ON THE COMPACTNESS OF CONNECTED SETS

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ABSTRACT: Let $X$ be a space and $U$ be an open cover of $X$. Then $X$ is $U$-chainable if for each $x \in X$, $\text{st}^U(x, U) = X$ and it is $U$-uniformly chainable if there is an $n \in \mathbb{N}$ such that for each $x \in X$, $X = \text{st}^n(x, U)$. In this paper we characterise connected spaces in terms of $U$-chainability, connected spaces satisfying finite discrete chain condition in terms of $U$-uniform chainability and obtain several results which are analogues of the known results for metric spaces.

KEY WORDS AND PHRASES: Chainable, uniformly chainable, strictly chainable, connected, countably compact, compact.

MATHEMATICS SUBJECT CLASSIFICATION (1985)

Primary: 54D05, 54D30

Secondary: 54D10, 54D18, 54E45.
1- **INTRODUCTION**: Connectedness and compactness are widely studied in Topology. In 1883 Cantor defined connectedness in metric spaces with the help of \( \varepsilon \)-chains. At present, however, the Riesz-Hausdorff definition, using the idea of separated sets, is universally accepted. On the other hand, a lot of experimentation has led to several forms of compactness. Compactness and several of its generalizations are defined in terms of open covers; e.g., compact, countably compact, paracompact, Lindeloff etc. Chainability characterizes connected sets among compact sets in the setting of metric spaces. In the same setting Beer [Be] has characterized compact sets among the connected sets.

In this paper, the concept of \( \varepsilon \)-chainability in metric spaces is generalized by the use of open covers of a topological space. This generalization yields a simple characterization of connectedness in terms of open covers. Several results of Beer [Be] as well as some of the earlier known results are then generalized.

2- **PRELIMINARIES**: In this section, we give some basic definitions and fundamental facts which are needed in proving the main results in the following sections. Let \((X, \tau)\) be a topological space.

Let \( \mathcal{U} = \) a family of open covers of \( X \); \( \mathcal{U} = \) a family of open covers of \( X \) which induces the fine uniformity on \( X \) (when \( X \) is Tychonoff); \( \mathcal{U}_0 = \) the family of all open covers of \( X \).

For each \( A \subseteq X \) and \( U \in \mathcal{U} \)

\[
\text{st}(A, U) = \text{st}^1(A, U) = \bigcup \{ \, u \in U : A \cap u \neq \emptyset \, \} ;
\]

\[
\text{st}(A, U) = \text{st}(\text{st}^{n-1}(A, U)) \text{ for } n > 1; \text{ and}
\]

\[
\text{st}(A, U) = \bigcup_{n=1}^{\infty} \text{st}(A, U) .
\]

We write \( \text{st}(x, U) \) for \( \text{st}(\{x\}, U) \).

(2.1) **Definition**: Let \( U \in \mathcal{U} \). Then \( X \) is \( U \)-chainable iff for each \( x \in X \),

\[
X = \text{st}(x, U) .
\]

\( X \) is \( \mu \)-chainable iff for each \( U \in \mathcal{U} \), \( X \) is \( U \)-chainable.
(2.2) Definition: For $U \in \mu$, $X$ is called $U$-locally uniformly chainable iff for each $x \in X$ there is an $n_x \in \mathbb{N}$ such that $\text{st}^{n_x}(x, U) = X$. If $n_x$ is independent of $x$, we say that $X$ is $U$-uniformly chainable. As in (2.1) we can obviously define $\mu$-locally uniformly chainable and $\mu$-uniformly chainable.

The following fact is well known.

(2.3) Theorem: If $X$ is a Hausdorff paracompact space, then $\mu_F = \mu_c$.

(2.4) Lemma: $X$ is $U$-chainable if and only if it is $U$-locally uniformly chainable.

Proof: If $X$ is $U$-locally uniformly chainable for $U \in \mu$, then for each $x \in X$, there is an $n_x \in \mathbb{N}$ such that $X = \text{st}^{n_x}(x, U)$. Now for each $p \in X$ and $n = 2n_x$, it is easy to see that $X = \text{st}^n(p, U)$ i.e. $X$ is $U$-uniformly chainable. Converse is obvious.

(2.5) Corollary: $X$ is $\mu$-uniformly chainable if and only if it is $\mu$-locally uniformly chainable.

(2.6) Lemma: Let $X$ be any topological space and $U \in \mu_c$. Then for each $x \in X$, $\text{st}^\infty(x, U)$ is an open and closed subset of $X$.

Proof: If $y \not\in \text{st}^\infty(x, U)$, the $\text{st}(y, U) \cap \text{st}^\infty(x, U) = \emptyset$. This shows that $\text{st}^\infty(x, U)$ is closed and it is obviously open.

The following result characterizes connectedness in terms of chainability which in turn depends upon covers.

(2.7) Theorem: A space $X$ is connected if and only if $X$ is $\mu_c$-chainable.

Proof: The result follows from the facts (a) $X$ is connected if and only if it has no proper open and closed subset and (b) for each $x \in X$ and $U \in \mu_c$, $\text{st}^\infty(x, U)$ is open and closed.

(2.8) Corollary: A Hausdorff paracompact space $X$ is connected if and only if it is $\mu_F$-chainable.
(2.9) Corollary: A compact Hausdorff space $X$ is connected if and only if it is $\mu$-chainable.

(2.10) Remarks: Since a compact Hausdorff space has a unique compatible uniformity, the above results generalize the well-known result: a compact metric space is connected if and only if it is $\varepsilon$-chainable for each $\varepsilon > 0$ [Be].

3. COMPACTNESS OF CONNECTED SPACES:

At first we introduce a concept which is analogous to total boundedness in uniform spaces.

(3.1) Definition: $X$ is $\mu$-star compact if and only if for each $U \in \mu$, there is a finite subset $F$ of $X$ such that $X = \text{st}(F, U)$.

Fleischman [FI] introduced the concept of star compactness, which is $\mu_0$-star compact and showed that a $T_2$ space $X$ is star compact if and only if it is countably compact. In case $\mu$ is a compatible uniformity on a Tychonoff space $X$, then $\mu$-star compactness is equivalent to $\mu$-total boundedness.

In case $X$ is a (metric) uniform space with a compatible covering (metric) uniformity $\mu$, $\mu$-chainability plus $\mu$-total boundedness yields $\mu$-uniform chainability (see [Be] page 808). We analogously have the following result which can be easily shown to hold.

(3.1) Lemma: If $X$ is $\mu$-star compact and $\mu$-chainable, then $X$ is $\mu$-uniformly chainable.

We now characterize $\mu_0$ uniformly chainable spaces.

(3.2) Definition: $X$ is DFCC if and only if every discrete collection of open sets is finite.

(3.3) Theorem: A $T_3$ space $X$ is $\mu_0$ uniformly chainable if and only if it is connected and is DFCC.
Proof: Suppose $X$ is $T_3$ and $\mu$-uniformly chainable. By Theorem (2.7), $X$ is connected. If $X$ is not DFCC, then there is a countably infinite discrete collection of open sets $U = \{ U_n : n \in \mathbb{N} \}$. Choose $x_n \in U_n$ for each $n$ in $\mathbb{N}$.

For each $n$ in $\mathbb{N}$ define

$$x_n \in A^n_1 \subseteq A^n_2 \subseteq \cdots \subseteq A^n_m \subseteq \cdots \subseteq A^n_n \subseteq u_n$$

where $A^n_m$ is open for each $m$.

Define:

$$W = X - \cup \{ A^n_n : n \in \mathbb{N} \}.$$

$$V^n_1 = A^n_2$$

$$V^n_2 = A^n_3 - A^n_1$$

$$V^n_{n-1} = A^n_n - A^n_{n-2}$$

$$V^n_n = u_n - A^n_{n-1}.$$

Now let $V = \{ W \} \cup \{ V^n_m : m < n, n \in \mathbb{N} \}$. It is easy to see that it is a collection of nonempty open sets in $X$. It is a cover of $X$. For any $x$ in $X$, either $x$ is a member of $U_n$ for some $n$ and hence $x$ is in $V^n_n$ for some $m$ and $n$, or $x \notin \cup_{n \in \mathbb{N}} u_n$ in which case $x$ is in $W$. Consider $x$ in $X$ and $n$ in $\mathbb{N}$. Now there exist infinitely many $u_n$ which do not contain $x$. Pick $p > n$ such that $x$ is not in $u_p$. It is easy to see that $st^n(x_p, V) \subseteq U_p$, so $x_p \notin st^n(x, V)$. $x$ and $n$ are arbitrary; so $X$ cannot be $\mu$-uniformly chainable.

For the converse, assume that $X$ is $\mu$-chainable and has DFCC property but is not $\mu$-uniformly chainable. Then there exists an $x$ in $X$ and $U$ in $\mu$, such that $X \neq st^n(x, U)$ for each $n$ in $\mathbb{N}$.

Define

$$u_i = \begin{cases} \text{st}(x, U) & \text{for } i = 1 \\ \text{st}^i(x, U) - \text{CL}(\text{st}^{i-1}(x, U)) & \text{for } i > 1. \end{cases}$$
Since $X$ is $\mu_0$-chainable, $u_i \neq \emptyset$ for each $i$ in $\mathbb{N}$ and $\{u_i : i \in \mathbb{N}\}$ is a discrete collection of open sets. So, $X$ is not DFCC, a contradiction.

We now introduce a concept which will provide a characterization of countably compact connected spaces in terms of open covers.

(3.4) Definition: Let $U \in \mu_0$. $X$ is said to be strictly $U$-chainable if and only if there is a finite subset $F = \{x_i : 1 \leq i \leq n\}$ of $X$ such that $X = \text{st}(F, U)$ and $x_{i+1} \in \text{st}(x_i, U)$ for $1 \leq i \leq n-1$. $X$ is strictly $\mu$-chainable if and only if it is strictly $U$-chainable for each $U \in \mu$.

From our previous discussion, it is clear that a strictly $U$-chainable space is $U$-uniformly chainable.

(3.5) Theorem: A Hausdorff space $X$ is strictly $\mu$-chainable if and only if it is $\mu$-starcompact and $\mu$-chainable.

Proof: Suppose $X$ is $\mu$-starcompact and $\mu$-chainable. Let $U \in \mu$. Then there is a finite set $F = \{x_i : 1 \leq i \leq n\} \subset X$ such that $\text{st}(F, U) = X$. Since $X$ is $\mu$-chainable, for each $i \leq n-1$, there is a finite set $F_i = \{x_i^j : 1 \leq j \leq n_i\}$ and $\{u_i^j : 2 \leq j \leq n_i\} \subset U$ such that

(i) $x_i^1 = x_i \in u_i^1$
(ii) $x_i^{n_i} = x_{i+1} \in u_i^{n_i}$
(iii) $x_i^j \in u_i^j \cap u_{i+1}^j$ for $j = 2, 3, 4, \ldots, n_i - 1$.

Then $F^* = \bigcup \{F_i : 1 \leq i \leq n\}$ provides a strict $U$-chain as in (3.4). The converse is obvious.

Combining Theorem 3.5 with Fleishman's Theorem [FI], we can state the following.

(3.6) Corollary: A Hausdorff space $X$ is countably compact and connected if and only if it is strictly $\mu_0$-chainable.

We conclude by providing a sequential characterization of $\mu_0$-uniform chainability. For $U \in \mu_0$ we define $U$-chain distance function $\phi_U : X \times X \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ as follows:
\[ \phi_U(x, y) = n - 1 \text{ where } n \text{ is the smallest natural number such that} \]
\[ y \in s(n, x, U). \]
\[ = \infty \text{ if no such } n \text{ exists.} \]

From Theorem (2.7) it is clear that \( X \) is connected if and only if for each \( U \in \mu_\beta . \phi_U \) is finite. If \( X \) is connected and \( U \in \mu_\beta \) we say that:

(a) \( \phi_U \) is constant on a sequence \( (x_n) \) if and only if \( \{ \phi_U(x_n, x_m) : n \neq m \} \) is a singleton;

(b) \( \phi_U \) is bounded on \( (x_n) \) if and only if \( \{ \phi_U(x_n, x_m) : n \neq m \} \) is finite.

We shall need the following basic theorem of combinatorics.

(3.7) Ramsey's Theorem: Let \( r \in \mathbb{N} \) and \( \{ A_i : 1 \leq i \leq m \} \) a partition of the \( r \)-element subsets of \( \mathbb{N} \). Then there is an infinite subset \( S \) of \( \mathbb{N} \) and \( i \in \{ 1, 2, 3, \ldots, m \} \) such that each \( r \)-element subset of \( S \) belongs to \( A_i \).

For the proof see [3].

(3.8) Theorem: For a connected space the following are equivalent:

(a) \( X \) is \( \mu_\beta \)-uniformly chainable.

(b) For each \( U \in \mu_\beta \), each sequence \( (x_n) \) in \( X \) has a subsequence on which \( \phi_U \) is constant.

(c) For each \( U \in \mu_\beta \), each sequence \( (x_n) \) in \( X \) has a subsequence on which \( \phi_U \) is bounded.

Proof: (a) \( \rightarrow \) (b). Let \( U \in \mu_\beta \). Since \( X \) is \( \mu_\beta \)-uniformly chainable, there is an \( m \) in \( \mathbb{N} \) such that \( \phi_U(x, y) \leq m \) for all \( x, y \) in \( X \). If \( (x_n) \) is a finite sequence, there is nothing to prove. If \( (x_n) \) is an infinite sequence define:

\[ A_i = \{ \{ x_i, x_k \} : \phi_U(x_i, x_k) = 1 \}, \quad 0 \leq i \leq m. \]

Clearly, \( \{ A_1, A_2, A_3, \ldots, A_m \} \) is a partition of 2-element subsets of the infinite set \( \{ x_n : n \in \mathbb{N} \} \). By Ramsey's Theorem (3.7), there is an \( i \),
such that for some infinite subset $S$ of $\{ x_i : i \in \mathbb{N} \}$ all two element subsets of $S$ are in $A_i$. By arranging the elements $S$ in the natural order we have a subsequence on which $\phi_U$ is constant.

(b) $\rightarrow$ (c) is obvious.

(c) $\rightarrow$ (a). If $X$ is not $\mu_r$-uniformly chainable, there is a $U \in \mu_r$ and $y$ in $X$ such that $X \not= st^n(y, U)$ for each $n$ in $\mathbb{N}$. Let $x_1 = y$ and $x_n \in st^n(x_1, U) - st^{n-1}(x_1, U)$. Clearly, $(x_n)$ is an infinite sequence such that $\phi_U(x_n, x_m) \geq |n - m|$. Hence $\phi_U$ is unbounded on any subsequence of $(x_n)$.

References

