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PAYOFF MATRICES IN COMPLETELY MIXED BIMATRIX GAMES WITH ZERO–VALUE

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ABSTRACT. A completely mixed bimatrix game (A, B) has a unique equilibrium strategy. The values of this game for each player, are defined by $v_1 = x^T Ay$ and $v_2 = x^T By$ where (x, y) is an equilibrium strategy. We give a formula for computing the completely mixed equilibrium strategy when the bimatrix game has zero–value.

1. INTRODUCTION

For the zero-sum two-person games Kaplansky (1945) introduced the notion of completely mixed strategies and showed that in games where both players have only completely mixed optimal strategies, the payoff matrix is square and each player has a unique optimal strategy. Raghavan (1970) extended this result to the non-zero-sum bimatrix games. Also Kaplansky (1945) gave a necessary and sufficient condition on the payoff matrix for a game of value zero to be completely mixed. He showed that if the value of a game is different from zero, then the payoff matrix is nonsingular and gave a formula for computing this value. Jansen $(1981a, b)$ showed that in completely mixed bimatrix games with $A > 0$ and $B < 0$, the matrices A and B are nonsingular. He also extended the formulas for computing equilibrium strategies and the values for completely mixed bimatrix games.

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Completely mixed bimatrix games have unique equilibrium strategies. The value of these games are defined to be the payoffs that the player receive when they play equilibrium strategies. In this paper we try to see how far the results can be extended to bimatrix games with zero value.

2. GENERAL RESULTS

A bimatrix game with m pure strategies for player 1 and n pure strategies for player 2, where $1 \leq m, n < \infty$, is specified by two real $m \times n$ matrices A and B. If player 1 chooses pure strategy i and player 2 chooses pure strategy j, the payoffs to players 1 and 2 are $a_{i,j}$ and $b_{i,j}$ respectively, for $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Let

$$P_n = \left\{ x \in \Re^n : x_i \geq 0, i = 1, \ldots, n, \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}$$

and $P_n^+ = \{x \in P_n : x_i > 0, i = 1, \ldots, n\}$. Vectors are assumed to be column vectors, and ^{T} denotes transpose. The vectors in P_n are called mixed strategies and denoted by $x \geq \mathbf{0}$ where $\mathbf{0} = (0, \ldots, 0)$. The vectors in P_n^+ are called completely mixed strategies and denoted by $x > \mathbf{0}$. A pair (x, y), where $x \in P_m$ and $y \in P_n$ is defined to be a equilibrium strategy of the game specified by (A, B) if

$$x^T A y \geq \xi^T A y \qquad \text{for all } \xi \in P_m$$

$$x^T B y \geq x^T B \eta \qquad \text{for all } \eta \in P_n$$

Nash (1950) proved that this equilibrium strategy exists. Let $\mathcal{E}$ be the set of all pairs of equilibrium strategies. We say that $\mathcal{E}$ is completely mixed if the elements of $\mathcal{E}$ are completely mixed pairs. Let $(x, y) \in \mathcal{E}$ be $v(x, y, A) = x^T A y$ and $v(x, y, B) = x^T B y$ are called equilibrium values of the bimatrix game (A, B).

Let

$$S(y) = \{x \in P_m : (x, y) \in \mathcal{E}\}$$

$$T(x) = \{y \in P_n : (x, y) \in \mathcal{E}\}.$$

We say that $S(y)$ is completely mixed if all elements of $S(y)$ are in P_m^+. A similar definition applies for $T(x)$.

Theorem 1 If the set $\mathcal{E}$ is completely mixed and $v(x, y, A) = v(x, y, B) = 0$ then

i. A and B are square matrices and $rank(A) = rank(B) = n - 1$

ii. $A_{i,j}$, $B_{i,j}$ denotes the cofactor of $a_{i,j}$ and $b_{i,j}$. Then there exists an i with $1 \leq i \leq m$ such that $A_{i,1}, \ldots, A_{i,n}$ are different from zero and have the same sign. There exists a j with $1 \leq j \leq n$ such that $B_{1,j}, \ldots, B_{n,j}$ are different from zero and have the same sign.

iii. $\sum_{i,j} A_{i,j} \neq 0$, and $\sum_{i,j} B_{i,j} \neq 0$.

Proof. The necessity of (i) is an immediate corollary to Theorem 1 and 4 from the paper of Raghavan (1970).

Let (x, y) be completely mixed equilibrium strategy. Let $A_{i,j}$ be the cofactor of $a_{i,j}$. Since $x > \mathbf{0}$ and $(x, y) \in \mathcal{E}$ implies that

$$Ay = \mathbf{0}.$$

Then

$$\frac{y_1}{A_{i,1}} = \frac{y_2}{A_{i,2}} = \ldots = \frac{y_n}{A_{i,n}}. \tag{1}$$

Since $rank(A) = n - 1$ then there exists $\bar{\imath}, \bar{\jmath}$ such that $A_{\bar{\imath},\bar{\jmath}}$ is different from zero. As y is a completely mixed strategy this implies in (1) that for $i = \bar{\imath}$, and for all j, $A_{\bar{\imath},j}$ have the same sign. A similar remark applies to B. Hence the necessity of (ii) is proven.

Since $rank(B) = n - 1$ then $rank(cof(B)) = 1$ where $cof(B)$ is the matrix in which the (i, j) elements are the cofactors for $b_{i,j}$. Without loss of generality we assume that the cofactor of $b_{n,n}$, $B_{n,n} \neq 0$, then

$$B = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n-1} & \sum_{j=1}^{n-1} t_j b_{1,j} \\ \vdots & & \vdots & \vdots \\ b_{n-1,1} & \cdots & b_{n-1,n-1} & \sum_{j=1}^{n-1} t_j b_{n-1,j} \\ \sum_{i=1}^{n-1}(-\lambda_i) b_{i,1} & \cdots & \sum_{i=1}^{n-1}(-\lambda_i) b_{i,n-1} & \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-\lambda_i) t_j b_{i,j} \end{bmatrix}$$

and

$$cof(B) = \begin{bmatrix} -t_1\lambda_1 B_{n,n} & \cdots & -t_{n-1}\lambda_1 B_{n,n} & \lambda_1 B_{n,n} \\ \vdots & & \vdots & \vdots \\ -t_1\lambda_{n-1} B_{n,n} & \cdots & -t_{n-1}\lambda_{n-1} B_{n,n} & \lambda_{n-1} B_{n,n} \\ -t_1 B_{n,n} & \cdots & -t_{n-1} B_{n,n} & B_{n,n} \end{bmatrix}$$

where $\lambda_i = x_i/x_n$. If $\sum_{i,j} B_{i,j} = 0$, implies that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} B_{i,j} = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i(-t_j)B_{n,n} = B_{n,n}\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{n}(-t_j) = 0$$

where $t_n = -1$, then

$$\sum_{j=1}^{n}(-t_j) = 0 \qquad \text{or} \qquad \sum_{j=1}^{n-1}(-t_j) = 1.$$

Since the system

$$I : \begin{cases} w^T B & = \mathbf{0}^T \\ \quad w & \geq \mathbf{0} \end{cases}$$

has a solution x, we 'll show that the system

$$II : \begin{cases} v^T B & = \mathbf{1}^T \\ \quad v & \geq \mathbf{0} \end{cases}$$

(where $\mathbf{1}$ is the column–vector of length n with every element equal to 1) has a solution. In fact, if the system (II) has not a solution, by Alternative Theorems (see Mangasarian book, page 34, Table 2.4.1) then (by Theorem (6) Farkas) the system

$$II^{(1)} : \begin{cases} Bz & \leq \mathbf{0} \\ \mathbf{1}^T z & > 0 \end{cases}$$

has a solution $\bar{z}$. Therefore the system

$$II^{(2)} : \begin{cases} Bs & \geq \mathbf{0} \\ \mathbf{1}^T s & < 0 \end{cases}$$

has a solution $\bar{s} = -\bar{z}$.

It suffices to analyze three different case:

$a)$ If $\bar{s}$ fulfills that

$$B\bar{s} > \mathbf{0}$$

then (by Theorem 5 (Gordan)) the system (I) has not any solution. This is a contradiction.

$b)$ If $\bar{s}$ fulfills that

$$B\bar{s} = \mathbf{0}$$

since $rank(B) = n-1$ and $t^T = (t_1, \ldots, t_{n-1}, -1)$ fulfills that $Bt = \mathbf{0}$ and

$$\sum_{j=1}^{n} t_j = 0$$

then $t = c\bar{s}$, but

$$0 = \sum_{j=1}^{n} t_j = c\sum_{j=1}^{n} \bar{s}_j \neq 0.$$

This is a contradiction.

$c)$ From $a)$ and $b)$ there exists i_1 and i_2 such that

$$\sum_j b_{i_1,j}\bar{s}_j > 0 \qquad \text{and} \qquad \sum_j b_{i_2,j}\bar{s}_j = 0$$

Let

$$I_1 = \{i : \sum_j b_{i,j}\bar{s}_j > 0\} \qquad \text{and} \qquad I_2 = \{i : \sum_j b_{i,j}\bar{s}_j = 0\}$$

We denote by $B_{I_1}(B_{I_2})$ the submatrix of B formed by the row $i \in I_1 (i \in I_2)$. Then $\bar{s}$ is a solution of system

$$II^{(3)} : \begin{cases} B_{I_1}s & = \mathbf{0} \\ B_{I_2}s & > 0 \end{cases}$$

but (by Theorem 2 (Motzkim)) the systems

$$II^{(4)} : \begin{cases} v_{I_1}^T B_{I_1} + v_{I_2}^T B_{I_2} & = \mathbf{0} \\ v_{I_2} & \geq 0 \ (v_{I_2} \neq \mathbf{0}) \end{cases}$$

has not any solution. This is a contradiction since $\bar{v}_{I_1} = x_{I_1}, \bar{v}_{I_2} = x_{I_1}$ where $x_{I_1} = \{x_i : i \in I_1\}$ and $x_{I_2} = \{x_i : i \in I_2\})$ is a solution of this system.

From $a), b)$ and $c)$ the system (II) has a solution. Let $\bar{v}$ a solution of systems (II), and $\tilde{v} = \bar{v}/\sum_i \bar{v}_i$. It is clear that $\tilde{v} \in P_m$ and is different from x.

Finally, we are showing that $(\tilde{v}, y) \in \mathcal{E}$.

$$\tilde{v}^T Ay \geq \xi^T Ay \qquad \text{for all } \xi \in P_m$$

it holds true because $Ay = \mathbf{0}$.

$$\tilde{v}^T By \geq \tilde{v}^T B\eta \qquad \text{for all } \eta \in P_n$$

it holds true because $\bar{v}^T B = \mathbf{1}^T$.

Thus $(x, y), (\tilde{v}, y) \in \mathcal{E}$. This contradicts that a completely mixed bimatrix game has a unique equilibrium strategy. Hence $\sum_{i,j} B_{i,j} \neq 0$. □

Corollary 1 If the set $\mathcal{E}$ is completely mixed and $(x, y) \in \mathcal{E}$ then

$$v(x, y, A) = \frac{\det A}{\sum_{i,j} A_{i,j}} \qquad \textit{and } v(x, y, B) = \frac{\det B}{\sum_{i,j} B_{i,j}}$$

the denominator is always different from zero.

Proof. By Theorem 4 of Raghavan (1970) we easily see that the pair (x, y) is a unique equilibrium strategy. Let

$$v(x, y, A) = v_1 \qquad \text{and } v(x, y, B) = v_2$$

then the game (C, D) given by

$$c_{i,j} = a_{i,j} - v_1 \qquad \text{and } d_{i,j} = b_{i,j} - v_2$$

is completely mixed, (x, y) is equilibrium strategy and

$$v(x, y, C) = 0 \qquad \text{and } v(x, y, D) = 0$$

In particular by Theorem 1 $\det(C) = \det(D) = 0$.

$$\det(C) = \det(A) - v_1 \sum_{i,j} A_{i,j} \qquad \text{and } \det(D) = \det(B) - v_2 \sum_{i,j} B_{i,j}$$

By Theorem 1 $\sum_{i,j} C_{i,j} \neq 0$ and $\sum_{i,j} D_{i,j} \neq 0$. The case $v_1 = 0$ or $v_2 = 0$ obviously need not be considered. Hence we have $\det(A) \neq 0$ and $\det(B) \neq 0$. □

Proposition 1 *If for the bimatrix game (A, B) there exist v_1, v_2 such that, for any $(x, y) \in \mathcal{E}$,*

$$Ay = v_1 \mathbf{1} \qquad \textit{and } x^T B = v_2 \mathbf{1}^T$$

and if (i) and (ii) of Theorem 1 hold true, then $\mathcal{E}$ is completely mixed.

Proof. By hiphotesis A is square, $rank(A) = n-1$ and there exists i such that $(A_{i,1}, \ldots, A_{i,n})$ have the same sign. Then the vector $\bar{y}$ ($\bar{y}_j = A_{i,j} / \sum_k A_{i,k}$) belongs to P_n^+. Similarly we choose $\bar{x} \in P_n^+$ ($\bar{x}_i = B_{i,j} / \sum_k B_{i,k}$). It is clear that $(\bar{x}, \bar{y}) \in \mathcal{E}$ and $v(\bar{x}, \bar{y}, A) = v(\bar{x}, \bar{y}, B) = 0$. Since $\bar{x} > \mathbf{0}$, it follows that if $y^* \in T(\bar{x})$ then $Ay^* = \mathbf{0}$. But $rank(A) = n-1$ assures us that $y^* = \bar{y}$ and $y^* > \mathbf{0}$. Let $(x, y) \in \mathcal{E}$, then there exists v_1, v_2 such that

$$Ay = v_1 \mathbf{1} \text{ and } x^T B = v_2 \mathbf{1}^T.$$

Thus $(\bar{x}, y), (x, \bar{y}) \in \mathcal{E}$. By the argument above we have $x = \bar{x}, y = \bar{y}$ and $x > \mathbf{0}, y > \mathbf{0}$. □

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