A Riemannian Characterization of Extrinsic 3-Symmetric Spaces

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Abstract

This paper contains a new characterization of extrinsic 3-symmetric spaces. It involves, besides the induced almost complex structure associated to the 3-symmetric space, only elements related to the Riemannian metric and the isometric imbedding.

1 Introduction

This paper contains a characterization of extrinsic 3-symmetric submanifolds of $R^N$ in terms of the Riemannian metric and the almost complex structure which can be canonically defined in any 3-symmetric space. In fact our result shows that the knowledge of the almost complex structure and its relationship with the metric and second fundamental form are sufficient to recover the 3-symmetric structure which makes them extrinsic 3-symmetric.

Some 3-symmetric spaces have been identified in [2], [1] and [7] as "twistor spaces" over Riemannian symmetric spaces of inner type. It turns out that these spaces are just the extrinsic 3-symmetric ones.

The interest of the theorem included in this note lies on the fact that it is an strictly Riemannian result i.e. it characterizes extrinsic 3-symmetric spaces in terms geometric invariants arising from the metric and the almost complex structure.

The result is the following.

Theorem 1 Let $M^{2n}$ be a compact simply connected almost Hermitian manifold with almost complex structure $J$. Assume that $i : M^{2n} \to R^{2n+q}$ is a full isometric imbedding with second fundamental form $\alpha$. Let $\nabla$ be the Riemannian connection and $R(X,Y,Z,W)$ the Riemannian curvature tensor. Then $M^{2n}$ is an extrinsic 3-symmetric submanifold of $R^{2n+q}$ with symmetry tensor $S = \left(\frac{\sqrt{3}}{2}\right) J - \left(\frac{1}{2}\right) I$ if and only if the following conditions are satisfied.

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i) $S$ preserves $\nabla J$ and $\nabla^2 J$.


iii) $(\nabla_U R)(X,Y,Z,W) + (\nabla_U R)(JX,JY,JZ,JW) = 0$.

iv) $\alpha(JX,JY) = \alpha(X,Y)$.

v) $\nabla^2_{JU} (\alpha(X,Y)) = \sqrt{3} \nabla^d_{JU} (\alpha(X,Y)) \Box$

This result extends a theorem due to D. Ferus [3] which characterizes the canonical imbedding of Hermitian Symmetric spaces which, as it is probably well known, are extrinsic k-symmetric for each $k \geq 2$. The next section contains preliminary definitions; the proof of Theorem 1 is contained in section 3.

2 Section

Let $M^{2n}$ be a connected Riemannian manifold, as in [5] we say that $M$ has an s-structure if for each point $p \in M$ there is an isometry $\theta_p$ of the Riemannian manifold $M$ for which $p$ is an isolated fixed point. The s-structure is of order $k \geq 2$ if $\theta_p^k = id_M$ for each point $p$ and $k$ is the minimum natural number with this property. The s-structure is called regular if $\theta_p \circ \theta_q = \theta_{tp(q)}$, where $r = \theta_p(q)$, for every $p$ and $q$ in $M$.

If we have an imbedding $i : M^{2n} \to R^{2n+q}$ we say that $M$ is an extrinsic k-symmetric submanifold of $R^{2n+q}$ if each $\theta_p$ extends to an isometry $\sigma_p : R^{2n+q} \to R^{2n+q}$ such that $\sigma_p (T_p(M)^\perp) = id_{T_p(M)^\perp}$. In this paper we consider only 3-symmetric spaces.

In our Riemannian regular s-manifold we may consider the canonical connection $\nabla^c$ defined by the formula of Graham-Ledger in terms of the tensor $S_p = (\theta_p)_* p$ and the Riemannian connection as follows. Let $D(X,Y)$ be the tensor field on $M$ defined by

$$D(X,Y) = \left[\nabla_{(I-S)^{-1} X} S\right] (S^{-1} Y)$$

then $\nabla^c$ is defined now as $\nabla^c_X Y = \nabla_X Y - D(X,Y)$. In this way $\nabla^c$ is uniquely determined as soon as we have the s-structure defined on $M$. It is important to indicate that the tensors $D$ and $S$, as well as the metric on $M$, are parallel with respect to $\nabla^c$.

Let us recall here a definition given in [8, (2.3)] and used in [6] to characterize R-spaces. The "canonical" covariant derivative of the second fundamental form of an isometrically imbedded k-symmetric space is defined by

$$(\nabla^c_X \alpha)(Y,Z) = \nabla^c_X (\alpha(Y,Z)) - \alpha(\nabla^c_X Y,Z) - \alpha(Y,\nabla^c_X Z)$$

where, as usual, $\nabla^\perp$ denotes the normal connection of the imbedding.
3 Section

First of all we observe that the conditions (i), (ii), (iii) and the hypothesis that $M$ is almost Hermitian with almost complex structure $J$ are precisely the assumptions of [4, p353, (4.5)]. The conclusion of [4] is that $M$ is a Riemannian locally $3$-symmetric space and $J$ is the canonical almost complex structure determined by the tensor $S$ defined by the local cubic isometries of $M$ as $J = \left( \frac{1}{3} \right) \left[ S + \left( \frac{1}{2} \right) I \right]$. We indicated in [4, p353, (4.5)] give a regular $3$-symmetric space.

As we indicated in the previous section we have on $M$ the uniquely defined canonical connection $\nabla^c$. In [8, (1.2)] it is shown that a compact $k$-symmetric space $M$, imbedded in $\mathbb{R}^n$, is extrinsic $k$-symmetric if and only if the following two conditions are satisfied

\begin{enumerate}
  \item $\alpha(SX,SX) = \alpha(X,X) \forall X \in T_p(M), p \in M.$
  \item $\alpha(SX,SX) = \alpha(X,X) \forall X \in T_p(M), p \in M.$
\end{enumerate}

We have to show that, in our situation, the hypothesis (iv) and (v) of (1) imply (i) and (ii) above.

Let us begin by proving (ii). From (iv) it follows that $\alpha(JX,Y) = -\alpha(X,JY)$ and therefore,

\[
\alpha(SX,SY) = \left( \frac{1}{4} \right) \alpha \left( \left( \sqrt{3} J - I \right) X, \left( \sqrt{3} J - I \right) Y \right) = \\
= \left( \frac{1}{4} \right) \left[ 3\alpha(JX,JY) - \alpha(\sqrt{3}JX,Y) - \alpha(X,\sqrt{3}JY) + \alpha(X,Y) \right] = \\
= \left( \frac{1}{4} \right) \left[ 3\alpha(JX,JY) - \alpha(\sqrt{3}JX,Y) + \alpha(\sqrt{3}JX,Y) + \alpha(X,Y) \right] = \\
= \left( \frac{1}{4} \right) \left[ 3\alpha(JX,JY) + \alpha(X,Y) \right] = \alpha(X,Y).
\]

As we indicated in Section 2, the symmetry tensor $S$ is canonically parallel and since $J = \left( \frac{1}{3} \right) \left[ S + \left( \frac{1}{2} \right) I \right]$ we have $\nabla^c J = 0$. Clearly from (iv) we obtain $\nabla^c J (\alpha(JX,Y)) = -\nabla^c J (\alpha(X,JY))$ and then $\alpha(\nabla^c J (JX),Y) = -\alpha(\nabla^c J (JX),Y)$. Finally $\alpha(JX,\nabla^c Y) = -\alpha(X,\nabla^c (JY))$ and these equalities add up to

\[
(\nabla^c J (JX,Y)) = -\alpha(\nabla^c J (X,JY)).
\]

By the expression of $J$ in terms of $S$ this becomes

\[
(\nabla^c J (SX,Y)) + \left( \frac{1}{2} \right) (\nabla^c J (X,Y)) = -\alpha(\nabla^c J (X,SY) - \left( \frac{1}{2} \right) (\nabla^c J (X,Y))
\]

and therefore

\[
(\nabla^c J (SX,Y)) = -\alpha(\nabla^c J (X,(S + I)Y)).
\]
But since $S^2 + S + I = 0$ we obtain
\[(\nabla^c_{SZ} \alpha)(S X, Y) = (\nabla^c_Z \alpha)(X, S^2 Y) . \quad (1)\]

Now we have the following

**Lemma 2** $(\nabla^c_{SZ} \alpha)(S X, Y) = (\nabla^c_Z \alpha)(X, S^2 Y)$.

**Proof.** By definition we have
\[(\nabla^c_{SZ} \alpha)(S X, Y) = \nabla^c_{SZ} (\alpha (S X, Y)) - \alpha (\nabla^c_{SZ} S X, Y) - \alpha (S X, \nabla^c_{SZ} Y) = \nabla^c_{SZ} (\alpha (S X, S^3 Y)) - \alpha (S \nabla^c_Z X, Y) - \alpha (S X, \nabla^c_{SZ} S^3 Y) . \]

Now by the condition (iv) of the theorem we have
\[\alpha (S \nabla^c_Z X, Y) = \alpha (\nabla^c_Z X, S^2 Y),\]
\[\alpha (S X, \nabla^c_{SZ} S^3 Y) = \alpha (S X, \nabla^c_{SZ} S^3 Y)\]
and also
\[\nabla^c_{SZ} (\alpha (S X, S^3 Y)) = \nabla^c_{SZ} (\alpha (X, S^3 Y)).\]

On the other hand, condition (v) can be written as follows
\[\left(\frac{\sqrt{3}}{2}\right) \nabla^c_{SZ} (\alpha (X, Y)) = \left(\frac{3}{2}\right) \nabla^c_Z (\alpha (X, Y)) \]
and this, by the definition of $S$, yields
\[\nabla^c_{SZ} (\alpha (X, Y)) = \nabla^c_Z (\alpha (X, Y)) .\]

From all these equalities we finally get
\[(\nabla^c_{SZ} \alpha)(S X, Y) = \nabla^c_Z (\alpha (X, S^2 Y)) - \alpha (\nabla^c_Z X, S^2 Y) - \alpha (X, \nabla^c_{SZ} S^2 Y) .\]

and, by definition, this is the right hand side of the identity that was to be proved. □

By equation 1 and the Lemma we obtain the identity
\[(\nabla^c_{SZ} \alpha)(S X, Y) = (\nabla^c_Z \alpha)(S X, Y) .\]

and by writing $X$ instead of $S X$ we transform this identity into
\[(\nabla^c_{SZ} \alpha)(X, Y) = (\nabla^c_Z \alpha)(X, Y) \]
which in turn may be written as
\[(\nabla^c_{(I-S)Z} \alpha)(X, Y) = 0 .\]
Since \((I - S)\) is non singular on \(M\) we obtain \((\nabla^2_x \alpha) = 0\) and this shows that the conditions of the theorem are sufficient.

Let us prove that the conditions are necessary.

In [4, (3.6), p349] there is a proof of the necessity of condition (i) and that (ii) and (iii) are necessary is proved in [4, (3.8)(i), p349] and [4, (3.10), p350] respectively. Notice that, by definition, each \(\theta_p\) is a holomorphic isometry and so the conditions \(\theta(R) = R\) and \(\theta(\nabla R) = R\) are satisfied.

That the condition (iv) is necessary, is proved easily as follows:

\[
\alpha(JX, JY) = \frac{1}{2} \alpha \left( \left( S + \left( \frac{1}{2} \right) I \right) X, \left( S + \left( \frac{1}{2} \right) I \right) Y \right) = \left( \frac{1}{2} \right) \left[ \alpha(SX, SY) + \left( \frac{1}{2} \right) \alpha(X, SY) + \left( \frac{1}{2} \right) \alpha(X, SY) + \left( \frac{1}{2} \right) \alpha(X, Y) \right] = \left( \frac{1}{2} \right) \left[ \left( \frac{1}{2} \right) \alpha(X, Y) + \left( \frac{1}{2} \right) \alpha((S^2 + S) X, Y) \right] = \left( \frac{1}{4} \right) \alpha(X, Y) - \frac{1}{2} \alpha(X, Y) = \alpha(X, Y).
\]

In order to finish the proof we first notice that condition (v) is equivalent to the following identity

\[
\nabla^L_{SU}(\alpha(SX, SY)) = \nabla^L_{SU}(\alpha(X, Y)).
\]

In fact, since the second fundamental form satisfies \(\alpha(SX, SY) = \alpha(X, Y)\), this last equality is just

\[
\nabla^L_{SU}(\alpha(X, Y)) = \nabla^L_{SU}(\alpha(X, Y)). \quad (2)
\]

Now replacing \(S = \left( \frac{\sqrt{3}}{2} \right) J - \left( \frac{1}{2} \right) I\) in the last equality we get

\[
\left( \frac{\sqrt{3}}{2} \right) \nabla^L_{SU}(\alpha(X, Y)) - \left( \frac{1}{2} \right) \nabla^L_{U}(\alpha(X, Y)) = \nabla^L_{U}(\alpha(X, Y))
\]

which is clearly equivalent to

\[
\nabla^L_{SU}(\alpha(X, Y)) = \sqrt{3} \nabla^L_{U}(\alpha(X, Y)).
\]

Now to prove the equation 2 we need to use that \(M\) is extrinsic 3-symmetric i.e. for each \(p \in M\) there exists an isometry \(\sigma_p : \mathbb{R}^{2n+q} \rightarrow \mathbb{R}^{2n+q}\) such that \(\sigma_p \mid T_p(M)^\perp = \text{Id}_{(T_p(M))^\perp}\) and \(\sigma_p \mid T_p(M) = \theta_p\). Since this is the case we have \(SU = (\theta_p)*pU = (\sigma_p)*pU\) and therefore

\[
\alpha(SX, SY) = \alpha((\theta_p)*pX, (\theta_p)*pY) = (\sigma_p)*p\alpha(X, Y).
\]

Then we may write

\[
\nabla^L_{SU}(\alpha(SX, SY)) = \nabla^L_{((\sigma_p)*pU)}((\sigma_p)*p\alpha(X, Y)) = \nabla^L_{U}(\alpha(X, Y)).
\]

This proves the validity of the equation 2 and completes the proof of Theorem 1. □
References


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