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THE NATURAL $\|.\|_\varphi$-APPROXIMANT IN ORLICZ SPACES

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Abstract: An Orlicz space L^{φ_0} is approached by spaces L^{φ_ε} with convex functions φ_ε tending to φ_0 in some specific way. For a function f we study convergence of the best L^{φ_ε} approximants to f as $\varepsilon \to 0$. Norm, pointwise and uniform convergence are considered.

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1. INTRODUCTION

In [**LR1**], Landers and Rogge introduce the concept of best natural approximant, say m, which can be obtained as follows. In a finite measure space, consider f_p as the best L^p approximant of the L^p function f. They proved that f_p converges in L^1 norm to a function m as p tends to one. This element m is a uniquely well determined function among the best L^1 approximants of f. Recently this result was partially extended to the set up of Orlicz spaces in [**ZF**]. We pursue here the study of the best natural approximant in these spaces.

Throughout this paper we will work on a finite measure space (X, Σ, μ) which will be denoted just by X. Set Φ for the set of continuous convex functions φ defined on $[0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(x) > 0$ for all large x, i.e. φ might be negative near 0. Given $\varphi \in \Phi$ we introduce the Orlicz space $L^\varphi(X)$ as usual, i.e., a measurable function f defined on X belongs to L^φ iff for some $\lambda > 0$ the integral $\int_X \varphi(\lambda|f|)$ is finite. Of course the integral above is always greater than $-\infty$, and it is understood that the integral is taken on whole space X with the measure μ. For general properties of Orlicz spaces see [**KR**], [**M**] or [**RR**].

Given an approximant class $C \subseteq L^1$ and $f \in L^\varphi$ we set

$$\mu_\varphi(f/C) = \{g \in C : \int \varphi(|f-g|) = \inf_{h \in C} \int \varphi(|f-h|)\}. \tag{1.1}$$

For some particulars φ and C it can be proved that $\mu_\varphi(f/C) \neq \phi$, see for example Lemma(3.6) in [**ZF**].

If $\varphi_0 \in \Phi$ is not a strictly convex function the set $\mu_{\varphi_0}(f/C)$ has, in general, more than one element. In order to select one element in $\widetilde{C} = \mu_{\varphi_0}(f/C)$ we approach the function

φ_0 with a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ of convex functions tending in a suitable way to φ_0 as ε tends to zero. It is proved in **[ZF]**, under some conditions, that given $m_\varepsilon \in \mu_{\varphi_\varepsilon}(f/C)$ the net $\{m_\varepsilon\}_{\varepsilon>0}$ converges in a norm on L^{φ_0} to some specific element in $\widetilde{C}$. This best approximant m_1 does not depend on the net $\{m_\varepsilon\}_{\varepsilon>0}$ but it does depend on the family $\{\varphi_\varepsilon\}_{\varepsilon>0}$, so we called it the natural best L^{φ_0} approximant adapted to the family $\{\varphi_\varepsilon\}_{\varepsilon>0}$.

Now, we introduce the best $\| \, \|_\varphi$ approximant analogous to the best L^φ-approximant introduced in **(1.1)**. To do so we consider the Luxemburg "norm "$\| \, \|_\varphi$, which is defined by

$$\|f\|_\varphi = \inf\{c > 0 : \int \varphi(|f|/c) \leq 1\}, \tag{1.2}$$

where φ is in Φ, (if $\varphi \geq 0$ then $\| \, \|_\varphi$ is actually a norm).

For an approximant class $C \subseteq L^1$ and $f \in L^\varphi$ we set

$$\mu_{\| \, \|_\varphi}(f/C) = \{g \in C : \|f - g\|_\varphi = \inf_{h \in C} \|f - h\|_\varphi\}. \tag{1.3}$$

Given a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ tending in a suitable way to φ_0 we shall prove that any net $\{\bar{m}_\varepsilon\}_{\varepsilon>0}$, $\bar{m}_\varepsilon \in \mu_{\| \, \|_{\varphi_\varepsilon}}(f/C)$, tends to an uniquely determined element $\bar{m}$. Here we shall deal with approximant classes C more particular than those given in **[ZF]**. In order to be self contained we present a short proof of the main result given there, of course we will use the additional properties that we have now on this particular approximant class C. Finally, when C is the class of monotone functions we give some pointwise convergence results.

2. NORM CONVERGENCE OF THE BEST APPROXIMANTS

We consider an approximant class $C \subseteq L^1(X)$ with the following compactness property. Given any sequence (f_n) in C such that

$$\int_X |f_n| \, d\mu \leq M, \tag{2.1}$$

for a finite constant M. Then there exists a subsequence $(f_{n'})$ of (f_n) that converges a.e. to a function f which is also in C.

If C has the above compactness property, clearly C is a closed set in L^1 and then $C \cap L^\varphi$ is closed under the "norm "convergence in L^φ. To see this fact use the following inequality

(2.2) For $\varphi \in \Phi$ there exists a non negative constant c such that $x \leq c\varphi(x)$, for all $x \geq c$.

Many classical examples of approximant classes C fulfill the compactness property asked above. Now we analyze some of them.

a)Let $X = [0,1]$, μ the Lebesgue measure and C be the set of integrable non-decreasing functions. In fact, **(2.1)** and the monotony of the function imply the pointwise boundedness of the sequence (f_n) and thus the Helly's selection Theorem can be used. This class has been widely used in approximation theory, see **[HL1]**, **[H]**, **[DH]** and **[HT]**.

Remark. We note that the Lebesgue measure in $[0,1]$ can be replaced by a finite Borel measure fulfilling the following condition, $\mu(I) > 0$ for every interval I of the form $[0,x]$ or $[x,1]$. Also, unbounded intervals can be used, for example take $X = [0,\infty)$ and μ a finite Borel measure on X such that $\mu([x,\infty)) > 0$ and $\mu([0,y)) > 0$ for large x and small positive y. We omit the easy proof of these facts.

b)Let C be the class of piecewise monotone functions on a fixed partition of $[0,1]$. The arguments given in a) can be used to prove the compactness property.

c)Set X for the unit n-cube $[0,1]^n$, μ the Lebesgue measure and let $C \subseteq L^1(X)$ be the set of all functions on X which are non-decreasing in each variable. The compactness of C follows as in **a)**, see **[HL]**, **[DH1]** for a proof of the Helly's selection theorem in this set up.

d)Let X be as in **c)** and C be the class of convex functions on X. Given a sequence (f_n) of functions in C fullfilling **(2.1)** it is easy to see that for any interior point x_0 the sequence $(f_n(x_0))$ is bounded and therefore we can use Theorem 10.9 of **[R]**, so the compactness property holds. See also **[HZ]**, **[HLT]**.

The classes in **a)**, **c)** and sometimes the one given in **d)** are those that appear more frequently in the literature.

We will assume the following conditions on φ_0 and on the approaching family $\{\varphi_\varepsilon\}_{\varepsilon>0}$, which were introduced in **[ZF]**.

(I) For every $\varepsilon > 0$, φ_ε belongs to Φ and $\varphi_\varepsilon(x) > 0$ for $x > 0$.

Note that we are not assuming that the functions in the family $\{\varphi_\varepsilon\}$ are strictly convex functions.

(II) For every $\varepsilon > 0$, there exists $0 < c_\varepsilon < 1$ and x_ε such that $\varphi_0(x) \leq c_\varepsilon \varphi_\varepsilon(x)$, for every $x \geq x_\varepsilon$.

(III) The following limit exists for every x

$$\beta(x) = \lim_{\varepsilon \to 0+} \frac{\varphi_\varepsilon(x) - \varphi_0(x)}{\varepsilon}.$$

(IV) For all $x \geq 0$, $\varepsilon > 0$

$$\varphi_\varepsilon(x) - \varphi_0(x) \geq \varepsilon \beta(x).$$

(V) There exists q a strictly convex function, $q \in \Phi$ with $(q(x)/x)$ tending to infinity as x tends to infinity such that $\beta(x) = q(\varphi_0(x))$ for $x \geq 0$.

(VI) For every $\varepsilon > 0$, $\psi_\varepsilon \in \Phi$, where

$$\psi_\varepsilon(x) = \sup_{0<\rho\leq\varepsilon} \frac{\varphi_\rho(x) - \varphi_0(x)}{\rho}.$$

Though condition **(VI)** can be somewhat weakened by assuming only that $\psi_\varepsilon \leq \psi_\varepsilon^*$, for some $\psi_\varepsilon^* \in \Phi$, we keep it in the way stated above.

Set $L_+^\psi = \bigcup_{\varepsilon>0} L^{\psi_\varepsilon}$. Then it follows rather easily that

(2.3) For every $\varepsilon \geq 0$ we have $L^{\varphi_\varepsilon} \subseteq L^1$, $L_+^\psi \subseteq L^\beta \subseteq L^1$ and $L_+^\psi \subseteq \cup_{0<\rho} \cap_{0<\varepsilon<\rho} L^{\varphi_\varepsilon}$.

(2.4) $x\varphi_\varepsilon(1) \leq \varphi_\varepsilon(x)$, for $x \geq 1$, $\varepsilon \geq 0$.

(2.5) $\varepsilon\psi_\varepsilon(x) \geq (1 - c_\varepsilon)\varphi_\varepsilon(x)$, for $x \geq x_\varepsilon$, where x_ε and c_ε have the meaning given in **(II)**.

In **[ZF]** were introduced the next two examples of $\{\varphi_\varepsilon\}_{\varepsilon>0}$.

(2.6) The functions defined by $\varphi_\varepsilon = {\varphi_0}^{1+\varepsilon}$ are the analogous to those given in **[LR1]**. In this case $\beta = \varphi_0 ln\varphi_0$ and $\psi_\varepsilon = {\varphi_0}^{1+\varepsilon} ln\varphi_0$.

(2.7) Given $q \in \Phi$ a strictly convex function and $(q(x)/x) \to \infty$ as $x \to \infty$ we set $\varphi_\varepsilon = \varphi_0 + \varepsilon q(\varphi_0)$. Now $\beta = q(\varphi_0)$ and $\psi_\varepsilon = \beta$.

It is easy to check that the families given in **(2.6)** and **(2.7)** fulfill the conditions **(I)-(VI)**. These φ_ε are strictly convex functions. It is possible to get a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ which fulfills the conditions above but the φ_ε are not strictly convex functions, see [**ZF**].

(2.8) Lemma. *Let $\varphi \in \Phi$ and let $C \subseteq L^1$ be an approximant class with the compactness property, we assume further $C \cap L^\varphi \neq \phi$. Then for any $f \in L^1(X)$, the set $\mu_\varphi(f/C)$ is non empty.*

Proof. Set $\alpha = \inf_{g \in C} \int \varphi(|f-g|)$. Clearly α is greater than $-\infty$ and we may assume that α is less than ∞, otherwise $\mu_\varphi(f/C) = C$. Let (f_n) be a minimizing sequence in C, i.e.,$\int \varphi(|f - f_n|) \to \alpha$. Now, by **(2.2)**, there exists a finite constant M such that $\int |f_n| \leq M$.

By the compactness property of the class C, there exist a subsequence $(f_{n'})$ of (f_n) and $g \in C$ such that $f_{n'} \to g$ *a.e.* as $n' \to \infty$. Using Fatou's Lemma, **(2.8)** follows ∎

Recall that a function $\varphi \in \Phi$ satisfies the Δ_2-condition for large x, if there exists a constant $c > 0$ such that $\varphi(2x) \leq c\varphi(x)$ for all x bigger than a fix number.

(2.9) Remark. If we assume in **Lemma (2.8)** that φ satisfies the Δ_2-condition for large x, then it is easy to see that $\mu_\varphi(f/C) = \mu_\varphi(f/C \cap L^\varphi)$, whenever $\int \varphi(|f-g|)$ is finite for some $g \in C \cap L^\varphi$.

(2.10) Remark. For every $f \in L^1(X)$, the non empty set $\mu_\varphi(f/C)$ has the compactness property. The proof of this fact is similar to **Lemma (2.8)**.

(2.11) Remark. Let C be a convex set in $L^1(X)$ with the compactness property, then for every $f \in L^1(X)$ we have that the set $\mu_\beta(f/\mu_{\varphi_0}(f/C))$ is either the set $\mu_{\varphi_0}(f/C)$ or has exactly one function, say $m_1(f) = m_1$. Indeed, if $\int \beta(|f-m|) = +\infty$ for every $m \in \mu_{\varphi_0}(f/C)$ then, $\mu_\beta(f/\mu_{\varphi_0}(f/C)) = \mu_{\varphi_0}(f/C)$. Otherwise, we will have the uniqueness property taking into account that the set $\mu_{\varphi_0}(f/C)$ is a convex set and β is a strictly convex function by property **(V)**. The uniqueness property is not hard to prove and it follows in the same way as the given in [**LR1**] for the case $\varphi_\varepsilon(x) = x^{1+\varepsilon}$.

In order to avoid the pathological situation $\int \beta(|f-g|)=+\infty$ for every $g \in \mu_{\varphi_0}(f/C)$ we shall assume from now on that all functions $\varphi_0, (\varphi_\varepsilon)$ and β have the Δ_2-condition for large x. Thus, $\mu_\beta(f/\mu_{\varphi_0}(f/C)) = \{m_1\}$. The function m_1 is called the natural best approximant of f adapted to the approaching family $\{\varphi_\varepsilon\}$ or simply the natural best approximant. Note that m_1 depends on f, thus we often denoted it by $m_1(f)$.

The following result was proved in **[ZF]** for a general closed set $C \subseteq L^1$, now the requirement we will impose on C is of a different nature, besides the proof of the following theorem is considerably shorter than the original one.

(2.12) Theorem. *Let f be in L_+^ψ and C be a convex set in L^1 with the compactness property. Further suppose $\mu_{\varphi_0}(f/C) \subseteq L_+^\psi$, with m_1 the natural best approximant and m_ε in $\mu_{\varphi_\varepsilon}(f/C)$. Then*

$$\int \varphi_0(|m_\varepsilon - m_1|) \to 0 \text{ as } \varepsilon \to 0.$$

Proof. By **(IV)**,

$$\int \beta(|f-m_\varepsilon|) \le \frac{1}{\varepsilon}\int \varphi_\varepsilon(|f-m_\varepsilon|) - \varphi_0(|f-m_\varepsilon|) \le \frac{1}{\varepsilon}\int \varphi_\varepsilon(|f-m|) - \varphi_0(|f-m|),$$

where $m \in \mu_{\varphi_0}(f/C)$. Since $f-m \in L_+^\psi$, the last integrals in the inequality above are uniformly bounded as $\varepsilon \to 0$. Therefore, there exists a constant M such that $\int \beta(|m_\varepsilon|) \le M$ for small ε.

Since $\beta \in \Phi$ the integral $\int |m_\varepsilon|$ is bounded for all ε near 0. By the compactness property we can select a sequence (m_{ε_j}) which converges a.e. to some function $\bar{m} \in C$. Again, using property **(IV)**, Fatou's Lemma and repeating the arguments given at the beginning of the proof we have $\int \beta(|f-\bar{m}|) \le \int \beta(|f-m|)$, for $m \in \mu_{\varphi_0}(f/C)$. Thus, by the uniqueness property, $\bar{m} = m_1$.

To end the proof we shall see that $\int \varphi_0(|m_{\varepsilon_j} - m_1|) \to 0$ *as* $\varepsilon_j \to 0$. Indeed, by property **(V)**, $\int \varphi_0(|m_{\varepsilon_j}|)$ is uniformly integrable. Hence by Egorov's theorem the result follows ∎

Now, we study the convergence of $\|\ \|_\varphi$ approximants. First we need a similar lemma to **(2.8)**.

(2.13) Lemma. *Let $\varphi \in \Phi$ and $f \in L^\varphi$. If $C \subseteq L^1$ satisfies the compactness property. Then $\mu_{\|\ \|_\varphi}(f/C) \neq \phi$.*

Proof. Let $\alpha = \inf_{h\in C} \|f - h\|_\varphi$ and (m_n) be a minimizing sequence, i.e. $\|f - m_n\|_\varphi$ converges to α. Therefore, for some $\lambda > 0$ we have

$$\int \varphi(\lambda|f - m_n|) = \int \varphi(\lambda\|f - m_n\|_\varphi \frac{|f - m_n|}{\|f - m_n\|_\varphi}) \leq \lambda\|f - m_n\|_\varphi \leq 1.$$

Thus, by **(2.2)** there exists M such that $\int |m_n| \leq M$. Therefore there exists a subsequence $(m_{n'})$ convergent a.e. to some $m \in C$ and the Lemma follows since we have

$$\|f - m\|_\varphi \leq \underline{\lim}\, \|f - m_{n'}\|_\varphi = \alpha \quad \blacksquare$$

(2.14) Lemma. *Let φ, $\psi \in \Phi$, with $\varphi \geq 0$ and $\psi(1) = 1$. Let $f \in L^{\psi\circ\varphi}(X)$ and assume $\mu(X) = 1$. Then $\|f\|_\varphi \leq \|f\|_{\psi\circ\varphi}$.*

Proof. For any $c > 0$ by Jensen's inequality we have

$$\psi[\int \varphi(|f|/c)] \leq \int \psi(\varphi(|f|/c)),$$

thus if $c = \|f\|_{\psi\circ\varphi}$, which always can be assumed bigger than zero, we have

$$\psi(\int \varphi(|f|/c)) \leq 1.$$

Since $\psi(1) = 1$ and $\psi \in \Phi$ it follows $\int \varphi(|f|/c) \leq 1 \quad \blacksquare$

For $f \in L^\varphi$ we set

$$e(\varphi) = e(f, \varphi, C) = \inf\{\|f - g\|_\varphi : g \in C\}.$$

The short notation $e(\varphi)$ is used since f and C are fixed, also we shall use e_ε for $e(\varphi_\varepsilon)$, $\varepsilon \geq 0$. We started dealing with a family $\{\varphi_\varepsilon\}$ approaching to a function φ_0. Now we assume that φ_ε is of the form $i_\varepsilon \circ \varphi_0$, $i_\varepsilon \in \Phi$ and the family $\{i_\varepsilon\}$ tends to the identity function when ε goes to zero. The conditions **(I)** to **(VI)** should be reformulated in the obvious way using i_ε. We replace **(II)** by **(II')** as follows

(II'). There exist a number x_0 and a constant c_0 such that for every $\varepsilon > 0$ and $x \geq x_0$,

$$\varphi_0(x) \leq c_0\varphi_\varepsilon(x).$$

The next lemma is an extension of a result given in **[O]** where the convergence is proved for bounded functions.

(2.15) Lemma. *Let f be in L_+^ψ, then $\|f\|_{\varphi_\varepsilon}$ converges to $\|f\|_{\varphi_0}$ as ε tends to* 0.

Proof. Let f be in L^{ψ_ε}, then $\int \psi_\varepsilon(|f|)$ is finite and $\varphi_\rho(|f|) - \varphi_0(|f|) \leq \rho\psi_\varepsilon(|f|)$ for $0 < \rho < \varepsilon$, thus $\varphi_\rho(|f|) \leq \rho\psi_\varepsilon(|f|)$. Therefore for every $c > 0$

$$\int \varphi_\rho(|f|/c) \to \int \varphi_0(|f|/c),$$

as $\rho \to 0$. Thus we have the next two statement:

(a) For c such that $0 < c < \|f\|_{\varphi_0}$ we have

$$\int \varphi_\rho(|f|/c) > 1,$$

for all small ρ.

(b) For $c > \|f\|_{\varphi_0}$ we have

$$\int \varphi_\rho(|f|/c) < 1,$$

for all small ρ.

By (a) and (b) given $\varepsilon > 0$ there exists ρ_0 such that

$$\int \varphi_\rho(|f|/(\|f\|_{\varphi_0} - \varepsilon)) > 1 > \int \varphi_\rho(|f|/(\|f\|_{\varphi_0} + \varepsilon))$$

for every $0 < \rho \leq \rho_0$. Therefore

$$\|f\|_{\varphi_0} - \varepsilon \leq \|f\|_{\varphi_\rho} \leq \|f\|_{\varphi_0} + \varepsilon \ \blacksquare$$

(2.16) Lemma. *Let f be in L_+^ψ, $\mu_{\varphi_0}(f/C) \subseteq L_+^\psi$ and C be a convex set in L^1 with the compactness property. Then $e(\varphi_\varepsilon) \to e(\varphi_0)$ when $\varepsilon \to 0$.*

Proof. For any $g \in C$ we have

$$\overline{\lim_{\varepsilon \to 0}}\, e(\varphi_\varepsilon) \leq \lim_{\varepsilon \to 0} \|f - g\|_{\varphi_\varepsilon} = \|f - g\|_{\varphi_0},$$

The last equality follows from **Lemma (2.15)**. Therefore

(a)$\overline{\lim}_{\varepsilon \to 0}\, e(\varphi_\varepsilon) \leq e(\varphi_0)$.

Moreover

(b) $e(\varphi_0) \leq e(\varphi_\varepsilon)$, for every $\varepsilon > 0$.

In fact we select $g_\rho \in C$ such that

$$\|f - g_\rho\|_{i_\varepsilon \circ \varphi_0} \leq e(\varphi_\varepsilon) + \rho,$$

where ρ is an arbitrary positive number. By **Lemma (2.14)**

$$e(\varphi_0) \leq \| f - g_\varepsilon \|_{\varphi_0} \leq \|f - g_\varepsilon\|_{i_\varepsilon \circ \varphi_0},$$

so (b) and the lemma follow ∎

We strength condition **(III)** as follows

(III') The limit

$$q(x) = \lim_{\varepsilon \longrightarrow 0+} \frac{i_\varepsilon(x_\varepsilon) - x_\varepsilon}{\varepsilon}$$

exists, where x_ε is any sequence tending to x.

From now on **(III')** is in force.

To prove the norm convergence result we need additional restrictions on φ_0 or f. For the following theorem we will assume one of the two cases

a) $\varphi_0(x) = |x|$ and any $f \in L^\psi_+$.

b) φ_0 as before and $f \in L^\infty$. C lattice, and the constants belong to C.

Note that under the condition b) we have: if $m \in \mu_\varphi(f/C)$ then $\|m\|_\infty \leq \|f\|_\infty$.

(2.17)Theorem. *With the same conditions as in* **Theorem (2.12)**, *and* $f \notin C_{\varphi_0} = C \cap L^{\varphi_0}$, *for any* $\bar{m}_\varepsilon \in \mu_{\| \|_{\varphi_\varepsilon}}(f/C)$ *we have* $\|\bar{m}_\varepsilon - \bar{m}_1\|_{\varphi_0} \rightarrow 0$ *as* $\varepsilon \rightarrow 0$, *where* $\bar{m}_1 = e_0 m_1(\frac{f}{e_0})$, *with* $m_1(\frac{f}{e_0})$ *the best natural approximant of* $\frac{f}{e_0}$ *in* $\mu_{\varphi_0}(\frac{f}{e_0}/\frac{C_{\varphi_0}}{e_0})$.

Proof. For any net $\{\bar{m}_\varepsilon\}_{\varepsilon>0}$, $\bar{m}_\varepsilon \in \mu_{\| \|_{\varphi_\varepsilon}}(f/C)$ we can find a sequence $\varepsilon_j \rightarrow 0$, and $\bar{m}_0 \in C$ such that $\bar{m}_{\varepsilon_j} \rightarrow \bar{m}_0$ pointwise. In fact, given $\varepsilon > 0$ it holds

$$\int \varphi_\varepsilon(\frac{|f - \bar{m}_\varepsilon|}{e_\varepsilon}) = 1$$

and using **(II')** we have for some constant M

$$\int \varphi_0\left(\frac{|f-\bar{m}_\varepsilon|}{e_\varepsilon}\right) \leq M.$$

Thus by **(2.16)** it follows easily that $\bar{m}_\varepsilon$ is bounded in L^1 for all ε small, and the existence of $(\bar{m}_{\varepsilon_j})$ and $\bar{m}_0$ is a consequence of the compactness property. Furthermore, the element $\bar{m}_0 \in \mu_{\|\,\|_{\varphi_0}}(f/C)$. In fact

$$\|f-\bar{m}_0\|_{\varphi_0} \leq \underline{\lim}_{\varepsilon_j} \|f-\bar{m}_{\varepsilon_j}\|_{\varphi_0} \leq \underline{\lim}_{\varepsilon_j \to 0} \|f-\bar{m}_{\varepsilon_j}\|_{\varphi_{\varepsilon_j}}.$$

The first inequality follows using a version of Fatou's theorem for the Luxemburg norm, the second one is obtained from **(2.14)** and the fact that $\varphi_\varepsilon = i_\varepsilon \circ \varphi_0$. Now from **Lemma(2.16)** the right hand side of the above inequality is e_0, and since $\bar{m}_0 \in C_{\varphi_0}$ we have $\bar{m}_0 \in \mu_{\|\,\|_{\varphi_0}}(f/C)$.

Using the equality $\mu_{\|\,\|_{\varphi_0}}(f/C_{\varphi_0}) = e_0\mu_{\varphi_0}(\frac{f}{e_0}/\frac{C_{\varphi_0}}{e_0})$, see **[LR]**, we have $\frac{\bar{m}_0}{e_0} \in \mu_{\varphi_0}(\frac{f}{e_0}/\frac{C_{\varphi_0}}{e_0})$.

For $m_j = \frac{\bar{m}_j}{e_{\varepsilon_j}}$ and $m_1 = m_1(\frac{f}{e_0})$ (the natural L^{φ_0} best approximant of f/e_0), we shall see that $\|m_j - m_1\|_{\varphi_0} \to 0$ as $j \to \infty$. We note that $m_j \to m_0 = \frac{\bar{m}_0}{e_0}$ pointwise. By **(III')** and Fatou's theorem we have

(1) $\int \beta(|\frac{f}{e_0} - m_0|) \leq \underline{\lim}_{j\to\infty} \frac{1}{\varepsilon_j} \int \varphi_{\varepsilon_j}(|\frac{f}{e_{\varepsilon_j}} - m_j|) - \varphi_0(|\frac{f}{e_{\varepsilon_j}} - m_j|)$.

If we take now n_j the best natural approximant in $\mu_{\varphi_0}(\frac{f}{e_{\varepsilon_j}}/\frac{C_{\varphi_{\varepsilon_j}}}{e_{\varepsilon_j}})$, i.e. the best natural φ_0-approximant of $\frac{f}{e_{\varepsilon_j}}$ adapted to the same approaching family φ_ε where the approximant class is $\frac{C_{\varphi_{\varepsilon_j}}}{e_{\varepsilon_j}}$, we can estimate **(1)** by

(2) $\underline{\lim}_{j\to\infty} \frac{1}{\varepsilon_j} \int \varphi_{\varepsilon_j}(|\frac{f}{e_{\varepsilon_j}} - n_j|) - \varphi_0(\frac{f}{e_{\varepsilon_j}} - n_j)$.

For the case $f \in L^\infty$, i.e. case b), we have that each function in $\mu_{\varphi_0}(\frac{f}{e_{\varepsilon_j}}/\frac{C_{\varphi_{\varepsilon_j}}}{e_{\varepsilon_j}})$ is bounded uniformly in j, (we are using that $e_{\varepsilon_j} \to e_0 \neq 0$).

We firstly show that there exists a subsequence (n_{j_l}) of (n_j) such converges a. e. to $m_1(f/e_0)$, the best natural approximant of f/e_0 with respect to the approximant class C_{φ_0}/e_0.

In fact we have

$$\int \beta(|\frac{f}{e_{\varepsilon_j}} - n_j|) \leq \int \beta(|\frac{f}{e_{\varepsilon_j}} - g_j|),$$

for all $g_j \in \mu_{\varphi_0}(\frac{f}{e_{\varepsilon_j}}/\frac{C_{\varphi_{\varepsilon_j}}}{e_{\varepsilon_j}})$. Since $f \in L^\infty$ we have $\sup_j|n_j| \in L^\infty$ and by the compactness property there exist a subsequence (n_{j_l}) and $n \in \frac{C_{\varphi_0}}{e_0}$ such that $n_{j_l} \to n$ a.e. as $l \to \infty$.

Now let h be in $\mu_{\varphi_0}(\frac{f}{e_0}/\frac{C_{\varphi_0}}{e_0})$. Since $h \in L^\infty$ we have $\frac{e_0}{e_{\varepsilon_j}}h \in \frac{C_{\varphi_{\varepsilon_j}}}{e_{\varepsilon_j}}$ and

$$\int \beta(|\frac{f}{e_{\varepsilon_j}} - n_j|) \leq \int \beta(|\frac{f}{e_{\varepsilon_j}} - \frac{e_0}{e_{\varepsilon_j}}h|).$$

By Fatou's theorem considering the subsequence (n_{j_l}) in the inequality above we have

$$\int \beta(|\frac{f}{e_0} - n|) \leq \int \beta(|\frac{f}{e_0} - h|),$$

for every $h \in \mu_{\varphi_0}(\frac{f}{e_0}/\frac{C_{\varphi_0}}{e_0})$. Thus $n = m_1(f/e_0)$. As $\sup_l|n_{j_l}| \in L^\infty$ we have

$$\frac{1}{\varepsilon_{j_l}} \int \varphi_{\varepsilon_{j_l}}(|\frac{f}{e_{\varepsilon_{j_l}}} - n_{j_l}|) - \varphi_0(\frac{f}{e_{\varepsilon_{j_l}}} - n_{j_l}) \to \int \beta(|\frac{f}{e_0} - m_1(f/e_0)|),$$

as $l \to \infty$. Then the limit in (2) is less or iqual than $\int \beta(|\frac{f}{e_0} - m_1(f/e_0)|)$.

For the case that f is not bounded, we have $\varphi_0(x) = x$ and it follows

(3) $\mu_{\varphi_0}(\frac{f}{e_{\varepsilon_j}}/\frac{C}{e_{\varepsilon_j}}) = \frac{e_0}{e_{\varepsilon_j}}\mu_{\varphi_0}(\frac{f}{e_0}/\frac{C}{e_0}) = \frac{1}{e_{\varepsilon_j}}\mu_{\| \|_{\varphi_0}}(f/C)$

and the set $\mu_{\varphi_0}(\frac{f}{e_0}/\frac{C_{\varphi_0}}{e_0})$ admits a minimum and a maximum, see **[LR]**. On the other hand, if $f \in L^{\psi_\varepsilon}$ the set $\mu_{\| \|_{\varphi_0}}(f/C)$ is also in L^{ψ_ε}. The proof of this last fact follows using the same argument of the proof given in **[LR1]** for the case $\varphi_{\varepsilon_j}(x) = x^{1+\varepsilon_j}$. Using that $\mu_{\| \|_{\varphi_0}}(f/C) \subseteq L^{\psi_\varepsilon}$, the equality **(3)** and recalling from **[LR]** that the set $\mu_{\varphi_0}(\frac{f}{e_0}/\frac{C}{e_0})$ has a maximum and a minimum, here we are using that C is a lattice, we can conclude that the function $\sup|n_j| \in L^{\psi_\varepsilon}$. In a similar way to the case $f \in L^\infty$ we can prove that the limit in **(2)** is bounded by $\int \beta(|\frac{f}{e_0} - m_1(f/e_0)|)$.

Therefore in both cases a) and b) we obtain

$$\int \beta(|\frac{f}{e_0} - m_0|) \leq \int \beta(|\frac{f}{e_0} - m_1(f/e_0)|).$$

Thus $m_0 = m_1(f/e_0)$.

It remains to prove the convergence in norm. Working as before we have that

$$\overline{\lim_{\varepsilon_j \to 0}} \int \beta(|\frac{f}{e_{\varepsilon_j}} - m_j|)$$

is bounded. Therefore $\varphi_0(|\frac{f}{e_{\varepsilon_j}} - m_j|)$ is uniformly integrable. Thus

$$\int \varphi_0(|m_j - m_1(f/e_0)|) \to 0$$

as $j \to \infty$. Then, since φ_0 is Δ_2, $\|\bar{m}_{\varepsilon_j} - e_0 m_1(f/e_0)\|_{\varphi_0} \to 0$ as $j \to \infty$ and the theorem follows ∎

3. POINTWISE CONVERGENCE

We have obtained in the previous section the best natural approximant as a limit in L^φ norm or in the mean, now we analyse the pointwise counterpart. We assume that all the functions are defined on a subset of $\mathbf{R}^n$ and we work with the Lebesgue measure, which on a set A will be denoted by $|A|$. In this section the following conditions are required for the approximant class C.

A.(The Almost Continuity). Each function f in C is continuous almost everywhere.

B.(The Helly's Selection Principle). Given a sequence (f_n) in C bounded in L^1 and a finite set of points K, there exists a subsequence $(f_{n'})$ which converges a.e. to a function $f \in C$ and it converges for every point of K.

C.(Separation Criteria). Given $f, g \in C$ and f continuous at a point y. If $f(y) \neq g(y)$ then $|\{f \neq g\}| > 0$.

All the examples introduced in section **2** fullfill the three conditions above. The following lemma has its antecedents in **[HL]**, we believe that our presentation here somewhat clarifies the essential properties required for the pointwise convergence.

(3.1) Lemma. Let $\varphi \in \Phi$, $\varphi(x) > 0$ for $x > 0$, and $(f_t)_{0<t<1}$ be a net in $C \cap L^\varphi$ and $f \in C \cap L^\varphi$. If $\int \varphi(|f_t - f|) \to 0$ as $t \to 0^+$, then f_t converges a.e. to f as $t \to 0^+$.

Proof. Let E be the set where the net $(f_t)_{0<t<1}$ does not converges pointwise to f, and assume $|E| > 0$. Then by property **A** there exist a point y where f is continuous and a sequence (t_n) such that $(f_{t_n}(y))$ does not converge to $f(y)$ as $t_n \to 0^+$. We may assume, by taking as many subsequences as necessary, that for some subsequence $(f_{t_{n'}})$ we have

1. $f_{t_{n'}} \to f$ a.e. as $t'_n \to 0$.

2. $f_{t_{n'}} \to g$ a.e. with $g \in C$ and $f_{t_{n'}}(y) \to g(y)$ and $f(y) \neq g(y)$. Use B.

Using **C** we have $|\{f \neq g\}| > 0$, which contradicts 1. and 2. and the lemma follows ∎

As a direct consequence of the Lemma we obtain the next two pointwise convergence results.

(3.2) Theorem. *Let f and C be as in* **Theorem(2.12)**. *Let m_1 be the natural best approximant of f adapted to the approaching family $\{\varphi_\varepsilon\}$. Then for any m_ε in $\mu_{\varphi_\varepsilon}(f/C)$, the net $\{m_\varepsilon\}_{\varepsilon>0}$ converges almost everywhere to m_1 as ε tends to 0.*

(3.3) Theorem. *Let f and C be as in* **Theorem(2.12)**, *and $\bar{m}_1$ be as in* **Theorem(2.17)**, *then for $\bar{m}_\varepsilon$ in $\mu_{\|\,\|_\varepsilon}(f/C)$ we have that the net $\{\bar{m}_\varepsilon\}_{\varepsilon>0}$ converges a.e. to $\bar{m}_1$ as ε tends to 0.*

Thus the pointwise convergence result is an easy consequence of the norm convergence results. The interesting situation is that we can get in most of the cases uniform convergence results, with this aim we give some definitions. It is important to remark that for the last part of this paper the approximant class will be the class of monotone functions.

(3.4) Definition. Given $f \in L^1(0,1)$ and $x \in (0,1)$, we say that f is approximately continuous at x if there exists a real number A such that for every $\varepsilon > 0$, x is a point of metric density one for the set $A_\varepsilon = \{y/|f(y) - A| < \varepsilon\}$, i.e.

$$\frac{|A_\varepsilon \cap I|}{|I|} \to 1,$$

when the measure of the interval I tends to 0 and $x \in I$.

(3.5) Definition. We write $f \in bA$ iff $f \in L^\infty$ and f is approximately continuous at each point of $(0,1)$.

(3.6) Remark. The number A in **Definition (3.4)** is uniquely determined, and we assume, as it is customary, that it is $f(x)$.

(3.7) Remark. For $f \in L^\infty$, x is an approximately point of continuity for f iff x is a Lebesgue point of f. In other words we shall work with a bounded function where each point of $(0,1)$ is a Lebesgue point of the function.

(3.8) Theorem. *Let $\varphi \in \Phi$, $f \in bA$ and $g \in \mu_\varphi(f/C)$ then g is a bounded continuous function.*

Proof. It suffices to consider the continuity of g in the open interval $(0,1)$ since on the end points of the interval $(0,1)$ we can redefine the function g in such way that it is continuous there, thus let $0 < y < 1$ and assume $g(y^+) - g(y^-) > 0$. Then we have the next four cases: $(1) f(y) - g(y^+) > 0$, $(2) g(y^-) - f(y) > 0$, $(3) g(y^+) - f(y) > 0$, $(4) f(y) - g(y^-) > 0$.

We shall prove the first case, the other cases follow in a similar way. First we note that for $x > 0$, $\varphi(x) = \int_0^x p(t)\, dt$ where p is a non negative monotone function. Then we have the following properties

(I) $|\varphi(x) - \varphi(y)| \leq p(max\{x, y\})|x - y|, \quad x, y > 0$

(II) $\varphi(x) \leq \varphi(x + \sigma) - \sigma p(x), \quad \sigma > 0.$

Let $\varepsilon = f(y) - g(y^+)$ then by the approximately continuous property of f there exists $\delta > 0$ such that

(III) $|\{f > f(y) - \varepsilon\} \cap (y - \delta, y]| > q\delta$

where $0 < q < 1$. Thus we define

$$\bar{g}(x) = \begin{cases} g(x) + \eta & \text{if } x \in (y - \delta, y]; \\ g(x) & \text{otherwise.} \end{cases}$$

where $\eta = min\{g(y^+) - g(y^-), \varepsilon\}$.

Set $F = \{f > f(y) - \varepsilon\} \cap (y - \delta, y]$ and $F' = (y - \delta, y] - F$, then we have

$$\begin{aligned} \int_{F'} \varphi(|f - \bar{g}|) &\leq \int_{F'} \varphi(|f - g|) + p(2\|f\|_\infty)\eta|F'| \\ &\leq \int_{F'} \varphi(|f - g|) + p(2\|f\|_\infty)\eta(1 - q)\delta \end{aligned}$$

To obtain the inequalities above we have used I,III and

$$\|g\|_\infty \leq \|f\|_\infty,\ \|\bar{g}\|_\infty \leq \|f\|_\infty,\ g(y^-) < \bar{g} < g(y^+),$$

and

$$\int_F \varphi(|f-\bar{g}|) \leq \int_{F'} \varphi(|f-g|) - \eta p(\eta+\varepsilon)|F| \leq$$
$$\leq \int_{F'} \varphi(|f-g|) - \eta p(\varepsilon) q\delta,$$

where the last inequality follows by (III).

If $p(\varepsilon)q > (1-q)p(2\|f\|_\infty)$ we arrive to the contradiction

$$\int \varphi(|f-\bar{g}|) < \int \varphi(|f-g|) \ \blacksquare$$

We can prove, following the line of **[DH]**.

(3.9) Theorem. *Let $\{\varphi_t\}_{0\leq t\leq 1}$ be a family of convex functions such that for every $x > 0$ it is satisfied*

$$0 < \inf_t \varphi_t(x) \leq \sup_t \varphi_t(x) < \infty$$

and $f \in bA$. Let $g_t \in \mu_{\varphi_t}(f/C)$ then $\{g_t\}_{0\leq t\leq 1}$ is an equicontinuous family at each $0 \leq y \leq 1$.

Proof. We suppose, that the family $\{g_t\}_{0\leq t\leq 1}$ is not equicontinuous function at some point y, $0 < y < 1$. Therefore there exist $\varepsilon > 0$ and a sequence x_n tending to y and another sequence t_n such that

$$|g_{t_n}(x_n) - g_{t_n}(y)| > 8\varepsilon.$$

We consider the case $y < x_n$ for every n and we suppose, w.l.g., there exists α such that $|f_{t_n}(y) - \alpha| < \varepsilon$ for every n. We can have $f(y) \geq \alpha + 4\varepsilon$ or $f(y) < \alpha + 4\varepsilon$, we shall work only on the first case.

Since $f \in bA$, given any q, $0 < q < 1$, there exists δ such that

$$|\{|f - f(y)| < \varepsilon\} \cap I| > q|I|,$$

for every interval $I \subseteq (y-\delta, y+\delta)$.

Set $F = \{|f - f(y)| < \varepsilon\} \cap (y - \delta, y)$ and $\bar{g}_{t_n}$ for the function defined by $g_{t_n} + \varepsilon$ on $[y - \delta, y]$, $\bar{g}_{t_n}(y)$ on $[y, x_n)$ and $\bar{g}_{t_n} = g_{t_n} - \varepsilon$ on $[x_n, y + \delta]$. As in the proof of **Theorem (3.8)** we have

(1) $\int_F \varphi_{t_n}(|f - \bar{g}_{t_n}|) \leq \int_F \varphi_{t_n}(|f - g_{t_n}|) - \varepsilon p_{t_n}(2\varepsilon)q\delta.$

(2) $$\int_{F' \cap (y-\delta, x_n)} \varphi_{t_n}(|f - \bar{g}_{t_n}|) \leq \int_{F' \cap (y-\delta, x_n)} \varphi_{t_n}(|f - g_{t_n}|) + \\ + p_{t_n}(2\|f\|_\infty) 2\|f\|_\infty |\{|f - f(y)| \geq \varepsilon\} \cap (y - \delta, x_n)|,$$

where F' is the complement of F, and

(3) $\int_{F \cap (y, x_n)} \varphi_{t_n}(|f - \bar{g}_{t_n}|) \leq \int_{F \cap (y, x_n)} \varphi_{t_n}(|f - g_{t_n}|) + 2\|f\|_\infty (x_n - y).$

By (1),(2) and (3) we have

$$\int_{(y-\delta, x_n)} \varphi_{t_n}(|f - \bar{g}_{t_n}|) \leq \int_{(y-\delta, x_n)} \varphi_{t_n}(|f - g_{t_n}|) + 2\|f\|_\infty (x_n - y) + \\ + p_{t_n}(2\|f\|_\infty) 2\|f\|_\infty |\{|f - f(y)| \geq \varepsilon\} \cap (y - \delta, x_n)| \\ - \varepsilon p_{t_n}(2\varepsilon) q\delta \leq \int_{(y-\delta, x_n)} \varphi_{t_n}(|f - g_{t_n}|) + 2\|f\|_\infty (x_n - y) + \\ + p_{t_n}(2\|f\|_\infty) 2\|f\|_\infty (1 - q)(x_n - y + \delta) - \varepsilon p_{t_n}(2\varepsilon) q\delta.$$

Thus

(4) $\int_{y-\delta}^{x_n} \varphi_{t_n}(|f - \bar{g}_{t_n}|) \leq M(x_n - y) - Nq + R(1 - q) + \int_{y-\delta}^{x_n} \varphi_{t_n}(|f - g_{t_n}|),$

where R is a number depending on f and δ, and n is large enough in such away that $x_n < y + \delta$, we have used here the hypothesis on the family $\{p_t\}$, $M = 2\|f\|_\infty$ and N is a positive number depending on ε and δ. We have used the hypothesis on the family $\{p_t\}$ near 0.

In (4) we take q near 1 such that $-Nq + R(1-q) < 0$. Now if we take n large, the inequality in (4) is less than $\int_{y-\delta}^{x_n} \varphi_{t_n}(|f - g_{t_n}|)$ and this is a contradiction ∎

(3.10) Theorem. *Let* $\{\varphi_t\}_{0 \leq t \leq 1}$ *a family with properties* **(I)-(VI)** *of section 2 and moreover assume that the condition of* **Theorem (3.9)** *is in force. Then, for every* $g_t \in \mu_{\varphi_t}(f/C)$ *the net* (g_t) *converges uniformely to the natural best approximant* m_1, *for* t *tending to* 0.

Proof. This theorem is a direct consequence of **Theorems (3.8), (3.9), (2.12)** and Arzela Ascoli's theorem ∎

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