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ON REAL ANALYTIC POINCARÉ SERIES OF EXPONENTIAL GROWTH

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ABSTRACT. Let G be the universal cover of Sl(2,**R**) , Γ a discrete subgroup of G of finite covolume and χ a unitary character of Γ. Using an analogue of the Eisenstein series, we define a family $\mathbf{M}(\nu, g, \chi)$ of automorphic functions of non-moderate growth on G, holomorphic in the half-plane $\{\nu \in \mathbf{C} \mid \operatorname{Re}\nu > 1\}$. We prove that this family admits a meromorphic continuation to $\mathbf{C}$ and satisfies a functional equation which relates a linear combination of $\mathbf{M}(\nu, g, \chi)$ and $\mathbf{M}(-\nu, g, \chi)$ with a linear combination of Eisenstein series. We also prove that, although the family is generically of exponential growth, the residues of $\mathbf{M}(\nu, g, \chi)$ in $\{\nu \mid \operatorname{Re}\nu \geq 0\}$ define square integrable automorphic forms.

§0. INTRODUCTION.

Poincaré Series are an important object in the theory of automorphic forms. They were originally introduced as holomorphic functions on the upper half plane satisfying a certain transformation rule with respect to a discrete subgroup Γ of Sl(2,**R**) of finite covolume, and a multiplier system v. Their importance stems from the fact that they generate the space of holomorphic cusp forms of each given weight. More recently, non-holomorphic Poincaré series have been studied, originated in the ideas of Selberg (see [S], [N], [He], [Br], [Br2]) and have been used in the study of Fourier coefficient of real analytic automorphic forms. There are also generalizations in higher dimensions (see for instance [CLPS] for the group $SO(n,1)$ and [MW] for Lie groups of real rank one with finite center).

The purpose of this paper is to show that the method in [MW] applies in the study of $L^2(\Gamma\backslash G, \chi)$, where G is the universal cover of Sl(2,**R**) and χ is a unitary character of Γ, a discrete subgroup of G of finite covolume. We shall define a family $\mathbf{M}(\nu, \chi)$ of functions on G, via an analogue of the Eisenstein series, but using a matrix entry of a principal series on G which transforms according to a character η of the unipotent subgroup N. This series defines a one parameter family of automorphic forms of non-moderate (i.e. exponential) growth in the half-plane $\{\nu \in \mathbf{C} \mid \operatorname{Re}\nu > 1\}$. We prove that this family admits a meromorphic continuation to $\mathbf{C}$ and satisfies a functional equation which asserts that a suitable combination of $\mathbf{M}(\nu, \chi)$ and $\mathbf{M}(-\nu, \chi)$ is a linear combination of Eisenstein series (see Theorem 7.5). Although the family $\mathbf{M}(\nu, \chi)$ is generically of exponential growth, certain special values yield automorphic forms in the standard sense, namely classical holomorphic cusp forms associated to a certain multiplier system.

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Several of the results in this paper are already present in [He] (see Chapter 9, §7) and in [Br2]. However, our methods are very different, and follow the representation theoretic approach in [MW], yielding, at times, more explicit information.

§1. PRELIMINARIES.

Let G denote the universal cover of G_0=Sl(2,**R**) , and let π :G→ G_0 denote the canonical projection. G is generated by elements $\{n(x), a(y), k(\theta)\,|x, \theta \in \mathbf{R}, y > 0\}$ such that

$$\pi(n(x)) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \pi(a(y)) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y^{-1}} \end{pmatrix} \quad \pi(k(\theta)) = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix} \tag{1.1}$$

and furthermore

$$n(x_1)n(x_2) = n(x_1 + x_2) \qquad x_1, x_2 \in \mathbf{R} \tag{1.2}$$
$$a(y_1)a(y_2) = a(y_1 y_2) \qquad y_1, y_2 \in \mathbf{R}^+ \tag{1.3}$$
$$k(\theta_1)k(\theta_2) = k(\theta_1 + \theta_2) \qquad \theta_1, \theta_2 \in \mathbf{R} \tag{1.4}$$
$$a(y)n(x) = n(yx)a(y) \qquad x \in \mathbf{R},\ y \in \mathbf{R}^+, \tag{1.5}$$
$$k(\theta)p(z) = p\,(\pi(k(\theta)) \cdot z)k(\theta - \arg|_{(-\pi,\pi]}\, e^{i\theta}(-z \cdot \operatorname{sen}\theta + \cos\theta)) \tag{1.6}$$

where $p(x+iy) = n(x)a(y)$ and $\pi(g) \cdot z$ denotes the action of G_0 on $\mathcal{H} = \{\operatorname{Im} z > 0\}$ by Moebius transformations. We will often write $\pi(g) = \overline{g}$.

Set $N = \{n(x)|\, x \in \mathbf{R}\}$, $A = \{a(y)|\, y > 0\}$, $K = \{k(\theta)|\, \theta \in \mathbf{R}\}$ and let $M = \{k(m\pi)|\, m \in Z\}$, the center of G , and $P = N \cdot A \cdot M$. Any element $g \in G$ decomposes uniquely $g = nak$, with $n \in N$, $a \in A$ and $k \in K$. We choose invariant measures dn, da, dk on N, A and K corresponding respectively to dx, $\frac{dy}{y}$, $\frac{1}{2\pi}d\theta$ where dx, dy, $d\theta$ denote Lebesgue measure. If $a = a(y)$, set $a^\rho = \sqrt{y}$. On G we will normalize the Haar measure so that

$$\int_G f(g)\,dg = \frac{1}{2\pi}\int_N \int_A \int_K f(nak)a^{-2\rho} dn\, da\, dk \qquad f \in C_c(G) \tag{1.7}$$

We identify $\mathfrak{g}$, the Lie algebra of G , with the Lie algebra of G_0. Let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Set $\mathfrak{g}_c = \mathfrak{g} \otimes_R \mathbf{C}$ and let $E^+ = H + i(X+Y)$, $E^- = H - i(X+Y)$, $W = X - Y$. Thus X , H and W span $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ respectively, the Lie algebras of N, A and K. If $a = \exp(H') \in A$ with $H' \in \mathfrak{a}$, we will write $a^\nu = e^{\nu(H')}$ for $\nu \in \mathfrak{a}^*$. Also we set $\rho = \frac{\alpha}{2}$. For $t \in \mathbf{R}$, we set $a_t = \exp(tH) = a(e^{2t})$, so that $a_t^\rho = e^t$. We will usually identify $\mathfrak{a}_{\mathbf{C}}^*$ with $\mathbf{C}$ via the map $\nu \mapsto \nu(H)$.

Denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}_c$ and let $\mathcal{U}(\mathfrak{g})$ act on $C^\infty(G)$ by left-invariant differential operators. If $X \in \mathfrak{g}_c$, $f \in C^\infty(G)$, as usual let $Xf(g) = \left.\frac{d}{dt}\right|_0 f(g \exp tX)$. The following formulas describe the action of $E^\pm, W$ with respect to the coordinates (x,y,θ): $E^{\pm} = e^{\pm 2i\theta}(2iy\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \mp i\frac{\partial}{\partial\theta})$ and $W = \frac{\partial}{\partial\theta}$. The Casimir element of G is given by $C = \frac{1}{2}(XY + YX + \frac{1}{2}H^2) = \left(\frac{H}{2}\right)^2 - \left(\frac{H}{2}\right) + XY$ which as a differential operator on G acts by $y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}$. Hence C acts on $f \in C^\infty(G/K)$ by $(\check{C}f)(x+iy) = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\check{f}(x+iy)$, where $\check{f}(g\cdot i) = f(g)$ and $g \cdot i = \pi(g)\cdot i = x+iy$. Furthermore we have $\int_{G/K} f(\tilde{g})d\tilde{g} = \int_N\int_A f(na)a^{-2\rho}\,dn\,da = \int_{\mathcal{H}} \check{f}(x+iy)\,dx\frac{dy}{y^2}$

From now on we let Γ be a discrete subgroup of G which contains M and such that $\Gamma\backslash G$ is non compact and has finite invariant measure. Let Q be a parabolic subgroup of G . Then $Q = kPk^{-1}$ for some $k \in K$, thus $Q = N_Q A_Q M$, where $N_Q = kNk^{-1}$ and $A_Q = kAk^{-1}$. Q is said to be Γ-*cuspidal* if $\Gamma \cap N_Q \neq 1$. Γ acts by conjugation on the set of such subgroups and the action has finitely many orbits. An orbit of this action is called a *cusp* of Γ.
Let $Q = kPk^{-1}$ be as above, $\mathfrak{a}_Q = Ad(k)\mathfrak{a}$. If $\nu \in \mathbf{C}$, $H \in \mathfrak{a}_Q$, we write $k\cdot\nu = \nu(Ad(k^{-1})H)$

Fix $\omega \subset N_Q$ a compact subset, such that $\omega \mapsto \Gamma \cap N_Q\backslash N_Q$ is surjective. If $y > 0$, let $A^+_{Q,y} = \{a \in A_Q | a^{k\rho} > y\}$. A Q-Siegel set for Γ is a set of the form $\mathcal{S}_{Q,\omega,y} = \omega \times A^+_{Q,y} \times K$. We next recall a basic result of reduction theory of $\Gamma\backslash G$.

Theorem 1.1. *Let Γ be as above. If $P_j = N_j A_j M$, $j = 1\ldots s$ is a complete system of representatives of Γ-cuspidal parabolic subgroups, and if for each j, $\omega_j \subset N_j$ is a compact subset such that $\pi(\omega_j) = \Gamma \cap N_j\backslash N_j$, then $\Gamma\backslash G - \left(\cup_{j=1}^{r} \mathcal{S}_{P_j,\omega_j,t_j}\right)^o$ is compact.*

§2. CHARACTERS AND MULTIPLIER SYSTEMS.

Let χ be a unitary character of Γ and fix $\tau \in (-1,1]$ such that $\chi(k(n\pi)) = e^{in\pi\tau}$ for any $k(n\pi) \in M$. We then write $\chi = \chi_\tau$. Let $r \in \mathbf{R}$; if $z \in \mathbf{C} - \{0\}$ we write $z^r = \exp(r\log z)$, where log stands for the principal branch of the logarithm (corresponding to $-\pi < \arg z \leq \pi$).

If $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_0$, set $j(\gamma,z) = cz + d$. Let $\overline{\Gamma} = \Pi(\Gamma)$. For g, $g_1 \in G_0$, we set

$$w(\gamma,\gamma_1) = \frac{1}{2\pi}\left\{\arg|_{(-\pi,\pi]}\, j(\gamma,\gamma_1\cdot z) + \arg|_{(-\pi,\pi]}\, j(\gamma_1,z) - \arg|_{(-\pi,\pi]}\, j(\gamma\gamma_1,z)\right\} \tag{2.1}$$

Let $r \in \mathbf{R}$. A function $v : \overline{\Gamma} \mapsto S^1$ is called a *multiplier system* of weight r if $v(\gamma\gamma_1) = v(\gamma)v(\gamma_1)\exp(2\pi i r w(\gamma,\gamma_1))$, for any γ, $\gamma_1 \in \overline{\Gamma}$, where $w(\gamma,\gamma_1)$ is as in (2.1) (see [Rn], chapter 3). In the notation of §1, let $\sigma : G_0 \mapsto G$ be the section given by:

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = p(\frac{ai+b}{ci+d})k(\arg|_{(-\pi,\pi]}(-ci+d)) \tag{2.2}$$

Then for g, $g_1 \in G_0$, $\sigma(gg_1) = \sigma(g)\sigma(g_1)k(2\pi w(g,g_1))$, where $w(g,g_1)$ is as in (2.1).

We now recall the correspondence between characters of Γ and multiplier systems of $\overline{\Gamma}$. (c.f. [Br], chapter 4). Let $\chi = \chi_\tau$, $\tau \in (-1,1]$ be a character of Γ, and let $r \equiv \tau$ mod 2. Then the function $v = v_\chi : \overline{\Gamma} \mapsto \mathbf{C}$ given by $v(\overline{\gamma}) = \chi(\sigma(\overline{\gamma}))$ is a multiplier system on $\overline{\Gamma}$ of weight r. Conversely, if v is a multiplier system on $\overline{\Gamma}$ of weight r, then the function $\chi = \chi_v : \Gamma \mapsto \mathbf{C}$ given by $\chi(\sigma(\overline{\gamma})k(n\pi)) = v(\overline{\gamma})e^{in\pi\tau}$ is a character of Γ. The maps $\chi \mapsto v_\chi$, $v \mapsto \chi_v$ are inverse to each other.

§3. AUTOMORPHIC FORMS.

Fix χ a unitary character of Γ, $\chi = \chi_\tau$, as above. A function $f : G \mapsto \mathbf{C}$ is *(χ,Γ)-equivariant* if $f(\gamma g) = \chi(\gamma) f(g)$, $\gamma \in \Gamma$, $g \in G$. For the next results we refer to [Br] or [Ro].

Theorem 3.1. *Let $\overline{\Gamma} = \pi(\Gamma)$, and let $\chi = \chi_\tau$ be a character of Γ, with $\tau \in (-1,1]$. If $r \in \mathbf{R}$ with $r \equiv \tau \bmod 2$ and $f : G \mapsto \mathbf{C}$ is a (χ,Γ)-equivariant function of weight r, then the function $F = \phi(f) : \mathcal{H} \mapsto \mathbf{C}$ defined by $F(z) = f\big(p(z)\big) y^{-\frac{r}{2}}$ satisfies*

$$F(\overline{\gamma} z) = v(\overline{\gamma}) j(\overline{\gamma}, z)^r F(z) \tag{3.1}$$

for any $\overline{\gamma} \in \overline{\Gamma}, z \in \mathcal{H}$, where $v = v_\chi$ is as in §2.

Conversely, if v is a multiplier system in $\overline{\Gamma}$ of weight r and $F : \mathcal{H} \mapsto \mathbf{C}$ satisfies (3.1), then the function f_1 on G defined by

$$f_1\big(\sigma(\overline{g}) k(n\pi)\big) = F(\overline{g} \cdot i) j(\overline{g}, i)^{-r} e^{irn\pi}$$

is (χ,Γ)-equivariant, with $\chi = \chi_v$ as in §2. The maps $f \to F$ and $F \to f_1$ are inverse to each other.

Definition 3.2. *Let $\chi = \chi_\tau$ be a character of Γ , $\tau \in (-1,1]$ as above. Let $r \in \mathbf{R}$. A function $f \in C^\infty(G)$ is said to be (χ,Γ)-automorphic of weight r if*

(1) $f(\gamma g) = \chi(\gamma) f(g) \qquad \gamma \in \Gamma,\ g \in G$
(2) *for any $k(\theta) \in K$, $g \in G$ $f\big(gk(\theta)\big) = f(g) e^{ir\theta}$*
(3) *f is a finite sum of eigenfunctions of C,*
(4) *f is of moderate growth. That is, if $Q = kPk^{-1}$ is a parabolic subgroup, then for any $X \in \mathfrak{g}_c$, there exist $C_X > 0$ and $d \in \mathbf{N}$ such that $|Xf(kg)| \le C_X a(g)^{d\rho}$*

We denote by $\mathcal{A}(\Gamma\backslash G, \chi)$ the space of Γ-automorphic forms and we observe that if $\chi\big(k(n\pi)\big) = e^{in\pi\tau}$ for any $k(n\pi) \in M$, then $\mathcal{A}(\Gamma\backslash G, \chi)$ is generated by functions of weight r, $r \equiv \tau \mod 2$.

As is well known, the correspondence in Theorem 3.1 gives an isomorphism of the space $\mathcal{A}(\Gamma\backslash G, \chi)$ with the space of real analytic automorphic forms on $\mathcal{H}$, of weight r and multiplier system v.

Definition 3.3. *Let Q be a cuspidal parabolic subgroup. Q is said to be regular (resp. irregular) with respect to χ if $\chi|_{\Gamma\cap N_Q} \equiv 1$ (resp. $\chi|_{\Gamma\cap N_Q} \not\equiv 1$).*

If $f \in \mathcal{A}(\Gamma\backslash G, \chi)$ and Q is regular, then the *Q-constant term* of f is defined by:

$$f_Q(g) = \int_{\Gamma\cap N_Q\backslash N_Q} f(ng)\, dn \tag{3.2}$$

If $f_Q \equiv 0$ for any regular cuspidal parabolic subgroup Q, f is said to be a *cusp form.* Denote by $\mathcal{A}_0(\Gamma\backslash G, \chi)$ the space of cusps forms in $\mathcal{A}(\Gamma\backslash G, \chi)$.

Let $f,\ g \in L^2(\Gamma\backslash G,\chi)$, the space of square integrable (χ,Γ)-equivariant functions on G with respect to the canonical inner product $\langle f,g\rangle = \int_{\Gamma\backslash G} f(x)\overline{g(x)}\,d\overline{x}$, where $d\overline{x}$ is the canonical measure on $\Gamma\backslash$G. $L^2(\Gamma\backslash G,\chi)$ defines a unitary representation of G , the action of G given by right translations.

We will make use of the spectral decomposition of $L^2(\Gamma\backslash G,\chi)$ (see [Ro], or [Br], Chapter 10). We first recall the definition of the Eisenstein series.

Definition 3.4. *Let Q be a regular cuspidal parabolic subgroup of G . Let $\xi = \chi\big|_M$ and let $\nu \in \mathbf{C}$. If $\phi \in C^\infty(K)$, $g \in G$, define the Eisenstein series by*

$$E(Q,\nu,g,\phi,\chi) = \sum_{\gamma\in\Gamma_Q\backslash\Gamma} a(\gamma g)^{\nu+\rho}\phi(k(\gamma g))\chi(\gamma)^{-1} \tag{3.3}$$

It is well known ([Ro], [Br]) that the series defining $E(Q,\nu,g,\phi,\chi)$ converges uniformly on compacta if $\mathrm{Re}\,\nu > \rho$ and lies in $\mathcal{A}(\Gamma\backslash G,\chi)$. Furthermore $E(Q,\nu,g,\phi,\chi)$ admits a meromorphic continuation to $\mathbf{C}$.

If V is a Banach representation of G we let V_∞, V_K and V_ω respectively denote the space of C^∞, (C^∞,K)-finite and analytic vectors in V. We will use the following theorem of Selberg and Roelcke.

Theorem 3.5. *(Spectral decomposition) If $\chi = \chi_\tau$, τ as above, then $L^2(\Gamma\backslash G,\chi) = L^2_d(\Gamma\backslash G,\chi) \oplus L^2_c(\Gamma\backslash G,\chi)$, where $L^2_d(\Gamma\backslash G,\chi)$ (resp. $L^2_c(\Gamma\backslash G,\chi)$) decomposes discretely (resp. continuously) under the action of G .*

Furthermore, let $\{\psi_j\}_{j=1}^\infty$ in $L^2_d(\Gamma\backslash G,\chi)_K$ (c.f. [W], ch. 1) be a complete orthonormal system of eigenfunctions, $C\psi_j = \mu_j\psi_j$. Fix a complete system of representatives of regular cuspidal parabolic subgroups $\{P_1,\cdots,P_s\}$. Then, if $f \in L^2(\Gamma\backslash G,\chi)$ is of weight r, $r \equiv \tau$ mod 2, we have

$$f = \sum_{i=1}^{\infty}\langle f,\psi_i\rangle\psi_i + \sum_{i=1}^{s}\frac{2\pi i}{m_i}\int_{\mathrm{Re}\,\lambda=0} f_i(\lambda)\,E(P_i,\lambda,\phi_r,\chi)\,d\lambda \tag{3.4}$$

in $\|\;\|_2$ with $f_i : i\mathbf{C} \mapsto \mathbf{C}$ measurable functions, $1 \le i \le s$, such that $\int_{\mathrm{Re}\,\nu=0} |f_i(\nu)|^2\,|d\nu| < \infty$.

§4. REPRESENTATION THEORY.

If $\xi \in \hat{M}$, then $\xi(k(\pi)) = e^{i\pi\tau}$, for some $\tau = \tau_\xi \in (-1,1]$. Set

$$V^\xi = \{f \in C(K) \mid f(mk) = \xi(m)f(k) \quad m \in M,\ k \in K\}$$

with norm $\|\cdot\|$ given by $\|f\|^2 = \int_{M\backslash K} |f(k)|^2\,dk$ for $f \in V^\xi$. Let $H^{\xi,\nu}$ be the completion of V^ξ with respect to the above norm, with action of G given by

$$\pi_{\xi,\nu}(x)f(u) = a(ux)^{\nu+\rho}f\big(k(ux)\big) \qquad u \in K,\ x \in G \tag{4.1}$$

where $g = n(g)a(g)k(g)$, $n(g) \in N$, $a(g) \in A$, $k(g) \in K$. When restricted to K, $H^{\xi,\nu}_K$ decomposes as follows: $H^{\xi,\nu}_K = \bigoplus_{r\equiv\tau_\xi \bmod 2} \mathbf{C}\phi_r$, where $\phi_r(k(\theta)) = e^{ir\theta}$ for $k(\theta) \in K$.

As before, let $H^{\xi,\nu}_{\infty}$ (resp. $H^{\xi,\nu}_{K}$) be the space of C^{∞}-vectors (resp. (C_{∞},K)-finite vectors) in $H^{\xi,\nu}$; $H^{\xi,\nu}_{K}$ is a $(\mathfrak{g},K)$-module, with the action of $\mathfrak{g}_c$ and K given by: $E^{\pm}\cdot\phi_r = (1+\nu\pm r)\phi_{r\pm 2}$, $W\cdot\phi_r = ir\phi_r$ and $k(\theta)\cdot\phi_r = e^{ir\theta}\phi_r$.

We next recall some standard facts on the Kunze-Stein intertwining operator.

(1) If $v\in V^{\xi}_{\infty}$, the integral $A(\xi,\nu)v(k) = \int_N \pi_{\xi,\nu}(snk)v(1)\,dn$ converges absolutely and uniformly on compacta for $k\in K$ and $\operatorname{Re}\nu > 0$.

(2) the map $\nu \mapsto \dfrac{1}{\Gamma(\nu)}A(\xi,\nu)$ analytically continues to a holomorphic function : $\mathbf{C} \mapsto$ {continuous endomorphisms of V^{ξ}_{∞}}

(3) If $r\equiv \tau_{\xi} \mod 2$ one has that $A(\xi,\nu)\phi_r = c_r(\nu)\phi_r$ where

$$c_r(\nu) = \frac{2^{1-\nu}\pi\, i^r\, \Gamma(\nu)}{\Gamma\left(\frac{1+\nu+r}{2}\right)\Gamma\left(\frac{1+\nu-r}{2}\right)} \tag{4.2}$$

(4) If $\nu\in$ is generic, one has $A(\xi,\nu)\circ\pi_{\xi,\nu}(x) = \pi_{\xi,-\nu}(x)\circ A(\xi,\nu)$, for $x\in G$.

It follows from (3) that $A(\xi,-\nu)\circ A(\xi,\nu) = \frac{1}{\mu_{\xi}(\nu)}\cdot \mathrm{Id}$, where

$$(\mu_{\xi}(\nu))^{-1} = c_r(-\nu)c_r(\nu) = -4\pi\,\frac{\cos\frac{\pi(\nu+r)}{2}\cos\frac{\pi(\nu-r)}{2}}{\nu\,\mathrm{sen}(\pi\nu)}$$

We note that this expression does not depend on r but only on ξ. In particular, if $r=0$ (resp. $r=1$) we get that $\mu_{\xi}(it) = \frac{t}{2\pi}\tanh\left(\frac{\pi t}{2}\right)$ (resp. $\mu_{\xi}(it) = \frac{t}{2\pi}\coth\left(\frac{\pi t}{2}\right)$).

For generic $\nu\in\mathbf{C}$, the $(\mathfrak{g}_c,K)$-space $H^{\xi,\nu}_{K}$ is irreducible. Reducibility occurs exactly when $(1+\nu)\equiv\tau \mod 2$ or $(1+\nu)\equiv -\tau \mod 2$. By using the intertwining operator $A(\xi,\nu)$ one can determine the composition series of $H^{\xi,\nu}_{K}$ at each ν. For such a description, together with the classification of irreducible $(\mathfrak{g},K)$-modules of G, we refer to [Pu] or [Br], Chapter 3.

§5. WHITTAKER VECTORS.

Let (V,π) be a representation of $\mathfrak{g}$. Let $\eta\big(n(x)\big) = e^{i\alpha x}$ be a non trivial character of N. A vector $w\in V$ is a *Whittaker vector* associated to the character η if

$$X\cdot w = d\eta(X)w \tag{5.1}$$

Let $\nu\in\mathbf{C}$. A vector $u\in V$ is said to be *conical* of weight ν if $X\cdot u = 0$ and $H\cdot u = \big(\nu-1\big)u$. Let $\mathfrak{b} = \mathbf{C}X\oplus\mathbf{C}H$ a Borel subalgebra of $\mathfrak{g}$. Let $\mathcal{U}(\mathfrak{b})$ be the universal enveloping algebra of $\mathfrak{b}$. Given $\mu\in\mathbf{C}$, we denote by $M(\mu)$ the Verma module $M(\mu) = \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\mathbf{C}$ where the action of $\mathcal{U}(b)$ is given by

$$X\cdot(1\otimes 1) = 0 \qquad H\cdot(1\otimes 1) = \big(\mu-1\big)(1\otimes 1) \tag{5.2}$$

Clearly $1\otimes 1$ is a conical vector in $M(\mu)$ of weight μ.

Given η, a non trivial character of N, we now sketch the construction of a Whittaker vector associated to η. For this purpose, we will work formally with power series in $M(\mu)$.

Assume $w = \sum_{k=0}^{\infty} a_k(\mu)(Y^k \otimes 1)$ is such that $X \cdot w = d\eta(X)w$. Then $\sum_{k=0}^{\infty} a_k(\mu)X \cdot (Y^k \otimes 1) = \sum_{k=0}^{\infty} d\eta(X)a_k(\mu)(Y^k \otimes 1)$. To compute the coefficients $a_k(\mu)$ we use the relations $[X, Y^k] = k\,Y^{k-1}(H - k + 1)$ and $[X, Y^k] \otimes 1 = X(Y^k \otimes 1) - Y^k X(1 \otimes 1) = X(Y^k \otimes 1)$, for $k \geq 0$. One obtains that

$$\sum_{k=0}^{\infty} a_k(\mu)X \cdot (Y^k \otimes 1) = \sum_{k=0}^{\infty} a_{k+1}(\mu)(k+1)\big(\mu - (k+1)\big)(Y^k \otimes 1) \tag{5.3}$$

It follows from (5.1) and (5.3) that $a_{k+1}(\mu) = a_k(\mu)\frac{d\eta(X)}{(k+1)\left(\mu-(k+1)\right)}$ for $k \geq 0$ or, putting $a_0(\mu) = \frac{1}{\Gamma\left(-\mu+1\right)}$, we obtain the recursion

$$a_k(\mu) = \frac{(-1)^k (d\eta)^k(X)}{k!\Gamma\big(-\mu + k + 1\big)} \tag{5.4}$$

Let $\bar{\mathfrak{n}} = \mathbf{R}Y$. Then $\mathrm{T}(\mu) = \sum_{k=0}^{\infty} a_k(\mu)Y^k$ can be viewed as an operator mapping the vector $1 \otimes 1$ of $M(\mu)$ to a Whittaker vector in the $\bar{\mathfrak{n}}$-completion of $M(\mu)$, (see [GW], §4). Similarly, if V is a $U(\mathfrak{g})$-module and $v \in V$ is such that $X \cdot v = 0$ and $H \cdot v = (\mu - 1)v$, then there exists a Whittaker vector $\tilde{w} = \mathrm{T}(\mu)v$ in the $\bar{\mathfrak{n}}$-completion of $\mathbf{V}$. We will now apply this theory to the principal series representations.

If $\nu \in \mathbf{C}$, $\xi \in \hat{M}$ consider the representation $(H^{\xi,\nu}, \pi_{\xi,\nu})$ of G . We denote $H_{\omega}^{\xi,\nu}$ the space of analytic vectors. Thus we have the following inclusions: $H_K^{\xi,\nu} \subset H_{\omega}^{\xi,\nu} \subset H_{\infty}^{\xi,\nu} \subset H^{\xi,\nu}$, with each space dense in the next.

Let $s = k(\pi/2)$ and set as in [GW]

$$\delta(\xi,\nu)f = f(1) \tag{5.5}$$

$$\delta_s(\xi,\nu)f = \int_N \pi_{\xi,\nu}(sn)f(1)\,dn = \big(A(\xi,\nu)f\big)(1) = \delta(\xi,-\nu) \circ A(\xi,\nu)(f) \tag{5.6}$$

for $f \in H_{\infty}^{\xi,\nu}$, $\mathrm{Re}\,\nu > 0$. It is easy to check that $\delta(\xi,\nu)$ and $\delta_s(\xi,\nu)$ are conical vectors in $(H_{\infty}^{\xi,\nu})'$, of weight $-\nu$ and ν respectively. We fix a character η of N and we set

$$w(\xi,\nu) = \mathrm{T}(-\nu)\,\delta(\xi,\nu) \tag{5.7}$$

$$w_s(\xi,\nu) = \mathrm{T}(\nu)\,\delta_s(\xi,\nu) \tag{5.8}$$

the corresponding Whittaker vectors. We will write also $w_1(\nu)$ y $w_s(\nu)$ when ξ is clear from the context. It turns out that the vectors $w_1(\nu)$ and $w_s(\nu)$ are continuous functionals on the space $H_{\omega}^{\xi,\nu}$ (see [GW]).

§6. RELATIONSHIP BETWEEN WHITTAKER VECTORS AND WHITTAKER FUNCTIONS.

For $k \in \mathbf{R}$, $s \in \mathbf{C}$, the Whittaker differential equation is given by

$$f''(y) + \left\{-\frac{1}{4} + \frac{k}{y} + \frac{\frac{1}{4} - s^2}{y^2}\right\} f(y) = 0 \qquad y > 0 \tag{6.1}$$

Equation (6.1) has a solution, $M_{k,s}(y)$, defined for $s \in \mathbf{C} - (-\frac{1}{2})\mathbf{N}$, given by

$$M_{k,s}(y) = y^{s+\frac{1}{2}} e^{-\frac{y}{2}} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}+s-k\right)_n}{n!(1+2s)_n} y^n \tag{6.2}$$

with $(a)_0 = 1$ and $(a)_n = \prod_{j=0}^{n-1}(a+j)$ for $n \geq 1$. It has exponential growth as $y \to \infty$ and $|M_{k,s}(y)| \ll y^{\frac{1}{2}+\mathrm{Re}\, s}$ as $y \to 0$.
Another solution of equation (6.1) is given by

$$W_{k,s}(y) = \frac{\Gamma(2s)}{\Gamma\left(\frac{1}{2}+s-k\right)} M_{k,-s}(y) + \frac{\Gamma(-2s)}{\Gamma\left(\frac{1}{2}-s-k\right)} M_{k,s}(y) \tag{6.3}$$

This function has exponential decay when $y \to \infty$, in fact $|W_{k,s}(y)| \ll y^k e^{-\frac{y}{2}}$ as $y \to \infty$, and exponential growth as $y \to 0$. The following integral formula will also be needed:

$$W_{k,s}(y) = \pi^{-1} 2^{2s-1} \Gamma(1/2+s+k)\, y^{\frac{1}{2}-s} \int_{-\infty}^{\infty} e^{-\frac{1}{2}ixy} (1+x^2)^{-\frac{1}{2}-s} e^{2ik \arctan x}\, dx \tag{6.4}$$

We will now look at certain matrix entries of the principal series representations. Let $\xi \in \hat{M}$, $\nu \in \mathbf{C}$, η a non trivial character of N and let $w \in (H_\omega^{\xi,\nu})'$ be a Whittaker vector associated to η. Denote by $\psi(g)$ the matrix entry $\psi(g) = w\big(\pi_{\xi,\nu}(g)\phi_r\big)$, for $g \in G$, $\phi_r \in H^{\xi,\nu}$. We observe that, if $g = nak$, with $n \in N$, $a \in A$ and $k \in K$, $\psi(g) = \eta(n)^{-1}\psi(a)\phi_r(k)$. If $\mathcal{C}$ denotes the Casimir operator, then ψ satisfies the differential equation $\mathcal{C}\psi = \frac{\nu^2-1}{4}\,\psi$

Proposition 6.1. *Let $t \in \mathbf{R}$, $a_t = \exp(tH)$. Let $\eta\big(n(x)\big) = e^{i\alpha x}$, with $\alpha > 0$. Let $w \in (H_\omega^{\xi,\nu})'$ be a Whittaker vector associated to η^{-1}. If $z = 2\alpha e^{2t}$, let $F(z) = w\big(\pi_{\xi,\nu}(a_t)\phi_r\big)$. Then*

(1) $F(z)$ satisfies the Whittaker differential equation (6.1) with $k = \frac{r}{2}$ and $s = \frac{\nu}{2}$,
(2) if $F_1(z) = w_1(\nu)\big(\pi_{\xi,\nu}(a_t)\phi_r\big)$ and $F_s(z) = w_s(\nu)\big(\pi_{\xi,\nu}(a_t)\phi_r\big)$ then

$$F_1(z) = \frac{(2\alpha)^{-\frac{\nu+1}{2}}}{\Gamma(\nu+1)} M_{\frac{r}{2},\frac{\nu}{2}}(z) \qquad F_s(z) = \frac{(2\alpha)^{\frac{\nu-1}{2}}}{\Gamma(-\nu+1)} c_r(\nu) M_{\frac{r}{2},-\frac{\nu}{2}}(z) \tag{6.5}$$

where $M_{k,s}$ is as in (6.2) and $c_r(\nu)$ is as in (4.2).

Proof. We have

$$\frac{d}{ds}\bigg|_{s=0} w\big(\pi_{\xi,\nu}(a_t \cdot \exp(sH))\phi_r\big) = \frac{d}{ds}\bigg|_{s=0} w\big(\pi_{\xi,\nu}(a_{t+s})\phi_r\big) = \psi'(a_t) = 2F'(z)z$$

Similarly, $\left(\frac{d}{ds}\right)^2\Big|_{s=0} w\big(\pi_{\xi,\nu}(a_t \cdot \exp(sH))\phi_r\big) = \frac{1}{4}\psi''(a_t) = z(F''(z)z + F'(z))$. Now, since $Y = X - W$, we compute

$$\frac{d^2}{ds_1 ds_2}\bigg|_{s_1=s_2=0} w\left(\pi_{\xi,\nu}(a_t)\pi_{\xi,\nu}\big((s_1+s_2)X\big)\phi_r\right) = \left(i\alpha e^{2t}\right)^2 \psi(a_t)$$

$$\left.\frac{d^2}{ds_1 ds_2}\right|_{s_1=s_2=0} w\left(\pi_{\xi,\nu}(a_t)\pi_{\xi,\nu}(s_1X)\pi_{\xi,\nu}(s_2W)\phi_r\right) = (i\alpha e^{2t})(ir)\psi(a_t)$$

Since $\mathcal{C} = \left(\frac{H}{2}\right)^2 - \left(\frac{H}{2}\right) + XY$ and $\mathcal{C}\psi(a_t) = w\big(\pi_{\xi,\nu}(a_t)\pi_{\xi,\nu}(\mathcal{C})(\phi_r)\big)$ we get that $\psi(a_t)$ satisfies $\frac{1}{4}\psi''(a_t) - \frac{1}{2}\psi'(a_t) + \left(-(\alpha e^{2t})^2 + r(\alpha e^{2t})\right)\psi(a_t) = \frac{\nu^2-1}{4}\psi(a_t)$, which in terms of $F(z)$ becomes $F''(z) + \left\{-\frac{1}{4} + \frac{r/2}{z} + \frac{\frac{1-\nu^2}{4}}{z^2}\right\} F(z) = 0$, the Whittaker's differential equation with $k = \frac{r}{2}$, $s = \frac{\nu}{2}$.

We briefly verify (2). By (5.7), $w_1(\nu) = \sum_{k=0}^{\infty} a_k(-\nu)Y^k(\delta(\xi,nu))$ with $a_k(-\nu) = \frac{(-1)^k(d\eta^{-1})^k(X)}{k!\Gamma\left(\nu+k+1\right)}$. Using that $(-Y)\cdot\delta(\xi,\nu)(f) = \delta(\xi,\nu)(Y\cdot f)$, we have that:

$$F_1(z) = \sum_{k=0}^{\infty} a_k(-\nu)\delta(\xi,\nu)\big(\pi_{\xi,\nu}(Y)^k\pi_{\xi,\nu}(a_t)\phi_r\big) =$$

$$\sum_{k=0}^{\infty} \frac{(i\alpha)^k}{k!\Gamma\left(\nu+k+1\right)} \left.\frac{d^k}{ds^k}\right|_{s=0} \pi_{\xi,\nu}\left(\exp(sY)a_t\right)\phi_r(1)$$

A computation shows that if we set $\theta(s) = \arg|_{(-\pi,\pi]}(1-is)$, then

$$\left.\frac{d^k}{ds^k}\right|_{s=0} \left(\pi_{\xi,\nu}(\exp(sY)a_t\right)\phi_r(1) = \left(e^{2t}\right)^k \left.\frac{d^k}{ds^k}\right|_{s=0} a_t^{\nu+\rho} a\left(\frac{1}{1+s^2}\right)\phi_r\left(k(\theta(s))\right)$$

Thus $w_1(\nu)\big(\pi_{\xi,\nu}(a_t)\phi_r\big) = \frac{a_t^{\nu+\rho}}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\frac{i^k\left(-\alpha e^{2t}\right)^k}{k!(\nu+1)\cdots(\nu+k)}a_k(r,\nu)$ for some coefficients $a_k(r,\nu)$. Since $a_t^{\nu+\rho} = \left(\frac{z}{2\alpha}\right)^{\frac{\nu+1}{2}}$ and $a_0(r,\nu) = 1$, we arrive at

$$F_1(z) = \frac{(2\alpha)^{-\frac{\nu+1}{2}}}{\Gamma(\nu+1)} z^{\frac{\nu+1}{2}}\left\{1 + \psi(z,\nu)\right\} \tag{6.6}$$

where $\psi(z,\nu) = O(z)$ as $z \to 0$. By using the asymptotic behavior at $z = 0$ we see that

$$F_1(z) = \frac{(2\alpha)^{-\frac{\nu+1}{2}}}{\Gamma(\nu+1)} M_{\frac{r}{2},\frac{\nu}{2}}(z) \tag{6.7}$$

To compute $F_s(z)$ we use (5.7) and (5.9):

$$F_s(z) = \mathrm{T}(\nu)\delta_s(\xi,\nu)\big(\pi_{\xi,\nu}(a_t)\phi_r\big) = \mathrm{T}(\nu)\big(\delta(\xi,-\nu)\big)\left(\pi_{\xi,-\nu}(a_t)c_r(\nu)\phi_r\right)$$

$$= c_r(\nu)w_1(-\nu)\big(\pi_{\xi,-\nu}(a_t)\phi_r\big) = \frac{(2\alpha)^{\frac{\nu-1}{2}}}{\Gamma(-\nu+1)} c_r(\nu) M_{\frac{r}{2},-\frac{\nu}{2}}(z)$$

The vectors $w_1(\nu)$ and $w_s(\nu)$ generate the space of Whittaker vectors in $(H_\omega^{\xi,\nu})'$.

Let $\eta(n(x)) = e^{i\lambda x}$ be a nontrivial character of N, let $\xi \in \hat{M}$ and $\nu \in \mathbf{C}$. If $f \in H_{\infty}^{\xi,\nu}$, $s = k(\pi/2)$, consider the integral

$$\mathcal{J}_{\xi,\nu}(f) := \int_N \eta(n)^{-1}\pi_{\xi,\nu}(sn)f(1)\,dn$$

In particular,

$$\begin{aligned}\mathcal{J}_{\xi,\nu}\big(\pi_{\xi,\nu}(a)\phi_r\big) &= a^{-\operatorname{Re}\nu+\rho}e^{i(\pi/2)r}\int_{-\infty}^{\infty} e^{-ix\lambda a^{2\rho}}(1+x^2)^{-\frac{\nu+1}{2}}e^{ir\arctan x}\,dx \\ &= \frac{\pi e^{i(\pi/2)r}}{\Gamma\left(\frac{1+\nu+r}{2}\right)}\Big(\frac{\lambda}{2}\Big)^{\frac{\nu-1}{2}}W_{\frac{r}{2},\frac{\nu}{2}}(2\lambda a^{2\rho})\end{aligned} \tag{6.8}$$

This integral is convergent for $\operatorname{Re}\nu > 0$ and analytically continues to a continuous functional on $H_{\infty}^{\xi,\nu}$, for $\operatorname{Re}\nu > 1$. $\mathcal{J}_{\xi,\nu}$ is a Whittaker vector associated to η^{-1}, the *Jacquet-Whittaker vector*. We will often write $\mathcal{J}(\nu)$ in place of $\mathcal{J}_{\xi,\nu}$. As a functional on $H_{\omega}^{\xi,\nu}$, $\mathcal{J}(\nu)$ can be expressed as a linear combination of $w_1(\nu)$ and $w_s(\nu)$ and this combination corresponds to the expression of the Whittaker function $W_{k,s}$ in terms of $M_{k,s}$ and $M_{k,-s}$. To conclude this section we recall some facts on the Jacquet-Whittaker vector (see [GW]).

Proposition 6.2. *Let $\eta\big(n(x)\big) = e^{i\alpha x}$ be a character of N with $\alpha > 0$. Let $\xi \in \hat{M}$, $\nu \in \mathbf{C}$ and $\mathcal{J}(\nu)$ as in (10.2). Then one has that*

$$\mathcal{J}(\nu) = a(\nu)w_1(\nu) + b(\nu)w_s(\nu) \tag{6.9}$$

with

$$a(\nu) = -2\alpha^{\nu}\pi e^{i(\pi/2)r}\frac{\cos\frac{\pi(\nu+r)}{2}}{\operatorname{sen}(\pi\nu)} \qquad \text{and} \qquad b(\nu) = \Gamma(-\nu+1) \tag{6.10}$$

Furthermore $\mathcal{J}(\nu)$ satisfies the functional equation

$$\mathcal{J}(-\nu)\circ A(\xi,\nu) = \gamma_{\xi}(\nu)\cdot\mathcal{J}(\nu) \tag{6.11}$$

where $\gamma_{\xi}(\nu) = 2\alpha^{-\nu}e^{i(\pi/2)r}\Gamma(\nu)\cos\frac{\pi(\nu-r)}{2}$

§7. THE SERIES $\mathbf{M}(\nu)$.

Let $\Gamma \subset G$ be a discrete subgroup as in §1 and let χ be a character of Γ. Let $P = N\cdot A\cdot M$ be a Γ-cuspidal parabolic subgroup. We write $\Gamma_P = \Gamma\cap P$, $\Gamma_N = \Gamma\cap N$. Fix η a character of N and ξ a character of M. We shall assume that we have the compatibility conditions $\eta|_{\Gamma_N} = \chi|_{\Gamma_N}$ and $\chi|_M = \xi$. If $\nu \in \mathbf{C}$, $v \in H_K^{\xi,\nu}$ and $g \in G$ we write

$$M(\xi,\nu,g,v) = w(\xi,\nu)\big(\pi_{\xi,\nu}(g)v\big) \tag{7.1}$$

where $w(\xi,\nu)$ denotes the Whittaker vector associated to η^{-1}, as in (5.7).

In what follows, when some variable is understood it will be omitted. For instance, if ξ, ν are fixed, we will write $M(\nu,g)$, $w(\nu)$ and π_ν for $M(\xi,\nu,g,v)$, $w(\xi,\nu)$ and $\pi_{\xi,\nu}$, respectively.

As seen in (6.6), if ω is a compact subset of $\{\nu \mid \operatorname{Re}\nu > 1\}$, there exists a constant $C_\omega > 0$ such that

$$|M(\xi,\nu,g,v)| \leq C_\omega a(g)^{\operatorname{Re}\nu+\rho} \tag{7.2}$$

for $\nu \in \omega$, $a(g)^\rho < T$. Now, if $X \in \mathcal{U}(\mathfrak{g})$, $XM(\xi,\nu,g,v) = M(\xi,\nu,g,Xv)$, hence XM satisfies a similar estimate. Thus, it follows from (7.2) and the convergence of the Eisenstein series, that the series

$$\mathbf{M}(\xi,\nu,g,v,\chi) = \sum_{\gamma\in\Gamma_P\backslash\Gamma} \chi(\gamma^{-1})M(\xi,\nu,\gamma g,v) \tag{7.3}$$

defines a C^∞ function in $\{\operatorname{Re}\nu > 1\} \times G$, holomorphic in ν (note that the summed function is invariant under $\Gamma_P = \Gamma \cap MN$). We shall call this series, the $\mathbf{M}$-series.

One of our goals will be to study the meromorphic continuation of the $\mathbf{M}$-series and to show that it satisfies a simple functional equation, connecting the values at ν and $-\nu$. We shall also study the poles of this meromorphic continuation for $\operatorname{Re}\nu \geq 0$.

We now define a truncation $\tilde{\mathbf{M}}(\xi,\nu,g,v,\chi)$, which will be very useful for our purposes.

Let $T \in \mathbf{R}$, and let $\phi \in C_c^\infty(G)$ be left N-invariant and right-K-invariant and such that $\phi(a) = 1$ if $a \in A_T^-$, $\phi(a) = 0$ if $a \in A_{T+1}^+$ and $0 \leq \phi \leq 1$. Here $A_T^- = \{\exp(tH) \mid t < T\}$ and $A_{T+1}^+ = \{\exp(tH) \mid t > T+1\}$. We define

$$\tilde{\mathbf{M}}(\xi,\nu,g,v,\chi) = \sum_{\gamma\in\Gamma_P\backslash\Gamma} \phi(\gamma g)M(\xi,\nu,\gamma g,v)\chi(\gamma^{-1}) \tag{7.4}$$

We observe that $\mathbf{M} - \tilde{\mathbf{M}}$ is locally a finite sum of translates of M. Hence $\tilde{\mathbf{M}}$ is defined and C^∞ in the same region as $\mathbf{M}$ (the parameter T will be usually understood).

Now let $\lambda_\nu = (\nu^2 - 1)/4$ be the eigenvalue of $\mathcal{C}$ in $H_\infty^{\xi,\nu}$. We now define the auxiliary function

$$\check{\mathbf{M}}(\xi,\nu,g,v,\chi) = (\mathcal{C} - \lambda_\nu I)\tilde{\mathbf{M}}(\xi,\nu,g,v,\chi) \tag{7.5}$$

Lemma 7.1. *(i) If* $\operatorname{Re}\nu > 1$ *and* $X \in \mathcal{U}(\mathfrak{g})$ *then* $X\tilde{\mathbf{M}}(\nu,g,\chi)$ *is bounded in absolute value and holomorphic in* ν.

(ii) $\check{\mathbf{M}}(\nu,g)$ *can be analytically continued to a* C^∞ *function on* $\mathbf{C} \times G$, *holomorphic in* ν. *Furthermore, the support of* $|\check{\mathbf{M}}(\nu)|$ *is a compact subset of* $\Gamma\backslash G$, *independent of* ν.

Proof. Similar to that of lemmas 2.2 and 2.3 in [MW].

Let $k \in K$ be such that $Q = kPk^{-1}$ is a (χ,Γ)-cuspidal parabolic subgroup. For $\nu \in \mathbf{C}$, let $(k\cdot\nu)H = \nu(Ad(k^{-1}H))$ if $H \in \mathfrak{a}_Q$. Let $L(k) : H^{\xi,\nu} \mapsto H^{\xi,k\nu}$ be the G-isomorphism given by $L(k)f(x) = f(k^{-1}x)$. Then $f \mapsto E(Q,k\nu,g,L(k)f,\chi)$ defines an intertwining operator from $H_K^{\xi,\nu}$ into $\mathcal{A}(\Gamma\backslash G,\chi)$.

We fix $\{P_1,\dots,P_s\}$, a complete system of representatives of regular cuspidal parabolic subgroups, with $P_i = k_iPk_i^{-1}$, $k_i \in K$, $(i = 1,2,\dots,s)$ and $P_1 = P$.

Now, if $\lambda \in \mathbf{C}$, $|\operatorname{Re}\lambda| \leq 1$, then $E(Q, k\lambda, L(k)f, \chi) \in L^1(\Gamma\backslash G, \chi)$ where defined (by the analog of [MW] A, 2.1 in our context). Thus, if $\operatorname{Re}\nu > 1$, the integrals defining $\langle \tilde{\mathbf{M}}(\nu, v, \chi), E(Q, -k\overline{\lambda}, L(k)f, \chi)\rangle$ and $\langle \check{\mathbf{M}}(\nu, v, \chi), E(Q, -k\overline{\lambda}, L(k)f, \chi)\rangle$ converge absolutely for $\operatorname{Re}\nu > 1$, off the poles of $E(Q, -k\lambda, g, L(k)f, \chi)$, by Lemma 7.1.

Lemma 7.2. *(i) Let Q be a regular-cuspidal parabolic subgroup, $Q = kPk^{-1}$. Then*

$$\langle \tilde{\mathbf{M}}(\nu, \phi_r, \chi), E(Q, -k\overline{\lambda}, L(k)\phi_r, \chi)\rangle$$

can be meromorphically continued to $\mathbf{C}\times\mathbf{C}$. The singularities lie in the set $\mathbf{C}\times D \cup \{(\nu, \lambda) \mid (\nu \pm \lambda) \in \{0, 2, 4, \dots\}\}$, where D is a discrete subset of $\mathbf{C}$ that contains the poles of $E(Q, -k\lambda, \phi_r, \chi)$, $\gamma_\xi(-\lambda)$ and $c_r(-\lambda)$.

(ii) The inner product

$$d_Q(\nu, \lambda) = \langle \check{\mathbf{M}}(\nu, \phi_r, \chi), E(Q, -k\overline{\lambda}, \phi_r, \chi)\rangle \tag{7.6}$$

has a meromorphic continuation to $\mathbf{C}\times\mathbf{C}$, with singularities containing in the set $\mathbf{C}\times D$ where D is a discrete set of $\mathbf{C}$ containing the poles of $E(Q, -k\lambda, g, \phi_r, \chi)$. Furthermore, if ν is not a pole of $\gamma_\xi(-\lambda)$, $E(Q, -k\lambda, \phi_r, \chi)$ nor of $c_r(-\lambda)$ then

$$d_Q(\nu, \nu) = -r_\eta(Q, -\nu)\,\alpha^{-\nu} e^{i(\pi/2)r} \cos\frac{\pi(\nu - r)}{2} \tag{7.7}$$

$$d_Q(\nu, -\nu) = -r_\eta(Q, \nu)\,\frac{2^{-\nu} e^{i(\pi/2)r}}{\Gamma\left(\frac{1+\nu-r}{2}\right)\Gamma\left(\frac{1+\nu+r}{2}\right)} \tag{7.8}$$

where $r_\eta(Q, \lambda)$ is the P-Fourier coefficient of $E(Q, k\lambda, g, \phi_r, \chi)$. In particular, if r is an odd integer, $d_Q(0,0) = 0$.

Proof. (compare [MW], Lemma 2.4)
The integral in question equals

$$\int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma_P\backslash\Gamma} \phi(\gamma g) M(\nu, \gamma g, \phi_r)\chi(\gamma^{-1})\overline{E(Q, -k\overline{\lambda}, g, L(k)\phi_r, \chi)}\, dg =$$

$$\int_{\Gamma_P\backslash G} \phi(x) M(\nu, x, \phi_r)\overline{E(Q, -k\overline{\lambda}, x, L(k)\phi_r, \chi)}\, dx =$$

$$\int_{\Gamma_N\backslash N}\int_A\int_{M\backslash K} \phi(a)\eta(n^{-1}) M(\nu, au, \phi_r)\overline{E(Q, -k\overline{\lambda}, nau, L(k)\phi_r, \chi)} a^{-2\rho}\, dn\, da\, du$$

now ([MW], Proposition A, 2.5) implies

$$\int_{\Gamma_N\backslash N} \eta(n) E(Q, -k\overline{\lambda}, nx, L(k)\phi_r, \chi)\, dn = r_\eta(Q, -\overline{\lambda})\mathcal{J}(-\overline{\lambda})\big(\pi_{-\overline{\lambda}}(x)\phi_r\big)$$

Thus, the inner product we are computing equals

$$r_\eta(Q, -\lambda)\int_A\int_{M\backslash K} a^{-2\rho}\phi(a) w(\nu)\big(\pi_\nu(a)\phi_r\big)\mathcal{J}(-\overline{\lambda})\big(\pi_{-\overline{\lambda}}(au)\phi_r\big)\, da\, du$$

Proposition 6.1 implies that $w(\nu)(\pi_\nu(a)\phi_r) = \frac{1}{\Gamma(\nu+1)} a^{\nu+\rho} \sum_{k=0}^{\infty} c_{\nu,k}(\phi_r) a^{k\alpha}$, with $c_{\nu,0} = 1$ the series converging absolutely and uniformly on compacta of $\mathbf{C} \times \operatorname{sop}\phi$. Using (6.9) we have that the above equals

$$r_\eta(Q,-\lambda)\left[\int_A \phi(a)a^{-2\rho}\frac{a(-\lambda)}{\Gamma(\nu+1)\Gamma(-\lambda+1)}\sum_{k,l=0}^{\infty} c_{\nu,k}\; c_{-\lambda,l}\; a^{\nu-\lambda+2\rho+(k+l)\alpha}da\; + \right.$$

$$\left.\int_A \phi(a)a^{-2\rho}\frac{b(-\lambda)c_r(-\lambda)}{\Gamma(\nu+1)\Gamma(\lambda+1)}\sum_{k,l=0}^{\infty} c_{\nu,k}\; c_{\lambda,l}\; a^{\nu+\lambda+2\rho+(k+l)\alpha}da\right]$$

Interchanging summation and integration, we obtain for (ν,λ) such that $\operatorname{Re}(\nu+\lambda) > 0$, $\operatorname{Re}(\nu-\lambda) < 0$,

$$\frac{r_\eta(Q,-\lambda)}{\Gamma(\nu+1)}\left\{\gamma_\xi(\lambda)\sum_{k,l=0}^{\infty} c_{\nu,k}(\phi_r)\; c_{-\lambda,l}(\phi_r)\left(\int_T^{T+1}\phi(a_t)\; e^{t\left((\nu-\lambda)+2(k+l)\right)}dt\; + \right.\right.$$

$$\left.\int_{-\infty}^{T} e^{t\left((\nu-\lambda)+2(k+l)\right)}dt\right) + c_r(-\lambda)\sum_{k,l=0}^{\infty} c_{\nu,k}(\phi_r)\; c_{\lambda,l}(\phi_r)\times$$

$$\left.\left(\int_T^{T+1}\phi(a_t)\; e^{t\left((\nu+\lambda)+2(k+l)\right)}dt + \int_{-\infty}^{T} e^{t\left((\nu+\lambda)+2(k+l)\right)}dt\right)\right\} =$$

$$\frac{r_\eta(Q,-\lambda)}{\Gamma(\nu+1)}\left\{\gamma_\xi(\lambda)\sum_{k,l=0}^{\infty} c_{\nu,k}(\phi_r)\; c_{-\lambda,l}(\phi_r)\left(\frac{e^{T\left((\nu-\lambda)+2(k+l)\right)}}{(\nu-\lambda)+2(k+l)} + F_{k,l}(\nu,-\lambda)\right)\right.$$

$$\left. + \; c_r(-\lambda)\sum_{k,l=0}^{\infty} c_{\nu,k}(\phi_r)\; c_{\lambda,l}(\phi_r)\left(\frac{e^{T\left((\nu+\lambda)+2(k+l)\right)}}{(\nu+\lambda)+2(k+l)} + F_{k,l}(\nu,\lambda)\right)\right\} \tag{7.9}$$

where $F_{k,l}(\nu,\lambda)$ is a holomorphic function in $\mathbf{C}\times\mathbf{C}$.

We briefly sketch the proof of the last assertion. By approximating suitably the Eisenstein series by C^∞ compactly supported functions, one concludes that

$$d_Q(\nu,\lambda) = \frac{\lambda^2-\nu^2}{4}\langle\tilde{\mathbf{M}}(\nu,\phi_r,\chi), E(Q,-k\bar{\lambda},\phi_r,\chi)\rangle$$

In particular, if ν is not a pole of $\gamma_\xi(\lambda)$, $r_\eta(Q,-\lambda)$ nor of $c_r(-\lambda)$ we get, by the proof of (i),

$$\begin{aligned} d_Q(\nu,\nu) &= \lim_{\lambda\to\nu}\frac{\lambda^2-\nu^2}{4}\langle\tilde{\mathbf{M}}(\nu,\phi_r,\chi), E(Q,-k\bar{\lambda},\phi_r,\chi)\rangle \\ &= \lim_{\lambda\to\nu}\frac{\lambda^2-\nu^2}{4}\frac{r_\eta(Q,-\lambda)}{\Gamma(\nu+1)}\gamma_\xi(\lambda)\frac{e^{T\left((\nu-\lambda)\right)}}{(\nu-\lambda)} = -r_\eta(Q,-\nu)\,\frac{\nu}{2}\,\frac{\gamma_\xi(\nu)}{\Gamma(\nu+1)} \end{aligned} \tag{7.10}$$

Similarly, one computes $d_Q(\nu,-\nu) = -r_\eta(Q,\nu)\,\dfrac{\nu}{2}\,\dfrac{c_r(\nu)}{\Gamma(\nu+1)}$. We observe that if $r \in \mathbf{Z}$ is odd, then $d_Q(0,0) = 0$. The lemma now follows from (6.10).

Theorem 7.3. *(i)* $\tilde{\mathbf{M}}(\nu,g,\chi)$ *and* $\mathbf{M}(\nu,g,\chi)$ *can be meromorphically continued to* $\mathbf{C}$. *They define* C^∞ *functions on* $(\mathbf{C}-D)\times G$, *(*D *a discrete subset). Moreover,* $\tilde{\mathbf{M}}(\nu,g,\chi)$ *has a pole if and only if* $\mathbf{M}(\nu,g,\chi)$ *does, and the principal parts coincide. Finally,* $\tilde{\mathbf{M}}(\nu,g,\chi)$ *has moderate growth where defined and lies in* $L^{2-\alpha}_\infty(\Gamma\backslash G,\chi)$, $\forall\alpha>0$ *if* $\mathrm{Re}\ \nu\geq 0$ *(respectively in* $L^2_\infty(\Gamma\backslash G,\chi)$ *if* $\mathrm{Re}\,\nu>0$, $\mathrm{Im}\ \nu\neq 0$*) and* ν *is not a pole.*

(ii) If $\nu_0 \neq 0$ *is a pole of* $\mathbf{M}(\nu,g,\phi_r,\chi)$, $\mathrm{Re}\,\nu_0 \geq 0$, *then* ν_0 *is a simple pole and* $\mathrm{Res}_{\nu=\nu_0}\mathbf{M}(\nu,g,\phi_r,\chi)$ *is a square integrable automorphic form.*

Furthermore, if $\nu_0 = 0$ *is a pole of* $\mathbf{M}(\nu,g,\phi_r,\chi)$ *then it is of order at most 2, and* $\lim_{\nu\to 0}\nu^2\ \mathbf{M}(\nu,g,\phi_r,\chi)$ *is a square integrable automorphic form. If it is a simple pole, then*

$$\mathrm{Res}_{\nu=0}\,\mathbf{M}(\nu,g,\phi_r,\chi) - 2\,e^{i(\pi/2)r}\cos\frac{\pi r}{2}\sum_{j=1}^{s} m_j\,r_\eta(P_j,0))\ E(P_j,0,\nu,g,\phi_r,\chi)$$

is a square integrable automorphic form.
If ν_0 *is a pole of* $\mathbf{M}(\nu,g,\phi_r,\chi)$ *and* $\mathrm{Re}\,\nu_0<0$, *then*

$$\begin{aligned}&\mathrm{Res}_{\nu=\nu_0}\{\mathbf{M}(\nu,g,\chi)\\ &-\frac{(2\alpha)^{\frac{-\nu}{2}}\,e^{i(\pi/2)r}}{\nu}\,\frac{\cos\frac{\pi(\nu-r)}{2}}{\Gamma\left(\frac{1+\nu+r}{2}\right)}\sum_{i=1}^{s} m_i\left[r_\eta(P_i,-\nu)\left(\frac{\alpha}{2}\right)^{\frac{-\nu}{2}}\Gamma\left(\frac{1+\nu+r}{2}\right)E(P_i,\nu,g,\phi_r,\chi)+\right.\\ &\qquad\left.r_\eta(P_i,\nu)\left(\frac{\alpha}{2}\right)^{\frac{\nu}{2}}\Gamma\left(\frac{1-\nu+r}{2}\right)E(P_i,-\nu,g,\phi_r,\chi)\right]\Big\}\end{aligned}$$

is a square integrable automorphic form. Here m_i *(resp.* $r_\eta(P_i,\nu)$*) is as in (3.7) (resp. (7.7)).*

Proof. If $\mathrm{Re}\,\nu > 1$ then $\tilde{\mathbf{M}}(\nu,g,\chi) = \tilde{\mathbf{M}}_d(\nu,g,\chi) + \tilde{\mathbf{M}}_c(\nu,g,\chi)$, where $\tilde{\mathbf{M}}_d(\nu,g,\chi)\in L^2_d(\Gamma\backslash G,\chi)$, $\tilde{\mathbf{M}}_c(\nu,g,\chi)\in L^2_c(\Gamma\backslash G,\chi)$ (see Theorem 3.6).

The meromorphic continuation of $\tilde{\mathbf{M}}_d$ and the fact that $\tilde{\mathbf{M}}_d$ lies in $L^2_\infty(\Gamma\backslash G,\chi)$ are proved by the same argument as in [MW] 2.5. We will thus concentrate on $\tilde{\mathbf{M}}_c(\nu,g,\chi)$.

As usual, we fix a complete system of cuspidal parabolic subgroups, $\{P_1,\ldots,P_r\}$ where P_i is regular (resp. irregular) if $1\leq i\leq s$ (resp. $s+1\leq i\leq r$). If $Q=kPk^{-1}$ is a regular cuspidal parabolic, we will write $E(Q,\lambda,g,\phi,\chi)$ in place of $E(Q,k\lambda,g,L(k)\phi,\chi)$.

As seen in §3, there exist a meromorphic function on $\mathbf{C}^2$, $d_i(\nu,\lambda)$, and $c_i(\nu,\lambda)$ holomorphic for $(\nu,\lambda)\in\{\nu\mid\mathrm{Re}\,\nu>1\}\times i\mathbf{R}$ such that

$$\tilde{\mathbf{M}}_c(\nu,g) = \sum_{i=1}^{s}\frac{m_i}{2\pi i}\int_{\mathrm{Re}\ \lambda=0} d_i(\nu,\lambda)E(P_i,\lambda,g,\phi_r,\chi)\,d\lambda$$

$$\tilde{\mathbf{M}}_c(\nu,g) = \sum_{i=1}^{s}\frac{m_i}{2\pi i}\int_{\mathrm{Re}\ \lambda=0} c_i(\nu,\lambda)E(P_i,\lambda,g,\phi_r,\chi)\,d\lambda$$

Using that $\check{\mathbf{M}}(\nu, g, \chi) = (C - \lambda_\nu)\tilde{\mathbf{M}}(\nu, g, \chi)$ one arrives at $d_i(\nu, \lambda) = \frac{\lambda^2 - \nu^2}{4} c_i(\nu, \lambda)$, hence $c_i(\nu, \lambda)$ also admits a meromorphic continuation to $\mathbf{C} \times \mathbf{C}$. Thus, to continue $\tilde{\mathbf{M}}_c(\nu, g, \chi)$ it is sufficient to continue each term

$$I_i(\nu, g) = \int_{\mathrm{Re}\,\lambda=0} \frac{d_i(\nu, \lambda)}{\lambda^2 - \nu^2} E(P_i, \lambda, \phi_r, \chi)\, d\lambda \tag{7.11}$$

The holomorphy of $d_i(\nu, \lambda)$ on $\mathbf{C} \times i\mathbf{R}$ implies that $I_i(\nu)$ is holomorphic off the imaginary axis. We claim that $I_i(\nu)$ is holomorphic for $\nu \in i\mathbf{R}$, $\nu \neq 0$. Denote by $I_{i,r}(\nu)$ (resp. $I_{i,l}(\nu)$) the holomorphic function $I_i(\nu)$ restricted to the halfplane $\{\nu \mid \mathrm{Re}\,\nu > 0\}$ (resp. $\{\nu \mid \mathrm{Re}\,\nu < 0\}$. Let $\epsilon > 0$ be such that $E(P_i, \lambda, \phi_r, \chi)$ has no poles in $\{\lambda \mid |\mathrm{Re}\,\lambda| \leq \epsilon\}$. Given $a > 0$, we modify the contour of integration along the imaginary axis, substituting the segment $[-ia, ia]$ by the three sides of the rectangle with vertices in $\{-ia; \epsilon - ia; \epsilon + ia; ia\}$. Let Γ_ϵ be this contour. Let

$$I_{i,\epsilon}(\nu, g) = \int_{\Gamma_\epsilon} \frac{d_i(\nu, \lambda)}{\lambda^2 - \nu^2} E(P_i, \lambda, \phi_r, \chi)\, d\lambda \tag{7.12}$$

thus $I_{i,\epsilon}(\nu)$ is holomorphic away from the set $C_\epsilon = \Gamma_\epsilon \cup -\Gamma_\epsilon$.
Let R_ϵ be the rectangle with contour C_ϵ. Then, if $\nu \in R_\epsilon$, $\mathrm{Re}\,\nu > 0$

$$I_{i,\epsilon}(\nu) - I_{i,r}(\nu) = \frac{\pi i}{\nu} d_i(\nu, \nu) E(P_i, \nu, \phi_r, \chi) \tag{7.13}$$

Thus (7.13) gives a meromorphic continuation of $I_{i,r}(\nu)$ to $\{\nu \mid \mathrm{Re}\,\nu > -\epsilon\}$, which is holomorphic on $i\mathbf{R} - \{0\}$. If $\nu \in R_\epsilon$, $\mathrm{Re}\,\nu < 0$

$$I_{i,\epsilon}(\nu) - I_{i,l}(\nu) = -\frac{\pi i}{\nu} d_i(\nu, -\nu) E(P_i, -\nu, \phi_r, \chi) \tag{7.14}$$

Hence

$$I_{i,l}(\nu) - \frac{\pi i}{\nu}\{d_i(\nu, \nu) E(P_i, \nu, \phi_r, \chi) + d_i(\nu, -\nu) E(P_i, -\nu, \phi_r, \chi)\} \tag{7.15}$$

is a meromorphic function in the halfplane $\{\nu \mid \mathrm{Re}\,\nu < 0\}$ which coincides in the strip $\{-\epsilon < \mathrm{Re}\,\nu < 0\}$ with the meromorphic continuation of $I_{i,r}(\nu)$ given by (7.13). This implies that $\tilde{\mathbf{M}}_c(\nu, g)$ has a meromorphic continuation to $\mathbf{C}$, holomorphic in the closed halfplane $\{\mathrm{Re}\,\nu \geq 0\}$ with the exception of a possible simple pole at $\nu = 0$. Also, $\tilde{\mathbf{M}}_c(\nu, g, \chi)$ is of class C^∞ in (ν, g), where defined (see [OW, pp 113-118]). We note that the explicit formula for $d_i(\nu, \nu)$ and $d_i(\nu, -\nu)$ are given in (7.7) and (7.8).

We now prove that $\tilde{\mathbf{M}}(\nu, , \chi)$ is of moderate growth. We have seen that $\tilde{\mathbf{M}}_d(\nu) \in L^2_\infty(\Gamma\backslash G, \chi)$, $\forall \nu \in \mathbf{C} - Q(\phi_r)$. To estimate $\tilde{\mathbf{M}}_c(\nu, g)$ we analyze the functions $I_i(\nu, g)$ defined in (7.11). Let D be a compact subset in $\{\nu \mid \mathrm{Re}\ \nu > 0\}$. Then there exists a constant $C_D > 0$ such that $\left|\dfrac{d_i(\nu, \lambda)}{\lambda^2 - \nu^2}\right| \leq C_D |d_i(\nu, \lambda)|$ for all $\nu \in D$, $\lambda \in i\mathbf{R}$. Thus

$$I_i(\nu, g) = \int_{\mathrm{Re}\ \lambda=0} \frac{d_i(\nu, \lambda)}{\lambda^2 - \nu^2} E(P_i, \lambda, g, \chi)\, d\lambda$$

lies uniformly in $L^2_c(\Gamma\backslash G, \chi)$, for $\nu \in D$. Since $C^k\check{\mathbf{M}}_d(\nu, g)$ and $C^k\check{\mathbf{M}}_c(\nu, g)$ lie in $L^2(\Gamma\backslash G, \chi)$, for any $k \geq 0$, it follows that $I_i(\nu, g) \in L^2_\infty(\Gamma\backslash G, \chi)$.

Now suppose $\nu_0 \in i\mathbf{R} - \{0\}$. We must then estimate

$$I_{i,\epsilon}(\nu, g) - \frac{\pi i}{\nu} d_i(\nu, \nu) E(P_i, \nu, g, \phi_r, \chi) \tag{7.16}$$

We choose $a > |\nu_0|$ and we write $\Gamma_\epsilon = \{it \mid |t| \geq a\} \cup \gamma_\epsilon$. Then

$$I_{i,\epsilon}(\nu, g) = \int_{\substack{\mathrm{Re}\,\lambda=0 \\ |\lambda|\geq a}} \frac{d_i(\nu,\lambda)}{\lambda^2 - \nu^2} E(P_i, \lambda, \phi_r, \chi)\, d\lambda + \int_{\gamma_\epsilon} \frac{d_i(\nu,\lambda)}{\lambda^2 - \nu^2} E(P_i, \lambda, \phi_r, \chi)\, d\lambda$$

Let D be a closed disc centered in ν_0 with radius $\delta = \min\{\frac{\epsilon}{2}, \frac{|\nu_0|}{2}, \frac{a-|\nu_0|}{2}\}$. By an argument similar to that in the case when $\mathrm{Re}\,\nu > 0$ one shows that the first integral lies in $L^2_\infty(\Gamma\backslash G, \chi)$. On the other hand, the second integral (over γ_ϵ) is estimated in absolute value by $\sup\{\left|\frac{d_i(\nu,\lambda)}{\lambda^2-\nu^2}\right| a(g)^{1+\epsilon} \mid \lambda \in \gamma_\epsilon,\ \nu \in D\}$ for any g in a Siegel set $\mathcal{S}_i$, associated to the parabolic P_i. Since the second summand is bounded, in absolute value, on any Siegel set associated to P_j, $j \neq i$, $1 \leq j \leq r$, one easily concludes that this term is uniformly in $L^{2-\alpha}_\infty(\Gamma\backslash G, \chi)$, $\alpha = \alpha(\epsilon)$. Letting ϵ tend 0 we get that, for $\nu_0 \in i\mathbf{R} - \{0\}$, $I_i(\nu_0, g) \in L^{2-\alpha}_\infty(\Gamma\backslash G, \chi)$, for any $\alpha > 0$.

Now the term $\frac{\pi i}{\nu} d_i(\nu,\nu) E(P_i, \nu, g, \chi)$ is estimated in absolute value by $C_D\, a(g)^{1+\epsilon}$, $\forall \nu \in D$, and also its derivatives. It then follows that $\tilde{\mathbf{M}}(\nu, g) \in L^{2-\alpha}(\Gamma\backslash G, \chi)$, $\forall \alpha > 0$ if $\mathrm{Re}\,\nu \geq 0$, $\nu \neq 0$.

Finally, if $\mathrm{Re}\,\nu < 0$, we must estimate

$$I_{i,l}(\nu, g) - \frac{\pi i}{\nu}\{d_i(\nu,\nu) E(P_i, \nu, g, \chi) + d_i(\nu, -\nu) E(P_i, -\nu, g, \chi)\}$$

The first term is holomorphic and lies in $L^2_\infty(\Gamma\backslash G, \chi)$ by the above argument, while the second term has moderate growth where defined. The assertion is now proved since any function in $L^p_\infty(\Gamma\backslash G, \chi)$ has moderate growth if $1 \leq p \leq \infty$.

We now study the principal parts of $\tilde{\mathbf{M}}(\nu, g)$ at the poles. Assume first that $\nu_0 \neq 0$ is a pole and $\mathrm{Re}\,\nu_0 \geq 0$. We have seen that $\tilde{\mathbf{M}}(\nu, g)$ is holomorphic in ν_0 and ν_0 is a simple pole of $\tilde{\mathbf{M}}_d(\nu, g)$.

Let $\mu = \lambda_{\nu_0} = \frac{\nu_0^2-1}{4}$. Then $\tilde{\mathbf{M}}_d(\nu, g) = \sum_{\mu_j=\mu} \frac{d_j(\nu)}{\mu_j - \lambda_\nu} \psi_j(g)$, where $d_j(\nu) = \langle \tilde{\mathbf{M}}(\nu_j), \psi_j \rangle$, and $\{\psi_j\}_1^\infty$ is as in (3.7). As $(\nu - \nu_0)\tilde{\mathbf{M}}_c(\nu, g) \mapsto 0$ pointwise, when $\nu \to 0$, one has ([MW], Theorem 2.5) that

$$\mathrm{Res}_{\nu=\nu_0} \tilde{\mathbf{M}}(\nu, g) = \lim_{\nu\to 0} (\nu - \nu_0) \tilde{\mathbf{M}}_d(\nu, g) = -\sum_{\mu_j=\mu} \frac{d_j(\nu_0)}{2\nu_0} \psi_j(g)$$

Hence the residue at ν_0 is a square integrable automorphic form.

If $\nu_0 = 0$ is a pole, it is at most a double pole. It is a double pole only if $d_j(0) \neq 0$ for some j such that $\mu_j = \lambda_0$. In this case, by an argument similar to the above we get that

$$\lim_{\nu \longrightarrow 0} \nu^2 \tilde{\mathbf{M}}(\nu, g) = - \sum_{\mu_j = \lambda_0} d_j(0) \psi_j(g)$$

is a square integrable automorphic form. If $\nu_0 = 0$ is a simple pole, then

$$\mathrm{Res}_{\nu=0} \tilde{\mathbf{M}}(\nu, g, \chi) = \lim_{\nu \longrightarrow 0} \nu \tilde{\mathbf{M}}(\nu, g) = - \sum_{\mu_j = \lambda_0} d_j'(0) \psi_j(g) - \sum_{i=1}^{s} 2\, m_i\, d_i(0,0)\, E(P_i, 0, g, \chi)$$

Since $\mathbf{M}(\nu, g, \chi) - \tilde{\mathbf{M}}(\nu, g, \chi)$ is given locally by a finite sum of entire functions, $\mathbf{M}(\nu, g, \chi)$ has a meromorphic continuation to $\mathbf{C}$, with the same poles as $\tilde{\mathbf{M}}(\nu, g, \chi)$ and the same principal part at each pole.

Proposition 7.4. *Let Q be a regular-cuspidal parabolic subgroup, $Q = kPk^{-1}$, $k \in K$. Then there exist meromorphic functions, $D_\eta(Q, P, \nu)$ and $D(Q, P, \nu)$ such that*

$$\int_{\Gamma_{N_Q} \backslash N_Q} \mathbf{M}(\nu, n', \phi_r)\, dn' = D_\eta(Q, P, \nu) c_r(\nu) \cdot \phi_r(k) \phi_r$$

$$\int_{\Gamma_N \backslash N} E(Q, k \cdot \nu, n, L(k)\phi_r)\, dn = vol(\Gamma_N \backslash N) \phi_r + \xi(k(-\pi))\ D(Q, P, \nu) c_r(\nu) \phi_r$$

If $\mathrm{Re}\, \nu > 1$

$$D_\eta(Q, P, \nu) = \frac{1}{\Gamma(\nu + 1)} \sum_{\delta \in S(P, N_Q)} \chi(\delta^{-1}) \eta(n_1(\delta k)) \xi(m(\delta k)) a_{\delta k}^{\nu + \rho}$$

$$D(Q, P, \nu) = \sum_{\delta \in S(P, N_Q)} \chi(\delta) \xi^{-1}(m(\delta k)) a_{\delta k}^{\nu + \rho}$$

where $S(P, N_Q)$ is a set of representatives of $\Gamma_P \backslash (\Gamma - \Gamma_P)/\Gamma_{N_Q}$.

Proof. Similar to that of Proposition 2.7 in [MW].

Theorem 7.5. (Functional equation) *$\mathbf{M}(\nu)$ satisfies the functional equation*

$$a(\nu)\mathbf{M}(\nu, \phi_r, \chi) + b(\nu) c_r(\nu) \mathbf{M}(-\nu, \phi_r, \chi) = c_r(\nu) c_r(-\nu) \sum_{i=1}^{s} \frac{\mathcal{D}(P_i, P, -\nu)}{vol(\Gamma_{N_i} \backslash N_i)} E(P_i, k_i \nu, L(k_i) \phi_r, \chi) \quad (7.17)$$

where $P_1, P_2, \ldots, P_s$ is a complete system of representatives of regular cuspidal parabolic subgroups, $P_i = k_i P k_i^{-1}$, $k_i \in K$ $(i = 1 \ldots s)$ and $P_1 = P$. Furthermore $\mathcal{D}(P_i, P, \nu) =$

$\Gamma(-\nu+1)D_\eta(Q,P,\nu)$ *with* $D_\eta(Q,P,\nu)$ *as in Proposition 7.4 and* $a(\nu)$ *and* $b(\nu)$ *are as in (6.10).*

Proof. The left hand side of (7.17) equals

$$a(\nu)\tilde{\mathbf{M}}_T(\nu,g,\phi_r,\chi)+b(\nu)c_r(\nu)\tilde{\mathbf{M}}_T(-\nu,g,\phi_r,\chi)$$
$$+\sum_{\gamma\in\Gamma_P\backslash\Gamma}(1-\phi_T(\gamma g))\mathcal{J}(\nu)\big((\pi_\nu(\gamma g)\phi_r\big)\chi(\gamma^{-1}) \quad (7.18)$$

Let us consider its growth on a Siegel set associated to P. If $g\in A^+_{t_0}$ and t_0 is sufficiently large, then

$$\sum_{\gamma\in\Gamma_P\backslash\Gamma}(1-\phi_{t_0}(\gamma g))\mathcal{J}(\nu)\big(\pi_\nu(\gamma g)\phi_r\big)=\mathcal{J}(\nu)\big(\pi_\nu(a)\phi_r\big) \quad (7.19)$$

Now, $\tilde{\mathbf{M}}(\nu,\chi)$ has moderate growth and $\mathcal{J}(\nu)\big(\pi_\nu(a)\phi_r\big)$ decays exponentially as $a^\rho\to\infty$. Using reduction theory, we conclude that (7.18) is an automorphic form. We compute the constant term along each regular parabolic subgroup $P_i=k_iPk_i^{-1}$. Let Q be such a P_i, and write k in place of k_i.

By Proposition 7.4, the constant term along Q is given by

$$\phi_r(k)\left\{a(\nu)D_\eta(Q,P,\nu)A(\xi,\nu)\phi_r+b(\nu)c_r(\nu)D_\eta(Q,P,-\nu)A(\xi,-\nu)\phi_r\right\}$$

where $\gamma_\xi(\nu)$ is as in (6.11). Let $a'\in A_Q$, then, applying (7.18) to $\pi_{\xi,\nu}(a')\phi_r$ we get

$$\phi_r(k)\left\{a(\nu)D_\eta(Q,P,\nu)(a')^{-k\nu+k\rho}c_r(-\nu)\phi_r+\right.$$
$$\left.b(\nu)c_r(\nu)D_\eta(Q,P,-\nu)(a')^{k\nu+k\rho)}c_r(-\nu)\phi_r\right\}$$

Now, the coefficient of $(a')^{k\cdot\nu+k\rho}$ of the constant term along Q of $E(Q,k\cdot\nu,a',L(k)\phi_r,\chi)$ is $vol(\Gamma_{N_Q}\backslash N_Q)$. Therefore, if we subtract

$$c_r(\nu)c_r(-\nu)\frac{b(\nu)D_\eta(Q,P,-\nu)}{vol(\Gamma_{N_Q}\backslash N_Q)}E(Q,k\cdot\nu,g,\phi_r,\chi)$$

from the left hand side of (7.17), the constant term of the resulting function will only involve $(a')^{-k\cdot\nu+k\rho}$.

It is now clear that if the second term of equation (7.17) is subtracted from the left hand side, we obtain an eigenfunction of moderate growth with eigenvalue $\frac{\nu^2-1}{4}$ for each $\nu\in U$, U an open set in $\{\nu\mid\mathrm{Re}\,\nu>1\}$. Now, since $\mathcal{C}$ has real eigenvalues in $L^2(\Gamma\backslash G,\chi)$, this difference must be identically zero in U, and by analytic continuation, in all of $\mathbf{C}$.

Writing explicitly $a(\nu)$, $b(\nu)$ and $c_r(\nu)$ the functional equation reads

$$\nu(2\alpha)^{\frac{\nu}{2}}\Gamma\left(\frac{1+\nu-r}{2}\right)\mathbf{M}(\nu,\phi_r,\chi)-\nu(2\alpha)^{\frac{-\nu}{2}}\Gamma\left(\frac{1-\nu-r}{2}\right)\mathbf{M}(-\nu,\phi_r,\chi)=$$
$$\left(\frac{2}{\alpha}\right)^{\frac{\nu}{2}}\frac{2\pi i^r}{\Gamma\left(\frac{1-\nu+r}{2}\right)}\sum_{i=1}^{s}\frac{\mathcal{D}(P_i,P,-\nu)}{vol(\Gamma_{N_i}\backslash N_i)}E(P_i,k_i\nu,L(k_i)\phi_r,\chi) \quad (7.20)$$

where $\eta(n(x))=e^{i\alpha x}$ is the character on N.

Remarks.
The family $\mathbf{M}(\nu, g, \phi_r, \chi)$ yields generically eigenfunctions of C on $C^\infty(\Gamma\backslash G, \chi)$ which are of exponential growth. However these are useful in the study of automorphic forms since, on the one hand, as we have seen, the residues in the closed right half plane do define automorphic forms (see also [Ne], [MW]). On the other hand one also get automorphic forms by taking certain special values of $\mathbf{M}(\nu, g, \chi)$. Actually one can verify that if $r \geq 2$, $\mathbf{M}(r-1, g, \phi_r, \chi)$ corresponds precisely to the classical Poincaré series of real weight r and multiplier system v_χ, in the notation of §2 (c.f. [Rn]).

The above family has also been studied by a different approach in [He], [Br], [Br2] and, previously, in the particular case when $r = 0$, in [Ne].

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