \left( \frac{\| b_k \|_\infty}{(2K)^{\beta(1 + \delta)} \lambda^{1/2}} \right).
\]
since \( \omega \) is of lower type \( l > 1/1 + \delta \), (2.25) is bounded by

\[
c \sum_{j \geq 1} (c2K)^{(1/(1+\delta))} \sum_k \mu(B_k) \omega \left( \frac{\|b_k\|_{\infty}}{\lambda^{1/l}} \right)
\]

\[
\leq c \sum_k \mu(B_k) \omega \left( \frac{\|b_k\|_{\infty}}{\lambda^{1/l}} \right).
\]

Therefore, using again that \( \omega \) is of lower type \( l \) and choosing \( \eta = c \), the sum (2.25) is less than or equal to 1, which implies

\[
(2.26) \quad \Lambda_2(\{T^k_j b_k\}_{j,k}) \leq c \Lambda_\infty(\{b_k\}).
\]

On the other hand, by (2.5) \( T^k_0 \) is a bounded operator on \( L^2 \), thus applying (1.8), (2.16), (2.17) and the fact that \( \omega(s)/s \) is nonincreasing, we get

\[
(2.27) \quad \sum_k \mu(B^k_0) \omega \left( \frac{\|T^k_0 b_k\|_2 \mu(B^k_0)^{-1/2}}{\lambda^{1/l}} \right)
\]

\[
\leq c \sum_k \mu(B_k) \omega \left( \frac{c \|b_k\|_{\infty}}{\lambda^{1/l}} \right)
\]

\[
\leq \sum_k \mu(B_k) \omega \left( c^{1/l} \frac{\|b_k\|_{\infty}}{\lambda^{1/l}} \right)
\]

Taking \( \eta = c \), and using (2.27), it follows that

\[
(2.28) \quad \Lambda_2(\{T^k_0 b_k\}_{k}) \leq c \Lambda_\infty(\{b_k\}).
\]

Collecting the estimates (2.21), (2.24), (2.26) and (2.28), we obtain that

\[
\|T f\|_{H_\omega} \leq c \|f\|_{H_\omega}
\]

In order to prove (2.22), let us first note that if \( \tilde{T} f \) is the operator of Lemma (2.6) associated to the kernel \( \tilde{k}_e(x, y) \), then \( \tilde{k}_e(x, y) \) is a function of bounded support belonging to \( \text{Lip}(\delta) \) for each \( x \in X \). Therefore

\[
\tilde{T}_e f = \sum_k \tilde{T}_e b_k, \text{ pointwise and in } (E^\circ)' .
\]
Moreover Lemma (2.6) implies that $\tilde{T}_\varepsilon f$ converges to $Tf$ in $L^2$. In consequence, if we are able to show

$$
(2.29) \quad \sum_k \tilde{T}_\varepsilon b_k \xrightarrow{\varepsilon \to 0} \sum_k T b_k \text{ in } H_\omega,
$$

then (2.22) holds immediately, completing the proof of the Theorem. Now, in order to prove (2.29), we decompose both operators, $\tilde{T}_\varepsilon$ and $T$, as in (2.23). Therefore, we have

$$
(2.30) \quad \sum_k (\tilde{T}_\varepsilon b_k - T b_k) = \sum_{k,j \geq 0} (\tilde{T}_{\varepsilon,j}^k b_k - T_j^k b_k) = \sum_{k,j \geq 0} \tilde{T}_{\varepsilon,j}^k b_k,
$$

where $T_{\varepsilon,j}^k$ is the operator associated to the kernel

$$
\tilde{K}_{\varepsilon,j}^k(x,y) = K(x,y)[\psi_1(\frac{d(x,y)}{\varepsilon})\psi_2(d(x,y)\varepsilon) - 1]\Phi_j^k(x,y) =: \tilde{K}_\varepsilon(x,y)\Phi_j^k(x,y).
$$

Since by (2.5) $\tilde{K}_\varepsilon(x,y)$ satisfies (2.1), (2.2) and (2.3) with a constant independent of $\varepsilon$, using Lemma (2.15) and proceeding as in estimates (2.25) and (2.27), we get that

$$
\sum_k \sum_{j \geq 0} \mu(B_j^k) \omega(\|T_{\varepsilon,j}^k b_k\|_2 \mu(B_j^k)^{-1/2}) < \infty,
$$

where $B_j^k \supset \text{supp}(T_{\varepsilon,j}^k b_k)$. Thus, given $0 < \beta \leq 1$, there exists $N = N(\beta)$ such that

$$
(2.31) \quad \sum_{|k| > N} \sum_{j > N} \mu(B_j^k) \omega(\|T_{\varepsilon,j}^k b_k\|_2 \mu(B_j^k)^{-1/2}) < \beta/2.
$$

This finishes the proof of the Theorem.
§ 3. CHARACTERIZATION OF THE ORLICZ-HARDY SPACES $H_\omega$

In this section we shall work, as before, on a normal space $X = (X, d, \mu)$ of order $\alpha$.

Let $\{b_i\}$, a sequence of multiples of $(\rho, q)$ atoms, $1 < q \leq \infty$, such that $\Lambda_q(\{b_i\}) < \infty$ and $\alpha_i = \|b_i\|_q \mu(B_i)^{-1/q} \omega^{-1}(\mu(B_i)^{-1})$, where $B_i \supset \text{supp}(b_i)$. Let $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$ and $\psi(x) \in \text{Lip}(\rho)$. Then

$$\left| \sum_i b_i(\psi) \right| \leq \|\psi\|_{\text{Lip}(\rho)} \sum_i \rho(r_i) \mu(B_i)^{1/q'} \|b_i\|_q \leq c \|\psi\|_{\text{Lip}(\rho)} \sum_i \alpha_i.$$  

In order to estimate the sum $\sum_i \alpha_i$ we shall need the following lemma whose proof can be found in [V], p. 410.

(3.2) LEMMA: Assume that $\rho(t), \{b_i\}$, and $\alpha_i$ are as above. Then there exists a constant $c$ independent of $\{b_i\}$, such that

$$\sum_i \alpha_i \leq c(\Lambda_q(\{b_i\}) + 1)^{1/2}.$$  

Using Lemma (3.2), by (3.1) it follows that the series $\sum_i b_i(\psi)$ is absolutely convergent for every $\psi \in \text{Lip}(\rho)$. Thus, if we define

$$f(\psi) = \sum_i b_i(\psi),$$  

we obtain a linear functional on $\text{Lip}(\rho)$ satisfying

$$|f(\psi)| \leq c \|\psi\|_{\text{Lip}(\rho)} [\Lambda_q(\{b_i\}) + 1]^{1/2}.$$  

(3.5) DEFINITION: Let $\omega$ be a growth function of positive lower type $l$. If $\rho(t) = t^{-1}/\omega^{-1}(t^{-1})$, we define $\tilde{H}^{p,q}(X) = \tilde{H}^{p,q}$, $1 < q \leq \infty$, as the linear space of all bounded linear functionals $f$ on $\text{Lip}(\rho)$ which can be represented as in (3.3), where $\{b_i\}$ is a sequence of multiples of $(\rho, q)$ atoms such that $\Lambda_q(\{b_i\}) < \infty$. For $f \in \tilde{H}^{p,q}$, we define

$$\|f\|_{\tilde{H}^{p,q}} = \inf \{\Lambda_q(\{b_i\})\},$$  

where the infimum is taken over all possible representations of $f$ of the form (4.3).

We now observe that, since every $\psi$ in $E^\alpha$ belongs to $\text{Lip}(\rho)$, we can define the linear transformation $R$ from $\tilde{H}^{p,q}$ into $H_\omega$ given by

$$R(f) = \tilde{f},$$  

where $\tilde{f}$ is defined as in (3.6).
where $\tilde{f}_1$ is the restriction of $f$ to $E^\circ$.

The next result states that $R$ is an isomorphism onto $H_\omega$. Its proof makes use of the atomic decomposition of $H_\omega$ and Lemma (5.5) in [V], and it follows the lines of (5.9) in [MS].

(3.7) *THEOREM:* Let $R$ be as in (3.6). Then $R$ defines a one to one linear mapping from $\tilde{H}^{p,q}$ onto $H^\omega$. Moreover, there exist two positive constants $c_1$ and $c_2$ such that

(3.8) \[ c_1 \| f \|_{\tilde{H}^{p,q}} \leq \| Rf \|_{H_\omega} \leq c_2 \| f \|_{\tilde{H}^{p,q}}. \]

PROOF: Let $f = \sum b_i$ in $\tilde{H}^{p,q}$. Theorem A implies that

\[ R(\tilde{H}^{p,q}) \subset H_\omega \text{ and } \| Rf \|_{H_\omega} \leq c \| f \|_{\tilde{H}^{p,q}}. \]

On the other hand, given $g \in H_\omega$, again by Theorem A, there exists a sequence $\{b_i\}$ of multiples of $(\rho, q)$ atoms such that

\[ g = \sum b_i \text{ in } (E^\circ)' \text{ and } \Lambda_q(\{b_i\}) \leq c \| g \|_{H_\omega}. \]

By (3.4), the sum $\sum b_i$ defines an element $f$ of $\tilde{H}^{p,q}$ whose restriction to $E^\circ$ coincides with $g$, that is $R(f) = g$. In order to show that $R$ is one to one, we need to prove that $f(\psi) = 0$ for every $\psi \in E^\circ$ implies $f(\psi) = 0$ for every $\psi$ in Lip$(\rho)$. This result is obtained in Lemma (5.5) of [V] as a consequence of lemma (3.2).

In what follows we will restrict our attention to the case $X = \mathbb{R}^n$ and we shall study the connection of the Hardy-Orlicz spaces $H_\omega(\mathbb{R}^n)$ with Riesz transforms. Using the boundedness result established in section 2, we shall obtain in Theorem (3.38) a characterization of $H_\omega(\mathbb{R}^n)$ in terms of these operators.

Let $P(x)$ be the Poisson kernel defined by $P(x) = c_n(1 + |x|^2)^{-n+1}$ and denote $P_t(x) = t^{-n}P(x/t)$. For $f \in L^2 \cap H_\omega(\mathbb{R}^n)$, we shall consider the $n + 1$ harmonic functions in $\mathbb{R}^{n+1}_+ = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$ defined by

\[ u_1(t, x) = P_t * R_1 f(x), \ldots, u_n(t, x) = P_t * R_n f(x), u_{n+1}(t, x) = P_t * f(x). \]
Let us denote by $F(x, t)$ the vector field associated to $f$ given by
\begin{equation}
F(x, t) = (u_1(t, x), \ldots, u_n(t, x), u_{n+1}(t, x)).
\end{equation}

The vector field $F$ satisfies the following generalized Cauchy-Riemann equations:
\begin{equation}
\text{div} F = \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}
\end{equation}
for every $j \neq k$; $j, k \in \{1, \ldots, n+1\}$, where $x_{n+1} = t$.

Let $x \in \mathbb{R}^n$ and $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1} : |x - y| < t\}$ the cone of aperture one and vertex in $x$. We define the non-tangential maximal function $f^{**}(x)$ of $f$ as
\begin{equation}
f^{**}(x) = \sup_{(y, t) \in \Gamma(x)} u(t, y) = \sup_{(y, t) \in \Gamma(x)} P_t f(y).
\end{equation}

We shall also consider the following maximal operator
\begin{equation}
f^*_M(x) = \sup_{\psi \in \text{supp} \psi} f(\psi)/A(\psi),
\end{equation}
where $A(\psi) = \int |\psi(t)| dt + |\text{supp} \psi|^{M+1} \int |\psi^{(M+1)}(t)| dt$ and the supremum is taken over all the functions $\psi \in C^\infty$ with compact support such that $\text{dist}(x, \text{supp} \psi) < |\text{supp} \psi|$.

For the case of $H^p$, $p \leq 1$, it is known that the norm $\| f^*_M \|_{L^p}$ is equivalent to that given by the atomic decomposition. On the other hand, in [V] (see Theorem A) the equivalence between the atomic Orlicz norm and the norm $\| f_\gamma \|_{L^\omega}$ is shown in the general context of spaces of homogeneous type.

For the case $L^p$, following the same argument given in Theorem A it can also be established that the norm $\| f^*_M \|_{L^\omega}$ is equivalent to that defined in the atomic Orlicz space $H^{p,q}$. Therefore, in the following we shall make use of the maximal $f^*_M$ instead of $f^*_\gamma$.

Moreover, following García-Cuerva - Rubio de Francia ([GC-RF] pag. 247) it is easy to see that
\begin{equation}
\| f^*_M \|_{L^\omega} \leq c \| f^{**} \|_{L^\omega} \quad \text{for} \quad M_1 > 1.
\end{equation}

On the other hand, the reverse inequality is a consequence of the following result whose proof is similar to that of Lemma (4.3) in [V].

(3.11) **LEMMA:** Let $\omega$ a growth function of positive lower type $l > \frac{n}{n+1}$. Assume that $b(x)$ is a function belonging to be $L^q(\mathbb{R}^n)$, $1 < q \leq \infty$, with support contained in
\[ B = B(x_0, r_0) \text{ and } \int b(x) dx = 0. \text{ Then, there exists a constant } c, \text{ independent of } b(x), \text{ such that} \]
\[ \int \omega(b^{**}(x)) dx \leq c |B| \omega\left( \|b\|_q |B|^{-1/q} \right). \]

Therefore, in the following we shall assume that there exist two positive constants \( 0 < c_1 \leq c_2, \) satisfying \( \|f\|_{H_\omega} \leq \|f^{**}\|_{L_\omega} \leq c_2 \|f\|_{H_\omega}. \)

We shall need the following technical lemma concerning the equivalence between growth functions.

(3.13) **Lemma:** Let \( \gamma \geq 1. \) Let \( \psi(t) \) be a continuous increasing function of lower type \( \alpha \) and upper type \( \beta \) such that \( \beta \geq \alpha > \gamma. \) Then, the function
\[ \Phi(t) = t^\gamma \int_0^1 \frac{\psi(s)}{s^{1+\gamma}} ds \]
is a continuous, increasing and convex function equivalent to \( \psi(t). \)

**Proof:** Since \( \alpha > \gamma, \) we get
\[ \Phi(t) = \int_0^1 \frac{\psi(ts)}{s^{1+\gamma}} ds \leq c \psi(t) \int_0^1 \frac{s^\alpha}{s^{1+\gamma}} ds = \frac{c}{\alpha - \gamma} \psi(t). \]

On the other hand, using the fact that \( \psi(t) \) is the upper type \( \beta, \) we have that
\[ \psi(st) \geq c s^\beta \psi(t) \quad \text{if} \quad s \leq 1. \]

Therefore, since \( \beta > \gamma, \) we obtain that
\[ \Phi(t) = \int_0^1 \frac{\psi(ts)}{s^{1+\gamma}} ds \geq c \psi(t) \int_0^1 \frac{s^\beta}{s^{1+\gamma}} ds = \frac{c}{\beta - \gamma} \psi(t). \]

To prove that \( \Phi \) is a convex function, it is enough to see that \( \Phi'(t) \) is increasing. Take \( t_1 < t_2. \) Since \( \psi \) is non-decreasing and \( \gamma \geq 1, \) it follows that
\[ \Phi'(t_2) - \Phi'(t_1) = \gamma t_2^{\gamma - 1} \int_{t_1}^{t_2} \frac{\psi(s)}{s^{1+\gamma}} ds + \gamma \left( t_2^{\gamma - 1} - t_1^{\gamma - 1} \right) \int_0^{t_1} \frac{\psi(s)}{s^{1+\gamma}} ds \]
\[ + \frac{\psi(t_2)}{t_2} - \frac{\psi(t_1)}{t_1} \]
\[ \geq \frac{t_2^{\gamma - 1} - t_1^{\gamma - 1}}{t_2^{\gamma - 1}} \psi(t_1) + \frac{\psi(t_2)}{t_2} - \frac{\psi(t_1)}{t_1} \]
\[ \geq \frac{\psi(t_2) - \psi(t_1)}{t_2} \geq 0. \]
which ends the proof of the lemma.

In the sequel, we shall assume that $\Phi(t)$ is a continuous strictly increasing non negative function of lower type greater than one and of finite upper type, such that $\lim_{t \to 0^+} \Phi(t) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$.

The following result, on harmonic majorization of subharmonic functions which are uniformly in an Orlicz space $L_\Phi$, is an extension to that of Theorem 4.10 in [GC-RF].

(3.14) THEOREM: Let $U(x,t)$ be a non-negative subharmonic function in $\mathbb{R}^{n+1}_+$ such that

$$\sup_{t > 0} \| U(.,t) \|_{L_\Phi} < \infty.$$ 

Then, $U(x,t)$ has a least harmonic majorant in $\mathbb{R}^{n+1}_+$. Moreover, this harmonic majorant is the Poisson integral of a function $h \in L_\Phi(\mathbb{R}^n)$, where $h$ is obtained as the limit of $U(x,t_j)$ for any sequence $t_j \downarrow 0$ ($j \to \infty$) in the weak * topology of $L_\Phi$.

For the proof of Theorem (3.14) we shall need the next result.

(3.15) LEMMA: Let $U(x,t)$ be a non-negative subharmonic function in $\mathbb{R}^{n+1}_+$ satisfying

(3.16) $$\sup_{t > 0} \| U(.,t) \|_{L_\Phi} = M < \infty.$$ 

Then, there exists a constant $c$ depending only on $\Phi$ and $n$, such that

(3.17) $$U(x,t) \leq c M \Phi^{-1}(1/t^n), \text{ for every } (x,t) \in \mathbb{R}^{n+1}_+.$$ 

Consequently, $U(x,t)$ is bounded in each proper sub-half-space $\{(x,t) \in \mathbb{R}^{n+1}_+ : t > 0\}$. Moreover, the following property holds:

$U(x,t) \to 0$ as $|x,t| \to \infty$ in each proper sub-half-space.

PROOF: Let $(x_0,t_0) \in \mathbb{R}^{n+1}_+$ and

$$B_0 = B((x_0,t_0),t_0/2) \subset B(x_0,t_0/2) \times (t_0 - t_0/2,t_0 + t_0/2) = B_0 \times (t_0/2,3t_0/2).$$

Since $U(x,t)$ is sub-harmonic, applying the Hölder inequality (1.12) with $\Psi$ the complementary function of $\Phi$, we have

(3.18) $$U(x_0,t_0) \leq \frac{1}{|B_0|} \int_{B_0} U(x,t)dxdt \leq \frac{c}{t_0^{n+1}} \int_{t_0/2}^{3t_0} \int_{B_0} x_{B_0}(x)U(x,t)dxdt$$

$$\leq \frac{c}{t_0^{n+1}} \int_{t_0/2}^{3t_0} \| U(.,t) \|_{L_\Phi} x_{B_0} \|_{L_\Phi} dt.$$
Taking \( \| x_{n_j} \|_{L^\Phi} \equiv |B_0|\Phi^{-1}(1/|B_0|) \), from (3.16) and (3.18), we get (3.17). On the other hand, given \( t_0 > 0 \) fix and \( \varepsilon > 0 \), since \( \lim_{s \to 0^+} \Phi^{-1}(s) = 0 \), there exists \( t_1 > t_0 \) such that \( \Phi^{-1}(1/t_1) \leq \varepsilon \). Thus, by (3.17) we obtain that

\[
U(x, t) \leq cM\varepsilon, \text{ for every } t \geq t_1 \text{ and } x \in \mathbb{R}^n.
\]

It only remains to prove that \( U(x, t) \leq \varepsilon \), for every \( t_0 \leq t < t_1 \) and \( |x| \) big enough. Let \( x \in \mathbb{R}^n \) and \( |x| > t_1 \). Take \( \tilde{B} = B((x, \tilde{t}), t_0/2) \) with \( t_0 \leq \tilde{t} < t_1 \). Proceeding as in the first part of the proof, we get

\[
(3.19) \quad U(x, \tilde{t}) \leq \frac{c}{t_0^{n+1}} \| x_B(x, t_0/2) \|_{L^\Phi} \int_{t_0/2}^{\tilde{t}_1} \| x_B(x, t_0/2)U(\cdot, t) \|_{L^\Phi} dt
\]

Now, let us observe that, for each \( t \), we have

\[
(3.20) \quad \int \Phi \left[ x_B(x, t_0/2)(y)U(y, t) \right] dy = \int_{B(x, t_0/2)} \Phi(U(y, t)) dy \\
\leq \int_{|y| \geq |x| - t_1/2} \Phi(U(y, t)) dy
\]

Since \( \Phi \) is of finite upper type, from (3.16) and (3.20) it follows that

\[
\| x_B(x, t_0/2)U(\cdot, t) \|_{L^\Phi} \to 0 \text{ as } |x| \to \infty \text{ for each } t.
\]

Therefore, using in (3.19) the Lebesgue dominated convergence Theorem, we obtain that

\[
U(x, \tilde{t}) \to 0 \text{ as } |x| \to \infty, \text{ uniformly for every } t_0 \leq \tilde{t} < t, \text{ completing the proof of the lemma.}
\]

**PROOF OF THEOREM (3.14)**: Let \( \{t_j\} \) be a sequence such that \( t_j \downarrow 0 \) and denote \( f_j(x) = U(x, t_j) \). Since \( \| f_j \|_{L^\Phi} < \infty \) for every \( j \), there exists a subsequence of \( \{f_j\} \), that we also denote \( \{f_j\} \), converging in the weak-* topology of \( L^\Phi \) (see Theorem 144 in [K]). That is, there exists a function \( f \in L^\Phi \), such that for every \( g \in L^\Phi \), \( \Phi \) being the complementary function of \( \Phi \), we have

\[
(3.21) \quad \int f_j(x)g(x)dx \xrightarrow{j \to \infty} \int f(x)g(x)dx.
\]

If we are able to prove that

\[
(3.22) \quad U(x, t + t_j) \leq \int_{\mathbb{R}^n} P_t(x - y)f_j(y)dy
\]
for every \( j \), then using (3.21), the conclusion of the Theorem follows immediately. Now, in order to prove (3.22) it is enough to see that the functions
\[
G_j(x, t) = U(x, t + t_j) \quad \text{and} \quad F_j(x, t) = P_t f_j(x)
\]
tend to zero when \( |(x, t)| \to \infty \). In fact, if this happens, given \( \varepsilon > 0 \), there exists \( R > 0 \) big enough satisfying
\[
D_j(x, t) = G_j(x, t) - F_j(x, t) \leq \varepsilon, \quad (3.23)
\]
for every \((x, t)\) such that \(|(x, t)| \geq R\), and in particular, (3.23) holds for every \((x, t)\) in the boundary of the region \( K_R = \{(x, t) \in \mathbb{R}^{n+1}_+ : |(x, t)| \leq R\} \). Since \( D_j(x, t) \) is subharmonic, it follows that
\[
D_j(x, t) \leq \varepsilon, \quad \text{for every } (x, t) \in K_R,
\]
which together with (3.23) proves (3.22). Finally, let us prove the convergence of the functions \( G_j \) and \( F_j \). Applying Lemma (3.15), we obtain,
\[
G_j(x, t) \to 0 \quad \text{as} \quad |(x, t)| \to \infty
\]
and
\[
f_j(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]
Using this fact and that \( f_j \in L^p \), by a standard argument, we may conclude that
\[
F_j(x, t) \to 0 \quad \text{as} \quad |(x, t)| \to \infty,
\]
which completes the proof of the Theorem.

We also need the following lemma which gives a norm inequality between the vector field \( F(x, t) \), defined in (3.9), and the function \( f(x) \).

(3.24) **Lemma:** Let \( F(x, t) \) be the function defined in (3.9). Then
\[
\sup_{t \geq 0} \| F(., t) \|_{L^\infty} \leq c \| f \|_{L^p}.
\]

**Proof:** Let \( \eta = \eta_1 + \eta_2 \) be a constant to be fixed later on. Let us estimate
\[
(3.25) \quad \sup_{t \geq 0} \int \omega \left[ \frac{|F(x, t)|}{(\eta \| f \|_{L^p})^{1/j}} \right] dx \leq \int \omega \left[ \sum_{j=1}^{n+1} \sup_t \frac{|u_j(t, x)|}{(\eta \| f \|_{L^p})^{1/j}} \right] dx
\]
An application of Theorem 2.20, together with (3.25) and the fact that $\omega(s)$ is lower type 1, imply by choosing $\eta_1 = \eta_2 = C_2$ and $\eta_2 = nC_2c$ with $C_2$ the constant appearing in (3.12). This finishes the proof of the lemma.

The next lemma provides the boundedness of the Poisson integral on $H_\omega$.

(3.26) LEMMA: Let $f \in H_\omega$. Then $u(t,x) = P_t*f(x)$ belongs to $L^q \cap H_\omega$, $1 < q \leq \infty$, and

$$\| u(t, \cdot) \|_{H_\omega} \leq c \| f \|_{H_\omega}.$$ 

PROOF: In view of Theorem (3.7), we have that $f \in \tilde{H}^{\rho, \infty}$ and there exists a sequence of multiples of $(\rho, \infty)$ atoms such that

$$f(\Psi) = \sum_j b_j(\Psi), \text{ for every } \Psi \in Lip(\rho).$$

Since $P_t(x) \in Lip(\rho)$ with $\| P_t \|_{Lip(\rho)} \leq c(t)$, we get

(3.27) $|u(t,x)| = |f(P_t(x - .))| = \sum_j b_j(P_t(x - .)) \leq c \| P_t(x - .) \|_{Lip(\rho)} \leq c(t).$

Therefore, $u(t,\cdot)$ is an $L^\infty$ function. Now, let us see that $u(t,\cdot) \in L^q, 1 < q < \infty$. Given $g \in S$, we have

$$u(t, \cdot)(g(\cdot)) = \int N \lim_{N \to \infty} \sum_{j=1}^N b_j*P_t(x)g(x)dx.$$
Using (3.27) and the Lebesgue dominated convergence Theorem, we obtain

\[ |u(t, \cdot)(g(\cdot))| = \lim_{N \to \infty} \sum_{j=1}^{N} b_j * P_t(x) g(x) dx = \lim_{N \to \infty} \sum_{j=1}^{N} b_j (P_t * g) = |f(P_t * g)| \leq c \| P_t * g \|_{L^{\infty}(\rho)} \]

In order to prove that \( u(t, \cdot) \in L^q \), it is enough to show

\[ (3.28) \quad \| P_t * g \|_{L^{\infty}(\rho)} \leq c(t) \| g \|_{L^{q'}}. \]

Let \( x, x' \in \mathbb{R}^n \) with \( |x - x'| > t/2 \). Then using the fact that \( \rho \) is of upper type \( m < 1 \), we have

\[ (3.29) \quad |P_t * g(x) - P_t * g(x')| \leq 2 \| P_t * g \|_{\infty} \leq 2 \| P_t \|_{L^1} \| g \|_{L^{q'}} \leq ct^{-n/q'} \rho \left( \frac{|x - x'|}{t} \right) \| g \|_{L^{q'}} \leq ct^{-n/q'} \max\{1/t, 1/m\} \rho(|x - x'|) \| g \|_{L^{q'}} = c(t) \rho(|x - x'|) \| g \|_{L^{q'}} \]

On the other hand, if \( |x - x'| < t/2 \), we obtain

\[ (3.30) \quad |P_t * g(x) - P_t * g(x')| \leq \int |P_t(x - y) - P_t(x' - y)| |g(y)| dy \]

\[ \leq \left( \int |P_t(x - y) - P_t(x' - y)|^q dy \right)^{1/q} \| g \|_{L^{q'}} \leq |x - x'| \| g \|_{L^{q'}} \left( \int |\nabla_x P_t((x - y) + \theta(x - x'))|^q dy \right)^{1/q} \leq c \frac{|x - x'|}{t} \| g \|_{L^{q'}} \left( \int_{|x - y| < t} dy + \int_{|x - y| > t} dy \right)^{1/q} \leq c \left( \frac{|x - x'|}{t} \right)^n \| g \|_{L^{q'}} \leq c \rho \left( \frac{|x - x'|}{t} \right) t^{-n/q'} \| g \|_{L^{q'}} \leq c(t) \rho(|x - x'|) \| g \|_{L^{q'}}. \]
because \( p \) is of upper type \( m < 1 \). Thus, from (3.29) and (3.30) we obtain (3.28). Next we prove that \( u(t, \cdot) \in H_\omega \). In fact,

\[
(3.31) \quad u(t, \cdot)^{(x)} = \sup_{|y-z|<s} |P_t^* f(y)| \leq \sup_{|y-z|<s+t} |P_{t+s}^* f(x)| \leq f^{**}(x).
\]

Therefore, we conclude that \( P_t^* f \in H_\omega \) with \( \|P_t^* f\|_{H_\omega} \leq c \|f\|_{H_\omega} \).

(3.32) REMARK: Let \( f \in (\text{Lip}(p))' \), then \( u(x, t) = f(P_t(x - \cdot)) \) is a harmonic function in \( \mathbb{R}^{n+1}_+ \). In fact, taking for example \( \frac{1}{h}[u(x, t + h) - u(x)] \), it can be proved that this incremental quotient tends to \( f(\frac{\partial}{\partial t} P_t(x - \cdot)) \), by showing that for each \((x, t)\) fixed

\[
\| \frac{1}{h} [P_{t+h}^*(x) - P_t^*(x)] - \frac{\partial}{\partial t} P_t(x - \cdot) \|_{\text{Lip}(p)} \rightarrow 0 \quad h \rightarrow 0.
\]

This, in turn, is a consequence of the mean value Theorem, and the fact that \( p \) is upper type \( m < 1 \).

(3.33) LEMMA: Let \( f \) be a distribution belonging to \( H_\omega \). Then

\[
\| u(t, \cdot) - f \|_{H_\omega} \rightarrow 0, \text{ as } t \rightarrow 0.
\]

PROOF: Let \( \varepsilon > 0 \). We first assume that \( f \in H_\omega \cap L^q \), \( 1 < q \leq \infty \). Thus, there exists a ball \( B = B(x_0, R) \) such that

\[
(3.34) \quad \int_{CB} \omega(f^{**}(x)) \, dx < \varepsilon/2.
\]

Since by (3.31) \( u(t, \cdot)^{(x)} \leq f^{**}(x) \), it follows that

\[
(3.35) \quad \int_{CB} \omega[(u(t, \cdot) - f(\cdot)^{(x}) \, dx \leq 2 \int_{CB} \omega(f^{**}(x)) \, dx < \varepsilon.
\]

On the other hand, if \( \lambda_t = \| u(t, \cdot) - f \|_{L^q \cap B}^{-1/q} \), using that \( \omega(s)/s \) is non increasing, we have

\[
\omega((u(t, \cdot) - f)^{(x)}) \leq \omega(M(u(t, \cdot) - f)(x)) \\
\leq \omega[(M(u(t, \cdot) - f)(x) + \lambda_t] \\
\leq \omega(\lambda_t) \left( M(u(t, \cdot) - f)(x) + 1 \right). 
\]

Integrating on \( B \), we obtain
From (3.35) and (3.36), since \( \omega \) is of finite upper type, we get

\[
\int_B \omega[(u(t,\cdot) - f)^\ast(x)]dx \leq C \omega(\lambda_t) |\lambda_t^{-1} M(u(t,\cdot) - f)| \|B\|^{1/q} + |B| \to_0
\]

\[
\leq C \omega(\lambda_t) |B| \to_0.
\]

which proves the lemma under the assumption \( f \in H_\omega \cap L^q \). Next, we shall remove that assumption. Let \( f \in H_\omega \). Given \( \varepsilon > 0 \), by the density of \( L^q \) in \( H_\omega \) (see Theorem (4.16) in [V]), there exists \( g \in L^q \) such that \( \|f - g\|_{H_\omega} < \varepsilon \). Hence, in view of Lemma (3.26), we have that there exists \( t_0 = t_0(\varepsilon) \) such that

\[
\|u(t,\cdot) - f\|_{H_\omega} \leq \|P_t^\ast(f-g)\|_{H_\omega} + \|P_t^\ast g - f\|_{H_\omega} + \|f - g\|_{H_\omega}
\]

\[
\leq \varepsilon + c \|f - g\|_{H_\omega} \leq c\varepsilon,
\]

for every \( t \leq t_0 \), as we wanted to prove.

Now we are in a position to prove the main theorem, which gives another characterization of the Hardy-Orlicz spaces.

THEOREM: Let \( \omega \) be a function of lower type \( l \) such that \( l > \frac{n}{n+1} \). Assume that \( \omega(s)/s \) is non increasing. Then there exist two constants \( c_1 \) and \( c_2 \) satisfying

\[
c_1 \|f\|_{H_\omega} \leq \|f\|_{L^\omega} + \sum_{j=1}^{n} \|R_j f\|_{L^\omega} \leq c_2 \|f\|_{H_\omega},
\]

for every \( f \in L^q \cap H_\omega(\mathbb{R}^n) \), \( 1 \leq q < \infty \), and

\[
c_1 \|f\|_{H_\omega} \leq \lim_{t \to 0} \|u(t,\cdot)\|_{L^\omega} + \sum_{j=1}^{n} \lim_{t \to 0} \|R_j(u(t,\cdot))\|_{L^\omega}
\]

\[
\leq c_2 \|f\|_{H_\omega}, \text{ for every } f \in H_\omega.
\]

PROOF: Let \( f \in L^q \cap H_\omega(\mathbb{R}^n) \). Let us first check the right inequality on (3.39). Since \( P_t^\ast f \) tends to \( f \) in \( L^q \), we have that

\[
|f(x)| \leq f^\ast(x) \text{ and } |R_j f(x)| \leq (R_j f)^\ast(x) \text{ for } a.e. x \in \mathbb{R}^n.
\]
Therefore,
\[
\int \omega \left[ \frac{|f(x)|}{(c \| f \|_{L^\infty})^{1/l}} \right] dx \leq \int \omega \left[ \frac{|f^{**}(x)|}{(c \| f \|_{L^\infty})^{1/l}} \right] dx \leq 1,
\]
and, applying Theorem (2.20),
\[
\int \omega \left[ \frac{|R_j f(x)|}{(c \| f \|_{L^\infty})^{1/l}} \right] dx \leq \int \omega \left[ \frac{R_j f^{**}(x)}{(c \| f \|_{L^\infty})^{1/l}} \right] dx \leq 1,
\]
for every \( j = 1, \ldots, n \), which implies that
\[
\| f \|_{L^\infty} + \sum_{j=1}^{n} \| R_j f \|_{L^\infty} \leq c_2 \| f \|_{L^\infty}.
\]

On the other hand, in order to prove the left inequality on (3.39), we shall consider the function
\[
(3.41) \quad U(y, t) = \left| F(y, t) \right|^{l'}
\]
with \( \frac{n-1}{n} < \frac{n}{n+1} < l' < l \), which is subharmonic in view of Lemma 4.14 in [GC, RF]. Now, we observe that Lemma (3.13) implies that the function \( \psi(t) = \omega(t^{1/l'}) \) is equivalent to a Young function \( \Phi(t) \) of low type \( l'/l' \) > 1 and of upper type \( 1/l' \). Then using Lemma (3.24), we get
\[
\sup_{t>0} \int \phi \left[ \frac{U(y, t)}{(c \| f \|_{L^\infty})^{1/l}} \right] dy \leq \sup_{t>0} \int \omega \left( \frac{|F(y, t)|}{(c \| f \|_{L^\infty})^{1/l}} \right) dy \leq 1.
\]
Therefore
\[
\sup_{t>0} \| U(\cdot, t) \|_{L^\infty} \leq c \| f \|_{L^\infty}^{l'/l'} < \infty.
\]
By Theorem (3.14), there exists a function \( h \in L^\Phi \) such that
\[
(3.42) \quad U(y, t) \leq P_t \ast h(y).
\]
Moreover, for \( t_j \downarrow 0 \) (\( j \to \infty \)) and \( g \in L^\psi \), with \( \psi \) the Young complementary function of \( \Phi \), we have
\[
(3.43) \quad \int h(x) g(x) dx = \lim_{j \to \infty} \int U(x, t_j) g(x) dx.
\]
Now, if \( G(x) = \sup_{(y, t) \in \Gamma(x)} |F(y, t)| \), by (3.41) and (3.42) we obtain that
\[
\int \omega \left[ G(x)/(c \| h \|_{L^\infty})^{1/l'} \right] dx = \int \omega \left[ \sup_{(y, t) \in \Gamma(x)} \frac{(U(y, t)/c \| h \|_{L^\infty})^{1/l'}}{1/l} \right] dx
\]
\[
\leq \int \omega \left( \frac{h^{**}(x)}{c \| h \|_{L^\infty}} \right)^{1/l} dx \leq \int \phi \left( \frac{Mh(x)}{c \| h \|_{L^\infty}} \right) dx,
\]
where \( Mh(x) \) is the Hardy-Littlewood maximal function. From the maximal operator theory in Orlicz spaces, it is known that \( M \) is bounded on \( L^q \). Therefore, it follows that

\[
\| G \|_{L^w} \leq c \| h \|_{L^w}^{1/f'},
\]

This implies, in particular, that \( F \) is non-tangentially bounded at almost every \( x \in \mathbb{R}^n \). Consequently, by Theorem 4.21 in [GC, RF]), there exists a function \( F_0(x) \) such that

\[
F_0(x) = \lim \text{non tang } F(y, t), \quad \text{for } a.e. \ x \in \mathbb{R}^n \quad (y, t) \to x
\]

In view of (3.43) and (3.45), we get

\[
h(x) = |F_0(x)|^{1/f'} \text{ for } a.e. \ x \in \mathbb{R}^n \text{ and } \| F_0 \|_{L^w} \approx \| h \|_{L^w}^{1/f'}
\]

Furthermore, since \( P_t * f \) converges to \( f \) in \( L^q \), we obtain

\[
\| F_0 \|_{L^w} \leq \| f \|_{L^w} + \sum_{j=1}^n \| R_j f \|_{L^w}
\]

Then, from (3.44), (3.46) and (3.47), we have

\[
\int \omega [f^{**}(x)/(c(\| f \|_{L^w} + \sum_{j=1}^n \| R_j f \|_{L^w})^{1/|f|})] \, dx \leq \int \omega \left[ \frac{G(x)}{c(\| F_0 \|_{L^w})^{1/|f|}} \right] \, dx \leq \int \omega \left[ \frac{G(x)}{(c \| h \|_{L^w})^{1/f'}} \right] \, dx \leq 1,
\]

which completes the proof of the Theorem for the case \( f \in L^s \cap H_w \). Now, we assume that \( f \in H_w \). Since Lemma (3.26) implies that \( u(t, \cdot) \in L^q \cap H_w \), applying (3.39) it follows that

\[
c_1 \| u(t, \cdot) \|_{H_w} \leq \| u(t, \cdot) \|_{L^w} + \sum_{j=1}^n \| R_j (u(t, \cdot)) \|_{H_w}
\]

From Lemma (3.26) and Remark (3.33), we may conclude that \( u(t, x) \) is harmonic and non-tangentially bounded function. Hence, there exists \( \lim_{t \to 0} u(t, x) \) for \( a.e. \ x \in \mathbb{R}^n \). Therefore, taking limit in (3.48) and applying Lemma (3.33) and the Lebesgue dominated convergence Theorem, we obtain (3.40) ending the proof of the Theorem. ///
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