ABSTRACT

In case of graphs without loops or parallel edges, Behzad-Radjavi [1] show that if $H$ is the total graph of a graph which is not a cycle or a complete graph then $H$ contains one and only one subgraph $G$, such that $H=T(G)$. In this work we show it is possible to extend this result to multigraphs and a constructive method for determining the primitive submultigraph is given.

1. INTRODUCTION

We consider finite, connected multigraphs with loops permitted. For simplicity we shall call these simply graph. A graph is trivial if it consists of a single vertex.

Given a graph $G$, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges.

With every graph $G$ is associated a graph $G^0$, called the LINE GRAPH of $G$, whose vertices are in a one to one correspondence with the edges of $G$ in such a way that two vertices $u',v'(u' \neq v')$ of $G^0$ are joined by as many edges as the corresponding edges $u,v$ of $G$ have common vertices, and also, if $u$ is a loop of $G$, then at the corresponding vertex $u'$ there is only one loop.

In a similar way we can associate with $G$ a graph $H = T(G)$ called the TOTAL GRAPH of $G$, such that $V(H) = V(G) \cup V(G^0)$
and $E(H) = E(G) \cup E(G^0) \cup E(C(H))$, where $C(H)$, called the MIDDLE GRAPH, has vertices $V(H) = V(G) \cup V(G^0)$ and edges $[x, u']$ where $x \in V(G)$ is an end vertex of $u \in E(G)$.

$H$ is a total graph if there is a graph $G$ such that $H = T(G)$. The preceding definitions have been considered in [2].

![Diagram of a graph](image)

**FIGURE 1**

**REMARKS**

If $H = T(G)$ then

1.1. $G$ is connected if and only if $H$ is connected.

1.2. The number of loops of $H$ is even.

1.3. If $u \in E(G)$, in the middle graph of $H$ there are one or two edges with vertex $u'$.

1.4. If loops or parallel edges are not admitted, $G$ is $k$-regular of order $n$ if and only if $H$ is $2k$-regular of order $n(1 + k/2)$.

1.5. If $H$ is the total graph of a cycle (in particular a loop or two parallel edges) or of a complete graph, $H$ cannot be the total graph of another type of graph. (Follows from 1.4 and the definition of total graph applied to a loop or a pair of two parallel edges).

1.6. If $H$ is the total graph of a path of order $n > 2$, $H$ cannot be a total graph of another type of graph.
1.7. If $x \in V(H)$ is the end vertex of at least two loops, then $x \in V(G)$.

1.8. If there are three or more parallel edges $[x,y]$, then $x, y \in V(G)$.

1.9. If $x \in V(H)$ is the end vertex of parallel edges and at least one loop, $x \in V(G)$.

If $x \in V(G)$ we shall denote by $d_x$ and $D_x$, respectively, the degree of $x$ in $G$ and in $T(G)$. For vertices $u' \in G^o$ we shall denote by $D_{u'}$ their degree in $H = T(G)$.

1.10. If $G$ is not trivial, $D_x - d_x \geq 1$.

1.11. If $x \in V(G)$ and there are $h_x$ loops, $h_x \geq 0$, with end vertex $x$, then $D_x = 2d_x - h_x$.

1.12. If $u = [x,x] \in E(G)$ and there are $h_x$ loops, $h_x \geq 1$, with vertex $x$, $D_u = d_x - h_x + 2 = D_x - d_x + 2$. If $d_x = 2$ then $D_u = D_x$, otherwise, $D_u < D_x$.

2. RESULTS

THEOREM 1

Let $H$ be a total graph. If $A$ is a non-trivial subgraph of $H$ and $A^o$ is its corresponding line graph in $H$ then there exists a unique graph $G$ such that $A \subseteq G$ and $H = T(G)$.

proof

If $A \subset H$ is not trivial and $A^o$ is its corresponding line graph in $H$, we can construct $T(A) \subseteq H$. It is obvious that if $T(A) = H$ then $A = G$.

If $T(A) \neq H$, let $\hat{A} = \{ u_1', u_2', \ldots, u_k' \}$ be the set of all vertices of $H = T(A)$ adjacent to some vertex of $A^o$. Now we consider the edges of $H$ with at least one vertex in $A$, the edges with the other vertex in $V(A^o) \cup \hat{A}$ are in $C(H)$. The remaining ones determine a well-defined subgraph $A_1 \subset H$ such that $A \subset A_1$, and its corresponding line graph $A_1^o$ is the graph induced by $V(A_1^o) \cup \hat{A}$. We can repeat the same argument with $A_1$ and its corresponding line graph $A_1^o$, and in this way determine a sequence

$A_1 \subset A_2 \subset \ldots \subset A_k$
such that
\[ H = T(A_k) \quad \text{and} \quad A_k = G. \]

Let \( A \) be the family of vertex subsets of a total graph that define cocycles without parallel edges. We say that a subgraph is an \( \alpha \)-subgraph if it is a graph induced by the minimal elements of \( A \) that have loops and parallel edges.

REMARKS

Let \( S \) be an \( \alpha \)-subgraph in \( H = T(G) \)

2.1. \( S \) is connected.

2.2. If \( S \) has no parallel edges, \( |V(S)| = 1 \) and its edges are loops.

2.3. If \( |V(S)| = 2 \), \( S \) has at least two parallel edges.

2.4. If \( |V(S)| \geq 3 \) and \( H \) has not three or more parallel edges \( [x,y] \) then \( S \subseteq G \).

LEMMA 1

If \( H = T(G) \) and \( A \subseteq G \) is an \( \alpha \)-subgraph, then it is possible to determine its corresponding \( A^0 \subseteq G^0 \).

proof

If \( H \) has a single subgraph \( A^0 \) whose vertices are adjacent to the vertices of \( A \) just as the total graph definition implies, the lemma is trivial. Otherwise, the corresponding \( A^0 \subseteq G^0 \) is the one whose vertices are not adjacent to the end vertex of a loop or parallel edges, except those of \( A \).

LEMMA 2 (Behzad-Radjavi)

Let \( G \) be a connected graph without loops or parallel edges, which is not a path, a cycle or a complete graph and let \( a_0 \) be a vertex of \( H = T(G) \) of maximal degree \( 2d \).

Then \( a_0 \in V(G) \) if and only if the graph induced by the neighborhood of \( a_0 \) has exactly one \( K_d \) as subgraph.

From the lemma it follows that the edges \( [a_0, x_i] \),
i \in \{1, 2, \ldots, d\} such that \( x_i \not\in K_d \) determine a subgraph \( A \subseteq G \) whose corresponding line graph \( A^0 \) is \( K_d \).

The part a) of the following theorem collects the results obtained in [1].

THEOREM 2

Let \( H \) be a total graph, connected, with loops or parallel edges permitted.

a) If \( H \) is the total graph of a cycle or a complete graph, \( H \) contains more than one subgraph \( G_i \) such that \( H = T(G_i) \).

b) Otherwise, \( H \) has exactly one subgraph \( G \) whose total graph is \( H \).

proof

a) If \( G \) is a cycle, \( G \neq K_3 \), (in particular a loop or two parallel edges) \( H \) contains two subgraphs \( G_i \) such that \( H = T(G_i) \).

For \( G = K_3 \), \( T(G) \) is the total graph of each of its eight triangles.

For a complete graph \( G \) of order \( p \), \( p > 3 \), there are exactly \( p \) other subgraphs \( G_i \) of \( T(G) \) with \( T(G_i) = T(G) \), \( i = 1, \ldots, p \).

b) We consider different cases and for each of them we shall show that there exists only one subgraph \( G \) such that \( H = T(G) \).

Certainly, if we assume that we know a certain subgraph \( A \subset H \) to which we can apply Lemmas 1 or 2, following the constructive method used in the proof of theorem 1 we can determine the desired subgraph \( G \).

That we can choose an adequate subgraph \( A \) in all cases is a consequence of the following considerations.

If \( H \) has no loops or parallel edges and is not the total graph of a cycle, a complete graph, or a path, to determine \( A \) we apply Lemma 2.
If \( H \) is the total graph of a path, we take \( A \) to be the smallest path which joins the unique vertices of degree 2. (see 1.6).

Otherwise \( H \) contains loops or parallel edges. If \( H \) contains \( \alpha \)-subgraphs such that their vertices satisfy 1.7, 1.8, 1.9 we choose \( A \) among them. If in \( H \) there are no such \( \alpha \)-subgraphs but there is an \( \alpha \)-subgraph \( S \) such that \( |V(S)| \geq 3 \) then \( S \) can be taken as \( A \).

If all \( \alpha \)-subgraphs \( S \) consist of only a loop or two parallel edges, we take \( A \) to be one of the loops whose vertex has maximum degree in \( H \) (see 1.12) or one of the pairs of parallel edges with the minimum number of vertices adjacent to both end vertices.

The following theorem is of interest in itself and generalizes the theorem of Behzad and Radjavi [1]. The proof is similar.

**THEOREM 3**

Let \( G_1 \) and \( G_2 \) be graphs. Then \( T(G_1) = T(G_2) \) if only if \( G_1 = G_2 \).

**proof**

It suffices to prove the result for connected graphs. If \( G_1 \) and \( G_2 \) are isomorphic so are their total graphs. The converse follows from 1.5, 1.6 and theorem 2.
REFERENCES


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