$L^p$ APPROXIMATION OF GENERALIZED BI-AXIALLY
SYMMETRIC POTENTIALS WITH FAST GROWTH

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ABSTRACT. The paper deals with growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called generalized bi-axially symmetric potentials (GBSP). We obtain the characterization of $q$-type and lower $q$-type of a GBSP, $H \in H_R$, $0 < R < \infty$, in terms of decay of approximation error $E_{n,p}(H, R_i), i = 1, 2.$

1 INTRODUCTION

Generalized bi-axially symmetric potentials (GBSP's) are the solutions of elliptic partial differential equation

\begin{equation}
\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{2\alpha + 1}{y} \frac{\partial H}{\partial y} + \frac{2\beta + 1}{y} \frac{\partial H}{\partial x} = 0, \quad \alpha, \beta > - \frac{1}{2},
\end{equation}

which are even in $x$ and $y$ cf. Gilbert [1]. A polynomial of degree $n$ which is even in $x$ and $y$ is said to be a GBSP polynomial of degree $n$ if it satisfies (1.1). A GBSP $H$ regular about origin can be expanded as

\begin{equation}
H \equiv H(r, \theta) = \sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha, \beta)}(\cos \theta),
\end{equation}

where $x = r \cos \theta, y = r \sin \theta$ and $P_n^{(\alpha, \beta)}(t)$ are Jacobi polynomials.

Let $D_R = \{(x, y) : x^2 + y^2 < R, 0 < R < \infty\}$ and $\overline{D}_R$ be the closure of $D_R$. A GBSP $H$ is said to be regular in $D_R$ if the series (1.2) converges uniformly on every compact subset of $D_R$. Let $H_R$ be the class of all GBSP's regular in $D_R$ for every $R' \leq R$ but for no $R' > R$. The functions in the class $H_\infty$ are called entire GBSP's.

McCoy [6] considered the approximation of an entire GBSP $H$ by GBSP polynomials and found the rate of decay of approximation error:

$$E_{n,p}(H, 1) = \inf \| H - g \|_{1,p} = \inf_{x \in \mathbb{R}} \left( \int_{D_1} \mu(x, y) \| H(x, y) \|^{p} dx \, dy \right)^{1/p},$$

in terms of growth parameters associated with the maximum modulus $M(r, H) = \max_{\theta} |H(r, \theta)|$, where $\mu$ is a weight function and $1 \leq p < \infty$.

Also, McCoy [7] considered the approximation of pseudo analytic functions, constructed as complex combination of real valued analytic functions to the Stokes-Beltrami system on the disc. These functions include the GBSP's. He obtained some coefficients and Bernstein type growth theorems on the disc in sup norm.
A GBSP $H$ is said to be regular in $\mathcal{D}_{R_0}$, the closure of $D_{R_0}$ if it is regular in $D_{R'}$ for some $R' > R_0$. Let $\mathcal{H}_{R_0}$ be the class of all GBSP’s regular on $\mathcal{D}_{R_0}$. For $H \in \mathcal{H}_{R_0}$, set
\[
\|H\|_{R_0} = \max_{(x,y) \in D_{R_0}} |H(x,y)|,
\]
and for $1 \leq p < \infty$,
\[
\|H\|_{R_0,p}^1 = \left( \int_{D_{R_0}} w(R_0, \theta) |H(R_0, \theta)|^p \, d\theta \right)^{1/p},
\]
\[
\|H\|_{R_0,p}^2 = \left( \int_{D_{R_0}} \overline{w}(x,y) |H(x,y)|^p \, dx \, dy \right)^{1/p},
\]
where the functions $w$ and $\overline{w}$ are positive and integrable (in the sense of Lebesgue) such that $\frac{1}{w}$ and $\frac{1}{\overline{w}}$ are bounded and $\|H\|_{R_0,p}$ and $\|H\|_{R_0,p}^p$ are $L^p$-norms on $\mathcal{H}_{R_0}$. For $H \in \mathcal{H}_{R_0}$ approximation errors $E_{n,p}^1(H, R_0)$ and $E_{n,p}^2(H, R_0)$ are defined as
\[
E_{n,p}^1(H, R_0) = \inf_{g \in \pi_n} \|H - g\|_{R_0,p}^1
\]
\[
E_{n,p}^2(H, R_0) = \inf_{g \in \pi_n} \|H - g\|_{R_0,p}^2
\]
where $\pi_n$ consists of all GBSP polynomials of degree at most $2n$. The concept of index $q$, the $q$-order $\rho(q)$ and lower $q$-order $\lambda(q)$ were introduced by Sato [8] in order to obtain a measure of growth of the maximum modulus when it is rapidly increasing. Thus, let $M(r, H) \to \infty$ as $r \to R$ and for $q = 2, 3, \ldots$, we define
\[
\rho(q)(H, R) = \limsup_{r \to R} \frac{\log^q M(r, H)}{\log M(r, H)}.
\]
where $\log M(r, H) = M(r, H)$ and $\log^q M(r, H) = \log(\log^{q-1} M(r, H))$. The GBSP $H \in \mathcal{H}_{R}$ is said to have the index $q$ if $\rho(q)(H, R) < \infty$ and $\rho(q-1)(H, R) = \infty$. If $q$ is the index of $H$ then $\rho(q)(H, R)$ is called the $q$-order of $H$. The notions of the index and $q$-order play a significant role in classifying the rapidly increasing functions analytic in $D_R$. To compare the growth of two functions analytic in $D_R$ that have same $q$-orders the distinct growth parameters are used.

We have the following definitions:

**Definition 1.** A GBSP $H \in \mathcal{H}_{R}$, $0 < R < \infty$ having $q$-order $\rho(q)(H, R)(\rho(q)(H, R) > 0)$ is said to have $q$-type $T_q(H, R)$ and lower $q$-type $t_q(H, R)$ if
\[
T_q(H, R) = \limsup_{r \to R} \frac{\log^{q-1} M(r, H)}{\log M(r, H)},
\]
\[
t_q(H, R) = \inf_{r \to R} \frac{\log^{q-1} M(r, H)}{\log M(r, H)},
\]
In this paper we study the growth and approximation of solutions (not necessarily entire) of certain elliptic partial differential equations. These solutions are called generalized bi-axially symmetric potentials (GBSP’s). The GBSP’s are taken to be regular in a finite hyperball and influence the growth of their maximum moduli on the rate of decay of their approximation.
errors in $L^p$-norm defined by (1.3) and (1.4). The results and methods employed are different from those of McCoy [7]. The text has been divided into three parts. Section 1 consists of introductory exposition of the topic and Section 2 includes some lemmas. Finally, we prove some theorems which characterize the $q$-type $T_q(H, R)$ and lower $q$-type $t_q(H, R)$ of a GBSP $H \in H_{R_0}$, $0 < R_0 < \infty$, in terms of rate of decay of approximation errors $E_{n,p}(H; R_0), 0 < R_0 < R < \infty, i = 1, 2$.

2 PRELIMINARY RESULTS

In this section we give some lemmas as preliminary results which have been used in the sequel.

Lemma 2.1. Let $H \in H_{R}, R > R_0$. Then there exist GBSP polynomials $g_n \in \pi_n$ such that

$$||H - g_n|| \leq KM(r, H)(n + 1)^{n+1/2}(R/r)^{2(n+1)}$$

for all $r$ sufficiently near to $R$ and all sufficiently large values of $n$. Here $K$ is a constant independent of $r$ and $n$ and $\eta = \max(\alpha, \beta)$.

Proof. The proof of this lemma follows from [4].

Lemma 2.2. Let $H \in H_{R}, R > R_0$. Then there exist GBSP polynomials $g_n \in \pi_n$ such that

$$E_{n,p}(H, R_0) \leq K(n + 1)^{n+1/2}(R_0/r)^{2(n+1)}M(r, H); i = 1, 2$$

for all $r$ sufficiently near to $R$ and all sufficiently large values of $n$. Here $K_i$ is a constant depending on $R_0, \alpha_p, \omega$ and $p$ only and $K_2$ a constant depending on $R_0, \omega$ and $p$.

Proof. Using (1.3), (1.4), (1.5), (1.6) and Lemma 2.1 we get the required result.

Lemma 2.3. Let $H \in H_{R_n}$. Then for $n \geq 1$,

$$|a_n|R_0^{\frac{n-1}{p}} \leq \frac{T^{1/p}(2\pi)^{1/2}((2n + \alpha + \beta + 1))P(n, \alpha, \beta)\Gamma(n + \eta + 1)}{(n + 1)(n + 1)}E_{n-1, p}(H, R_0),$$

where $P(n, \alpha, \beta) = \frac{\Gamma(n+1)(n+\alpha+\beta+1)}{(n+1)(n+\alpha+\beta+1)}$.

Proof. By (1.5), for $H \in H_{R_n}$ there exists a GBSP polynomial $g_{n-1}^* \in \pi_{n-1}$ such that

$$2E_{n-1}^1(H, R_0) \geq \|g_{n-1}^*\|_{R_0}^{1/p} \geq \frac{1}{T^{1/p}}\left(\int_0^{2\pi} |H(R_0, \theta) - g_{n-1}^*(R_0, \theta)|^p d\theta\right)^{1/p}.$$

since $1/w$ is bounded and we have $w \geq \frac{1}{T}, T > 0$. For $p > 1$ choose $r^*$ such that $1/p + 1/r^* = 1$. Using Holder’s inequality we get

$$2E_{n-1}^1(H, R_0) \geq \left(\int_0^{2\pi} |H(R_0, \theta) - g_{n-1}^*(R_0, \theta)|^p d\theta\right)^{1/p},$$

(2.4) Combining (2.3) and (2.4), we get

$$2E_{n-1, p}(H, R_0) \geq \frac{1}{(2\pi)^{1/p}}\int_0^{2\pi} |H(R_0, \theta) - g_{n-1}^*(R_0, \theta)|^p d\theta \leq \left(\int_0^{2\pi} |H(R_0, \theta) - g_{n-1}^*(R_0, \theta)|^p d\theta\right)^{1/p}.$$
for $p > 1$, since GBSP's $H$ and $g_{n-1}^*$ are even in $x$ and $y$. For $p = 1$, (2.4) is obvious with $v^* = 0$.

From the orthogonality of Jacobi polynomials [9] and uniform convergence of the series (1.2) on $D_{R_o}$, we have

$$
\frac{a_n R_o^{2n}}{(2n + \alpha + \beta + 1) P(n, \alpha, \beta)} = 2 \int_0^{\pi/2} H(R_o, \theta) p_{n}^{(\alpha, \beta)}(\cos 2\theta) \sin^{2\alpha+1} \theta \cos^{2\beta+1} \theta \, d\theta.
$$

Thus, for any $g \in \pi_{n-1}$ we have

$$
(2.5) \frac{a_n R_o^{2n} p(n, \alpha, \beta)}{(2n + \alpha + \beta + 1)} = 2 \int_0^{\pi/2} (H(R_o, \theta) - g(R_o, \theta)) p_{n}^{(\alpha, \beta)}(\cos 2\theta) \sin^{2\alpha+1} \theta \cos^{2\beta+1} \theta \, d\theta.
$$

From [9], we know that

$$
\max_{-1 \leq k \leq 1} |p_k^{(\alpha, \beta)}(t)| = \frac{\Gamma(k + n + 1)}{\Gamma(n + 1) \Gamma(k + 1)}, \quad \eta = \max(\alpha, \beta).
$$

Taking in particular, $g_{n-1}^*$ it follows that

$$
(2.6) \frac{a_n R_o^{2n}}{(2n + \alpha + \beta + 1)} \leq 2 \frac{\Gamma(n + \eta + 1)}{\Gamma(\eta + 1) \Gamma(n + 1)} \int_0^{\pi/2} |H(R_o, \theta) - g_{n-1}^*(R_o, \theta)| \, d\theta.
$$

Combining (2.5) and (2.7), the lemma follows.

**Lemma 2.4.** Let $H \in \overline{H}_{R_o}$. Then for $n \geq 1$, we have

$$
|a_n| R_o^{2n+2} \leq \frac{\hat{T}^{1/p}(\pi R_o^2)^{1/v^*}(2n + 2)(2n + \alpha + \beta + 1) P(n, \alpha, \beta) \Gamma(n + \eta + 1)}{\Gamma(\eta + 1) \Gamma(n + 1)} E_{n-1}^2(H, R_o).
$$

**Proof.** By (1.6), for $H \in \overline{H}_{R_o}$, there exists $g_{n-1} \in \pi_{n-1}$ such that

$$
2E_{n-1}^2(H, R_o) \geq \|H - g_{n-1}\|^2_{R_o, p} \geq \frac{1}{T^{1/p}} \left( \int_{D_{R_o}} |H(x,y) - g_{n-1}(x,y)|^p \, dx \, dy \right)^{1/p}
$$

$$
\geq \frac{1}{\hat{T}^{1/p}(\pi R_o^2)^{1/v^*}} \left( \int_{D_{R_o}} |H(x,y) - g_{n-1}(x,y)|^p \, dx \, dy \right)^{1/p},
$$

(2.8)

where $\hat{w} = \frac{1}{p}, \hat{T} > 0$ and $1/p + 1/v^* = 1$. From the orthogonality of Jacobi polynomials and uniform convergence of the series (1.2) on $D_{R_o}$, we have for $0 \leq r \leq R$,

$$
\frac{a_n r^{2n}}{(2n + \alpha + \beta + 1) P(n, \alpha, \beta)} = 2 \int_0^{\pi/2} (H(r, \theta) - g_{n-1}(r, \theta)) p_{n}^{(\alpha, \beta)}(\cos \theta) \sin^{2\alpha+1} \theta \cos^{2\beta+1} \theta \, d\theta.
$$

Using (2.6), we get

$$
\frac{a_n r^{2n}}{(2n + \alpha + \beta + 1) P(n, \alpha, \beta)} \leq \frac{\Gamma(n + \eta + 1)}{2\Gamma(n + 1) \Gamma(\eta + 1)} \int_0^{2\pi} |H(r, \theta) - g_{n-1}(r, \theta)| \, d\theta.
$$

Since $H$ and $g_{n-1}$ are even in $x$ and $y$. Multiplying both sides of the above inequality by $r \, dr$ and integrating from 0 to $R_o$, we get
\[
\frac{a_n r^{2n+2}(2n+2)^{-1}}{(2n+\alpha+\beta+1)P(n,\alpha,\beta)} \leq \frac{\Gamma(n+\eta+1)}{2\Gamma(n+1)|\eta+1|} \int_{\Delta_{R_0}} |H(x,y) - g_{n-1}(x,y)| \, dx \, dy.
\]

Combining (2.8) and (2.9) we obtain the required result.

**Lemma 2.5.** Let \( H \in H_R, 0 < R < \infty (R > R_0) \). Then

\[
M(r, H) \leq |a_n| + \frac{T^{1/(2\pi)^{1/\nu'}}}{\Gamma(n+1)} M(r, h) + \frac{r}{R_0} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)|\eta+1|} E_{-1,0}^2(H, R_0) \frac{r^{2n+2}(2n+\alpha+\beta+1)P(n,\alpha,\beta)}{R_0^{2n+2}}.
\]

Proof. Using (2.8) and Lemma 2.3 we get

\[
|\sum_{n=0}^\infty a_n r^{2n+2} p_n^{(\alpha,\beta)}(\cos 2\theta)| \leq |a_n| + \sum_{n=0}^\infty |a_n| r^{2n+2} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)|\eta+1|} E_{-1,0}^2(H, R_0) \frac{r^{2n+2}(2n+\alpha+\beta+1)P(n,\alpha,\beta)}{R_0^{2n+2}}.
\]

which corresponds to desired result.

**Lemma 2.6.** Let \( H \in H_R, 0 < R < \infty \). Then

\[
M(r, H) \leq |a_n| + \frac{T^{1/(2\pi)^{1/\nu'}}}{\Gamma(n+1)} R_0^2(1 - e^*) M(r, h^*) + \frac{r}{R_0} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)|\eta+1|} E_{-1,0}^2(H, R_0) \frac{r^{2n+2}(2n+\alpha+\beta+1)P(n,\alpha,\beta)}{R_0^{2n+2}}.
\]

Proof. Using Lemma 2.4 the proof has the same analysis as that of Lemma 2.5.

**Lemma 2.7.** Let \( f(z) = \sum_{n=0}^\infty a_n z^n \) be analytic in \(|z| < R\). Then the function \( f(z) \) is of q-order and q-type \( T(q) \) if and only if

\[
T(q) = B(q) V(q),
\]

where \( B(q) = (q + 1)^{\rho+1}/\rho^* \), \( A(q) = 1 \) for \( q = 2 \); \( B(q) = 1 \), \( A(q) = 0 \) for \( q = 3, 4, \ldots \) and

\[
V(q) = \limsup_{n \to \infty} \left( \log^{\eta_2} |a_n| \right)^{1/(q+1)} R^n \log^+ |a_n| R^n A(q).
\]

Proof. The lemma can be proved by simple manipulation of the results in [2] and [3].

**Lemma 2.8.** Let \( f(z) = \sum_{n=0}^\infty a_n z^n \) be analytic in \(|z| < R\) and have q-order \( \rho(q) \) \((\rho(q) > 0)\) and lower q-type \( t(q) \). If \( \psi(n_k) = \left| a_{n_k}/a_{n_{k+1}} \right|^{1/(n_{k+1} - n_k)} \) forms a nondecreasing sequence of \( k \) for \( k > k_0 \), then

\[
B(q) t(q) \leq \liminf_{k \to \infty} \left( \log^{\eta_2} |a_{n_k}| R^{n_k} \right)^{1/(n_{k+1})} R^{n_k+1} A(q)
\]

and

\[
B(q) t(q) \leq L(q) \liminf_{k \to \infty} \left( \log^{\eta_2} |a_{n_k}| R^{n_k} \right)^{1/(n_{k+1})} R^{n_k+1} A(q).
\]
where \( L(q) = \limsup_{k \to \infty} (\log^{[q-2]} n_k / \log^{[q-2]} n_{k-1}) \) and \( B(q) \) and \( A(q) \) have the same meaning as in Lemma 2.7.

Proof. The proof of this lemma is available in [2] and [3].

3 MAIN RESULTS

Theorem 3.1. Let \( H \in H_R \) and have q-order \( \rho_q(H, R) \) \( (0 < \rho_q(H, R) < \infty) \) and q-type \( T_q(H, R) \). Then

\[
G(q) = B(q, H) T_q(H, R),
\]

where

\[
G(q) = \limsup_{n \to \infty} (\log^{[q-2]} n) \left( \frac{\log^+ \mathcal{E} \mathcal{N},p(H, R_o)(\frac{R}{R_o})^{2n}}{n} \right)^{\rho_q(H, R) + A(q)}
\]

or

\[
B(q, H) = \frac{(\rho_q(H, R)+1)^{\rho_q(H, R)\rho_q(H, R)^2}}{\rho_q(H, R)\rho_q(H, R)^2}, \quad A(q) = 1 \text{ if } q = 2 \text{ and } B(q, H) = 1, \quad A(q) = 0 \text{ if } q = 3, 4 \ldots
\]

Proof. Let \( G(q) < \infty \). For given \( \epsilon > 0 \) and for all \( n > n_0(\epsilon) \), we have

\[
(\log^{[q-2]} n) \left( \frac{\log^+ \mathcal{E} \mathcal{N},p(H, R_o)(\frac{R}{R_o})^{2n}}{n} \right)^{\rho_q(H, R) + A(q)} < G(q) + \epsilon
\]

or

\[
\log^{[q-1]} n + (\rho_q(H, R) + A(q)) + (\log^+ \log^+ \mathcal{E} \mathcal{N},p(H, R_o) + 2n \log \frac{R}{R_o} - \log n < G(q) + \epsilon
\]

or

\[
\rho_q(H, R) + A(q) > \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+ \mathcal{E} \mathcal{N},p(H, R_o) - 2n \log \frac{R}{R_o}}
\]

Let \( 0 \leq T_q(H, R) < \infty \). For given \( \epsilon > 0 \) and \( r > r_o (1.7) \) implies

\[
\log M(r, H) < \exp^{(q-2)}\left\{ (T_q(H, R) + \epsilon)\left( \frac{R}{R-r} \right)^{\rho_q(H, R)} \right\}
\]

Using Lemma 2.2, we further have

\[
\log^+ \mathcal{E} \mathcal{N},p(H, R_o)(\frac{R}{R_o})^{2n} \leq \log M(r, H) + (\eta + \frac{1}{2}) \log(n + 1) + 2n \log \left( \frac{R}{r} \right) + \log K_i
\]

\[
(3.2) \leq \exp^{(q-2)}\left\{ (T_q(H, R) + \epsilon)\left( \frac{R}{R-r} \right)^{\rho_q(H, R)} \right\} + (\eta + \frac{1}{2}) \log(n + 1) + 2n \log \left( \frac{R}{r} \right) + O(1)
\]

Let \( r \) be given by the equation

\[
(3.3) \prod_{i=0}^{q-2} \exp\left\{ (T_q(H, R) + \epsilon)\left( \frac{R}{R-r} \right)^{\rho_q(H, R)} \right\} = \frac{2n(R-r)}{R\rho_q(H, R)}
\]
For $q = 2$, using (3.3) in (3.2) we have for sufficiently large values of $n$

$$
\log^+ \left( E_{n, p}(H, R_o) \frac{R}{R_o} \right)^{2n} \leq \left( \frac{\log^{(q-2)} n}{T_q(H, R) + \epsilon} \right)^{1/\rho_q(H, R)} (1 + \rho_q(H, R)) + o(1)
$$

On proceeding to limits, the above inequality yields $G(2) \leq B(2, H) T_2(H, R)$.

Next, for $q = 3, 4, \ldots$, (3.3) implies

$$
(\log^{(q-2)} n (1 + o(1))) \left( \frac{\log^+ \left( E_{n, p}(H, R_o) \frac{R}{R_o} \right)^{2n}}{n} \right)^{1/\rho_q(H, R)} < (T_q(H, R) + \epsilon)(1 + o(1)).
$$

Taking limits as $n \to \infty$, we observe that $T_q(H, R) \geq G(q)$ for $q \geq 3$.

To prove the reverse inequality we utilise Lemma 2.5 for the case $i = 1$ and Lemma 2.6 for $i = 2$ and then apply Lemma 2.7 to the functions $h(u)$ and $h^*(u)$.

**Theorem 3.2.** Let $H \in H_R$ and $H$ have $q$-order $\rho_q(H, R)$ and lower $q$-type $t_q(H, R)$. Let $n_k$ be an increasing sequence of natural numbers. Then

$$
B(q, H) t_q(H, R) \geq \liminf_{k \to \infty} \left[ (\log^{(q-2)} n_{k-1}) \left( \frac{\log^+ \left( E_{n, p}(H, R_o) \frac{R}{R_o} \right)^{2n_k}}{n_k} \right)^{1/\rho_q(H, R) + A(q)} \right].
$$

**Proof.** Let

$$
\liminf_{k \to \infty} \left[ (\log^{(q-2)} n_{k-1}) \left( \frac{\log^+ \left( E_{n, p}(H, R_o) \frac{R}{R_o} \right)^{2n_k}}{n_k} \right)^{1/\rho_q(H, R) + A(q)} \right] = \phi(q) \equiv \phi.
$$

First suppose that $0 < \phi < \infty$. Then, for $\phi > \epsilon > 0$ and $k > k_o$,

$$
\log^+ \left( E_{n, p}(H, R_o) \frac{R}{R_o} \right)^{2n_k} > n_k \left\{ \frac{(\phi - \epsilon) B(q, H)}{\log^{(q)} n_{k-1}} \right\}^{1/(\rho_q(H, R) + A(q))}
$$

Choose a sequence $\{r_{nk}\}$ such that

$$
2 \log \frac{R}{r_{nk}} = \frac{(\phi - \epsilon) C'(q)}{\log^{(q-2)} n_{k-1}},
$$

where $C'(q) = \rho_q(H, R)$ if $q = 2$ and $C'(q) = C'$, $0 < C' < 1$ if $q = 3, 4 \ldots$.

By Lemma 2.2, if $k > k_o$ and $r_k \leq r < r_{k+1}$, then denoting $\rho_q(H, R) + A(q)$ by $\rho^*$, we get
\[
\log M(r, H) \geq \log E_{n_k,p}^i (H, R_o) \left( \frac{R}{R_o} \right)^{2n_k} - \left( \eta + \frac{1}{2} \right) \log(n_{k+1}) - 2n_k \log \left( \frac{R}{R_o} \right) - \log K_i
\]

Using (3.4) we get

\[
\log M(r, H) > \exp^{[q-2]} \left\{ \frac{\phi - \epsilon}{\log(n_{k+1})} \right\} \left\{ B(q, H) \right\}^{1/q} - C' \left( \frac{B(q, H)}{C'(q)} \right)^{1/q} + O(1).
\]

For \( q = 2 \), we have

\[
\log M(r, H) > \frac{\exp^{[q-2]} \left\{ \frac{\phi - \epsilon}{\log(n_{k+1})} \right\} \left\{ B(q, H) \right\}^{1/q} - C' \left( \frac{B(q, H)}{C'(q)} \right)^{1/q} + O(1)}{2 \log B(q, H) - 1} + O(1).
\]

Proceeding to limits as \( r \to R \) we get \( t_q(H, R) \geq \phi \).

Now, for \( q = 3, 4, \ldots \),

\[
\lim_{r \to R} \frac{\log^{[q-1]} M(r, H)}{L(\rho_2(H,R))} \geq \phi C'.
\]

Since the above inequality holds for every \( C' \), making \( C' \to 1 \), we get \( t_q(H, R) \geq \phi \) for \( q = 3, 4, \ldots \).

If \( \phi = 0 \) the result follows trivially. If \( \phi = \infty \), the above inequality with an arbitrary large number in place of \( \phi - \epsilon \) gives \( t_q(H, R) = \infty \).

**Theorem 3.3.** Let \( H \in H_0 \) \( 0 < r < \infty (R_o < R) \) and \( H \) have \( q \)-order \( \rho_q(H, R) \) and lower \( q \)-type \( t_q(H, R) \). If \( \psi(n_k) = (E_{n_k, p}^i (H, R_o) / E_{n_k, p}^i (H, R_o))^{1/(n_{k+1} - n_k)} \) forms a nondecreasing function of \( k \) for \( k > k_o \) and \( \log^{[q-2]} n_k \approx \log^{[q-2]} n_{k+1} \) as \( k \to \infty \), then

\[
B(q, H) t_q(H, R) \leq \liminf_{k \to \infty} \left[ \log^{[q-2]} n_k \right] \left( \frac{\log E_{n_k, p}^i (H, R_o) \left( \frac{R}{R_o} \right)^{2n_k}}{n_k} \right)^{\rho_q(H,R) + A(q)}
\]

\[
\leq L(q) \liminf_{k \to \infty} \left[ \log^{[q-2]} n_{k-1} \right] \left( \frac{\log E_{n_k, p}^i (H, R_o) \left( \frac{R}{R_o} \right)^{2n_k}}{n_k} \right)^{\rho_q(H,R) + A(q)},
\]

where \( L(q) = \lim \sup_{k \to \infty} \left[ \log^{[q-2]} n_k / \log^{[q-2]} n_{k-1} \right] \).

Proof. The proof of the above theorem follows by using Lemma 2.2 and Lemma 2.7 for \( i = 1 \) to the function \( h(u) \) and for \( i = 2 \) to the function \( h^*(u) \).

On combining theorems 2 and 3 we have the following result:
Theorem 3.4. Let \( H \in H_R \) \( 0 < r < \infty (R_0 < R) \) and \( H \) have \( q \)-order \( \rho_q (H, R) \) and lower \( q \)-type \( t_q (H, R) \). If \( \psi(n_k) = (E_{n_k}^i (H, R_0) / (E_{n_k+1}^i (H, R_0)))^{1/(n_k+1-n_k)} \) forms a nondecreasing function of \( k \) for \( k > k_0 \) and \( \log^{[r-2]} n_k \approx \log^{[r-2]} n_{k+1} \) as \( k \rightarrow \infty \), then

\[
B(q, H)t_q (H, R) = \lim \inf_{k \to \infty} \left( \log^{[r-2]} n_{k-1} \left( \frac{\log^+ E_{n_k}^i (H, R_0) (R/R_0)^{2n_k}}{n_k} \right) \right)^{\rho_q (H, R) + A(\psi)}
\]

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