APPROXIMATION AND INTERPOLATION OF FUNCTIONS 
OF HYPERBOLIC COMPLEX VARIABLE

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ABSTRACT. We develop a theory of approximation and interpolation by polynomials of the functions of hyperbolic complex variable. In the class of the so-called pseudoholomorphic functions, a Jackson-type estimate is obtained and a result on mean convergence of Lagrange interpolation is proved. Then, estimates in approximation of continuous functions by areolar polynomials of Bernstein-type and of DeVore-Szabados-type are given.

1. INTRODUCTION

Let \( \alpha, \beta \in \mathbb{R} \) be fixed and \( q \) a solution of the equation \( q^2 = \alpha q + \beta \). An important result (see e.g. [11]) states that the algebraic ring \( C_q = \{ z = a + q b; a, b \in \mathbb{R} \} \) is ring-isomorphic with one of the following three:

(i) \( C_q, q^2 = -1 \), called the field of complex numbers, if \( \alpha^2 / 4 + \beta < 0 \);

(ii) \( C_q, q^2 = 0 \), called the ring of dual complex numbers, if \( \alpha^2 / 4 + \beta = 0 \);

(iii) \( C_q, q^2 = +1 \), if \( \alpha^2 / 4 + \beta > 0 \). A number in \( C_q, q^2 = +1 \) is called binary [11], or double [32], or perplex [10], or anormal-complex [2], or hyperbolic complex [6-8],[24]. Throughout in this paper we will use the term of hyperbolic complex number.

Suggested by the classical complex analysis, between 1935-1941 a theory of the functions of hyperbolic complex and dual complex variables was deeply investigated in e.g. [4-8],[24-31] and seems to have some applications in theoretical physics, as was pointed out in the recent papers [10],[13-14].

For all that, the theory of approximation of functions of hyperbolic complex variable by polynomials contains a single result obtained in 1936 in [8]. Because of this reasons, the purpose of the present paper is to give other contributions to this field of investigations. The main idea that can be derived is that, in contrast with what happens in the classical complex analysis, the properties are consequences of known results in the real approximation theory.
Section 2 contains some preliminaries. In Section 3 we firstly give an estimate for the approximation result in [8] and then, by using the Szabados’s polynomials in [20], we obtain a Jackson-type result.

Also, we consider the problem of approximation by some particular classes of areolar polynomials.

In Section 4 we deal with the interpolation of functions of hyperbolic complex variable.

2. PRELIMINARIES

We will consider some known concepts and results in [6-7] which will be used in the next sections.

Let $C_q$ be with $q^2 = +1$.

**DEFINITION 2.1.** The hyperbolic complex numbers $q_1 = (1+q)/2$, $q_2 = (1-q)/2$ are called isotropic units. If $z = x + qy \in C_q$ then $z = z'q_1 + z'q_2$ with $z' = x + y, z'' = x - y$ is called the isotropic form of $z$.

If $z = x + qy \in C_q$ then $\bar{z} = x - qy$ is the conjugate of $z$, $|z| = (x^2 + y^2)^{1/2}$ is the modulus of $z$ and $N_q(z) = z \cdot \bar{z} = x^2 - y^2$ is the hyperbolic norm of $z$.

If $z_n, z \in C_q, n \in N$ we say that $z_n \to z$ if $|z_n - z| \to 0$.

The number $z = x + qy = z'q_1 + z'q_2 \in C_q$ represents a point $(x, y)$ in the system of the axes XOY and on the other hand, a point $M_{iso}(z', z'')$ in the so-called isotropic system of coordinates composed by the first and second bisectrix and obtained from XOY by a rotation with $\pi / 4$ in trigonometric sense.

A rectangle having the sides parallel with the first and second bisectrix will be called isotropic rectangle.

If $a = a_1q_1 + a_2q_2, b = b_1q_1 + b_2q_2 \in C_q$ satisfy $N_q(b - a) \neq 0$, then by $R(a, b)$ will be denoted the isotropic rectangle having $M_{iso}^{(1)}(a_1, a_2), M_{iso}^{(2)}(a_1, a_2)$ (i.e. in the isotropic system of coordinates) as opposite sharp points. Let suppose, for example, that $a_1 < b_1$ and $a_2 < b_2$.

In this case, $R(a, b) = \{M_{iso}(z', z''); a_1 \leq z' \leq b_1, a_2 \leq z'' \leq b_2\}$.

Let us denote $R^*(a, b) = \{z = z'q_1 + z'q_2 \in C_q; M_{iso}(z', z'') \in R(a, b)\}$.

If $\alpha = A_1q_1 + A_2q_2, \beta = B_1q_1 + B_2q_2 \in C_q$ then

$\alpha + \beta = (A_1 + B_1)q_1 + (A_2 + B_2)q_2, \quad \alpha \cdot \beta = (A_1B_1)q_1 + (A_2B_2)q_2, \quad \alpha^* = A_1q_1 + A_2q_2,$

$\alpha / \beta = (A_1 / B_1)q_1 + (A_2 / B_2)q_2, \text{ for } B_1 \cdot B_2 \neq 0.$

If $a \in R$ then $a = a_1q_1 + a_2q_2$ and if $z = x + qy = Aq_1 + Bq_2$ then $\bar{z} = Bq_1 + Aq_2$.

The number $z = x + qy = Aq_1 + Bq_2 \in C_q$ is divisor of zero iff $N_q(z) = 0$ (or equivalently $A \cdot B = 0$). Also we have $|z| = [(A^2 + B^2) / 2]^{1/2} \leq |A| + |B| / \sqrt{2}.$

If $z_n = x_n + qy_n = A_nq_1 + B_nq_2, n \in N, z = x + qy = Aq_1 + Bq_2$ , then $z_n \to z$ iff $x_n \to x, y_n \to y$ or iff $A_n \to A, B_n \to B$. 
3. UNIFORM APPROXIMATION

Let \( \Omega \subset C_q, q^2 = +1, f: \Omega \to C_q \). We can write \( f(z) = u(x, y) + q \cdot v(x, y) \),
\( z = x + qy \in \Omega \) or the isotropic form \( f(z) = P_1(z_1, z_2) q_1 + P_2(z_1, z_2) q_2 \), \( z = z_1 q_1 + z_2 q_2 \in \Omega \).

The definition of the continuity of \( f \) at \( z_0 = x_0 + qy_0 = z'_0 q_1 + z''_0 q_2 \) being the same as in the classical complex analysis, can be proved that \( f \) is continuous in \( z_0 \) iff \( u(x, y), v(x, y) \) are continuous in \( (x_0, y_0) \) or iff \( P_1(z_1, z_2), P_2(z_1, z_2) \) are continuous in \( (z'_0, z''_0) \) (see [7]).

An important subclass consists in the so-called pseudoholomorphic functions, i.e. of the form \( f(z) = f_1(z') q_1 + f_2(z') q_2 \), \( z = z'_1 q_1 + z''_2 q_2 \), with \( f_1, f_2 \) continuous functions on their domains of definition.

As was proved in [7], the natural domains of definition for such functions \( f \) are of the form \( R^*(a, b) \).

The following result of approximation was proved in [8].

**THEOREM 3.1.** If \( f: R^*(0,1) \to C_q \) is pseudoholomorphic on \( R^*(0,1) \),
\( (0 = 0 \cdot q_1 + 0 \cdot q_2, 1 = 1 \cdot q_1 + 1 \cdot q_2) \) then the Bernstein polynomials

\[
B_n(f)(z) = \sum_{k=0}^{\infty} \binom{n}{k} f(k/n) (1-z/n)^k \text{ converge uniformly (when } n \to \infty) \text{ to } f(z) \text{ on } R^*(0,1).
\]

**REMARK.** In the system of the axes of coordinates \( XOY \), \( R(0,1) \) represents a quadrangle having the side equal with \( 1 \) and \( (0,0), (0,1) \) as opposite sharp points, while \( R^*(0,1) = \{z = z'_1 q_1 + z''_2 q_2 ; 0 \leq z' \leq 1, 0 \leq z'' \leq 1\} \) has as geometric image a quadrangle of side equal with \( \sqrt{2}/2 \) and with \( (0,0), (0,1) \) as opposite sharp points.

The proof of Theorem 3.1. is based on the following relation:

\[
(1) \quad f(a_k) z^k (1-z)^{n-k} = [f_1(a_k) q_1 + f_2(a_k) q_2] \left[ z^k q_1 + z^{n-k} q_2 \right] \left[ (1-z^k q_1 + (1-z)^{n-k} q_2 \right] = f_1(a_k) z^k (1-z)^{n-k} q_1 + f_2(a_k) z^k (1-z)^{n-k} q_2,
\]

\( z = z'_1 q_1 + z''_2 q_2 \), \( a_k \in R \).

We will give an estimate in the above theorem by using the following.

**LEMMA 3.2.** Let \( f(z) = P_1(z', z'') q_1 + P_2(z', z'') q_2 \), \( z = z'_1 q_1 + z''_2 q_2 \), \( z \in R^*(0,1) \), \( P_1, P_2 \) continuous for \( z \in R^*, \delta > 0 \). Denoting
\( \omega(f, \delta) = \sup \| f(z_1) - f(z_2) \|, z_1, z_2 \in R^*(0,1), |z_1 - z_2| \leq \delta \),
\( \omega(P_k, \delta, \delta) = \sup \| P_k(z'_1, z''_1) - P_k(z'_2, z''_2) \|, z'_1, z''_1, z'_2, z''_2 \in [0,1], k = 1,2 \),
the following inequality:

\[
\omega(f; \delta) + \omega(P_1; \delta, \delta) + \omega(P_2; \delta, \delta) \leq 2\sqrt{2} \omega(f; \delta), \quad \delta > 0
\]

holds.

**PROOF.** We have:

\[
|f(z_1) - f(z_2)| = \left| \left[ P_1(z'_1, z''_1) - P_1(z'_2, z''_2) \right] q_1 + \left[ P_2(z'_1, z''_1) - P_2(z'_2, z''_2) \right] q_2 \right|
\]

\[
= \left[ \left( P_1(z'_1, z''_1) - P_1(z'_2, z''_2) \right)^2 + \left( P_2(z'_1, z''_1) - P_2(z'_2, z''_2) \right)^2 \right]^{1/2}.
\]

From here it follows

\[
|f(z_1) - f(z_2)| \geq \left| P_1(z'_1, z''_1) - P_1(z'_2, z''_2) \right| \sqrt{2}, \quad |f(z_1) - f(z_2)| \geq \left| P_2(z'_1, z''_1) - P_2(z'_2, z''_2) \right| \sqrt{2}.
\]

Passing to supremum with \( |z_1 - z_2| \leq \delta, z_1, z_2 \in R^*(0,1) \), we get

\[
\omega(f; \delta) \geq \left( \sqrt{2} \right) \sup \left\{ |P_1(z'_1, z''_1) - P_1(z'_2, z''_2)|, |z_1 - z_2| \leq \delta, z_1, z_2 \in R^*(0,1) \right\}
\]
\[
= (1/\sqrt{2}) \sup \left\{ \left| P_k(z', z^r) - P_k(z^r, z') \right| \sqrt{(z^r - z')^2 + (z^r - z')^2} \leq \delta \sqrt{2}, z', z^r, z^r, z^r, z^r \in [0,1] \right\} \geq
\]
\[
\geq (1/\sqrt{2}) \sup \left\{ \left| P_k(z', z^r) - P_k(z^r, z') \right| \left| z^r - z' \right| \leq \delta, \right. \left. \left| z^r - z' \right| \leq \delta, z', z^r, z^r, z^r, z^r \in [0,1] \right\} =
\]
\[
= (1/\sqrt{2}) \omega (P_k; \delta, \delta), k = \frac{1}{2},
\]
which implies
\[
\omega (P_k; \delta, \delta) \leq 2\sqrt{2} \omega (f; \delta)
\]
and the lemma is proved.

**REMARK.** If \( f(z) = f_1(z') q_1 + f_2(z') q_2 \) is pseudoholomorphic on \( R^*(0,1) \) then entirely analogous we get
\[
\omega (f_1; \delta) + \omega (f_2; \delta) \leq 2\sqrt{2} \omega (f; \delta), \delta > 0.
\]

**THEOREM 3.3.** If \( f(z) = f_1(z') q_1 + f_2(z') q_2, \quad z = z' q_1 + z'' q_2 \) is pseudoholomorphic on \( R^*(0,1) \) then
\[
B_n (f)(z) - f(z) \leq 2k_0 \omega (f; 1/\sqrt{n}), \quad n \in \mathbb{N}, \quad z \in R^*(0,1), \text{ where } k_0 \text{ represent the Sikkema's constant in [18].}
\]

**PROOF.** By (1) (see [8, p.205, Theorem 1]) we have
\[
B_n (f)(z) - f(z) = \left[ B_n (f_1)(z') - f_1(z') \right] q_1 + \left[ B_n (f_2)(z'') - f_2(z'') \right] q_2,
\]
which by the Remark of Lemma 3.2. and by [18] implies
\[
\left| z_k (z^r) - f(z^r) \right| \leq \frac{k_0}{\sqrt{2}} \omega (f; 1/\sqrt{n}) + \omega (f; 1/\sqrt{n}) \leq 2k_0 \omega (f; 1/\sqrt{n}), n \in \mathbb{N}, \quad z \in R^*(0,1), \text{ and the theorem is proved.}
\]

**Theorem 3.3.** can be improved by the following result of Jackson-type.

**THEOREM 3.4.** If \( f(z) = f_1(z') q_1 + f_2(z') q_2, \quad z = z' q_1 + z'' q_2 \in R^*(0,1) \) is pseudoholomorphic on \( R^*(0,1) \), then there exists a sequence of polynomials \( \left( P_n (f)(z) \right) \) of degree \( P_n \leq n \), such that
\[
\left| f(z) - P_n (f)(z) \right| \leq C \cdot \omega (f; 1/\sqrt{n}), \forall n \in \mathbb{N}, \quad z \in R^*(0,1),
\]
where \( C > 0 \) is independent of \( n \) and \( f \).

**PROOF.** By [20], for any \( g \in C[0,1] = \{ g : [0,1] \to \mathbb{R}; g \text{ continuous on } [0,1] \} \), there exists a polynomial sequence of the form
\[
(2) \quad P_n (g)(x) = \sum_{k=0}^{\infty} g(k/n) \cdot s_k, x \in \mathbb{R}
\]
where \( s_k \) are polynomials of degree \( \leq n \) with real coefficients, independent of \( g \), such that
\[
\left| g(x) - P_n (g)(x) \right| \leq C \omega (g; 1/n), \quad x \in [0,1], \quad n \in \mathbb{N}
\]
with \( C > 0 \) an absolute constant.

Applying this last estimate, we get
\[
\left| f_1(z^r) - \sum_{k=0}^{\infty} f_1(k/n) \cdot s_k (z^r) \right| \leq C \omega (f_1; 1/n), \quad z^r \in [0,1], \quad n \in \mathbb{N}
\]
\[
\left| f_2(z^r) - \sum_{k=0}^{\infty} f_2(k/n) \cdot s_k (z^r) \right| \leq C \omega (f_2; 1/n), \quad z^r \in [0,1], \quad n \in \mathbb{N}.
\]
Let us denote \( P_n(f)(z) = \sum_{k=0}^{n} f(k/n) \cdot s_{k,n}(z) \). Since
\[
 a_p \cdot z^n = (a_1, a_2, \ldots, a_p) \cdot \left((z)^n q_1 + (z)^n q_2\right) = a_p (z)^n q_1 + a_p (z)^n q_2, \quad (a_p \in \mathbb{R}),
\]
and \( f(k/n) = f\left[(k/n)q_1 + (k/n)q_2\right] = f_k(k/n)q_1 + f_2(k/n)q_2 \), we easily get
\[
 s_{k,n}(z) = s_{k,n}(z)q_1 + s_{k,n}(z)q_2,
\]
and \( P_n(f)(z) = P_n(f_1)(z)q_1 + P_n(f_2)(z)q_2 \), \( z = z\cdot q_1 + z\cdot q_2 \in R^*(0,1) \).

Therefore, by the Remark of Lemma 3.2 we obtain
\[
 f(z) - P_n(f)(z) \leq \frac{1}{\sqrt{2}} \left[\left(\left|f_1(z') - f_1(z)\right| + \left|f_2(z') - f_2(z)\right|\right)\right] \leq 2 \cdot \mathcal{O}(f;1/n), \quad z \in R^*(0,1),
\]
which proves the theorem.

REMARK. By [8, Theorem IV], the possibility of uniform approximation by polynomials on isotropic rectangles characterize the pseudoholomorphic functions. Then a natural question arise: how can be approximated a function which is not pseudoholomorphic, i.e. for example, if \( f \) is continuous of general form
\[
f(z) = P_1(z_1, z_2)q_1 + P_2(z_1, z_2)q_2.
\]
\( z = z\cdot q_1 + z\cdot q_2 \in R^*(0,1) \)?

Firstly, we will introduce the following

**DEFINITION 3.5.** The functions \( \tilde{f}_1(z) = P_1(z_1, z_2)q_1 + P_1(z_2, z_1)q_1 \), \( \tilde{f}_2(z) = P_1(z_2, z_1)q_1 + P_2(z_1, z_2)q_2 \) will be called pseudoconjugates of the function \( f \).

The expressions
\[
 B_{n,k}^{(1)}(f)(z, \bar{z}) = \sum_{k=0}^{n} \sum_{j=0}^{k} \tilde{f}_1[(k/n)q_1 + (j/n)q_2] \cdot p_{k,n}(z) \cdot p_{k,n}(\bar{z})
\]
and
\[
 B_{n,k}^{(2)}(f)(z, \bar{z}) = \sum_{k=0}^{n} \sum_{j=0}^{k} \tilde{f}_2[(k/n)q_1 + (j/n)q_2] \cdot p_{k,n}(z) \cdot p_{k,n}(\bar{z})
\]
where \( p_{k,n}(z) = \binom{n}{k} z_k (1-z)^{n-k} \), \( p_{n,k}(z) = \binom{n}{j} z^j (1-z)^{n-j} \), are called areolar polynomials of the degree \( n \) of Bernstein-type.

**REMARKS.** 1). If we define the concept of areolar derivative of a function \( f: D \to C_q \), \( D \subset C_q \), \( q^2 = +1 \), by analogy with the classical complex analysis (see e.g. [17, p. 102]), i.e.
\[
 D(f)(z) = (1/2) \left(\left[\partial u / \partial x \left(x, y\right) - \partial v / \partial y \left(x, y\right)\right] + q \left[\partial v / \partial x \left(x, y\right) - \partial u / \partial y \left(x, y\right)\right]\right) \left(\partial f / \partial z\right)(z),
\]
\( f(z) = u(x, y) + qv(x, y), \quad z = x + qy \), then it is not difficult to see that the successive areolar derivative of order \( n+1 \) of \( B_{n,k}^{(1)}(f)(z, \bar{z}) \) and \( B_{n,k}^{(2)}(f)(z, \bar{z}) \) is null, which justifies the name of areolar polynomials of degree \( n \).

2). Let \( z = z_1 q_1 + z_2 q_2 \in C_q \). By \( \bar{z} = z_2 q_1 + z_1 q_2 \) and by (1) we obtain
\[
B_n^{(1)}(f)(z,\bar{z}) = \sum_{k=0}^{n} \sum_{j=0}^{n} \left[ P_k(\frac{k}{n}, j/n) q_1 + P_j(\frac{j}{n}, k/n) q_2 \right] \binom{n}{k} \left( \frac{1}{n} \right)^k \left( \frac{1}{n} \right)^{n-k} = \sum_{k=0}^{n} \sum_{j=0}^{n} P_k(\frac{k}{n}, j/n) p_{n,k}(z) p_{n,j}(\bar{z}) q_1 + \sum_{k=0}^{n} \sum_{j=0}^{n} P_j(\frac{j}{n}, k/n) p_{n,j}(z) p_{n,k}(\bar{z}) q_2,
\]

\[
B_n^{(2)}(f)(z,\bar{z}) = \sum_{k=0}^{n} \sum_{j=0}^{n} \left[ P_j(\frac{j}{n}, k/n) q_1 + P_k(\frac{k}{n}, j/n) q_2 \right] \binom{n}{j} \left( \frac{1}{n} \right)^j \left( \frac{1}{n} \right)^{n-j} = \sum_{k=0}^{n} \sum_{j=0}^{n} P_j(\frac{j}{n}, k/n) p_{n,j}(z) p_{n,k}(\bar{z}) q_1 + \sum_{k=0}^{n} \sum_{j=0}^{n} P_k(\frac{k}{n}, j/n) p_{n,k}(z) p_{n,j}(\bar{z}) q_2,
\]

i.e. \( B_n^{(1)}(f)(z,\bar{z}) = B_n^{(2)}(f)(z,\bar{z}) = B_n(f)(z_1, z_2) q_1 + B_n(P_2)(z_1, z_2) q_2 \), where \( B_n(f)(z_1, z_2) \) represents the usual Bernstein polynomial of two real variables \( z_1, z_2 \).

Therefore, let us denote \( B_n^{(0)}(f)(z,\bar{z}) = B_n^{(1)}(f)(z,\bar{z}) = B_n^{(2)}(f)(z,\bar{z}) = \bar{B}_n(f)(z,\bar{z}) \).

We can prove the

**THEOREM 3.6.** If \( f(z) = \sum_{k=0}^{n} \sum_{j=0}^{n} P_k(\frac{k}{n}, j/n) q_1 + P_j(\frac{j}{n}, k/n) q_2 \) is continuous on \( R^*(0,1) \) then for all \( n \in \mathbb{N} \) and \( z = z_1 q_1 + z_2 q_2 \in R^*(0,1) \) we have

\[
|f(z) - \bar{B}_n(f)(z,\bar{z})| \leq 4 \cdot \frac{a(f; 1/\sqrt{n})}{\sqrt{n}}, \quad \forall n \in \mathbb{N}, z_1, z_2 \in [0,1], k \in 1,2.
\]

PROOF. By a well-known Ipatov's result (see e.g. [19, p.339]) we have

\[
B_n(P_k)(z_1, z_2) - P_k(z_1, z_2) \leq 2a(P_1; 1/\sqrt{n}), \quad \forall n \in \mathbb{N}, z_1, z_2 \in [0,1], k \in 1,2.
\]

Then, by Lemma 3.2 we get

\[
|f(z) - \bar{B}_n(f)(z,\bar{z})| = \left| \left[ P_k(z_1, z_2) - B_n(P_k)(z_1, z_2) \right] q_1 + \left[ P_j(z_1, z_2) - B_n(P_j)(z_1, z_2) \right] q_2 \right| \leq \left[ \left| P_k(z_1, z_2) - B_n(P_k)(z_1, z_2) \right| q_1 + \left| P_j(z_1, z_2) - B_n(P_j)(z_1, z_2) \right| q_2 \right| / \sqrt{2} \leq 4 \cdot \frac{a(f; 1/\sqrt{n})}{\sqrt{n}},
\]

which proves the theorem.

Using the polynomials given by (2), we can introduce

\[
P_n(f)(z,\bar{z}) = \sum_{k=0}^{n} \sum_{j=0}^{n} P_k(\frac{k}{n}, j/n) q_1 + P_j(\frac{j}{n}, k/n) q_2 \bar{f}_k(z) \bar{f}_j(\bar{z}),
\]

called areolar polynomial of degree \( n \) of DeVore-Szabados-type.

Then by (3) and reasoning as in the proof Theorem 3.6, we immediately obtain the

**THEOREM 3.7.** If \( f(z) = \sum_{k=0}^{n} \sum_{j=0}^{n} P_k(\frac{k}{n}, j/n) q_1 + P_j(\frac{j}{n}, k/n) q_2 \), \( z = z_1 q_1 + z_2 q_2 \), is continuous on \( R^*(0,1) \) then

\[
|f(z) - P_n(f)(z,\bar{z})| \leq C a(f; 1/n), \quad \forall n \in \mathbb{N}, z \in R^*(0,1),
\]

where \( C > 0 \) is an absolute constant.

**REMARKS.** 1). Comparing, for example, Theorems 3.3 and 3.4 with the results regarding the approximation by Bernstein-type polynomials in the classical complex analysis (see e.g. [3], [12], [16], [23]) we see that they are essentially different.

2). Similar results with the Theorems 3.3 and 3.6 can be obtained if in the place of the Bernstein polynomials we consider, for example, the Meyer-König and Zeller's operator in [15], or the Baskakov's operator in [1], or the Szasz-Mirakyan's operator in [21].

3). By simple calculus we obtain

\[
\left( \frac{\partial B_n(f)}{\partial z} \right)(z,\bar{z}) = \left( \frac{\partial B_n(f)}{\partial z_1} \right)(z_1, z_2) q_1 + \left( \frac{\partial B_n(f)}{\partial z_2} \right)(z_1, z_2) q_2,
\]
where \( f(z) = P_1(z_1, z_2) \cdot q_1 + P_2(z_1, z_2) \cdot q_2, z = z_1 q_1 + z_2 q_2 \in R^*(0,1). \)

Now, if \( P_1 \) and \( P_2 \) have continuous partial derivative of order one, then we immediately get that \( (\partial \overline{B}_n(f) / \partial \overline{z})(z, \overline{z}) \) converges (when \( n \to +\infty \)) uniformly on \( R^*(0,1) \) to
\[
(\partial P_1 / \partial z_2)(z_1, z_2) \cdot q_1 + (\partial P_2 / \partial z_1)(z_1, z_2) \cdot q_2 = (\partial f / \partial \overline{z})(z),
\]
taking into account the formulas for \( \partial P_1 / \partial z_2, \partial P_2 / \partial z_1 \), in [7, chapter I, §6] and the formula for \( \partial f / \partial \overline{z} \) in the Remark 1 of Definition 3.5.

Also, since by the standard technique in approximation by real Bernstein polynomials we have
\[
(\partial B_n(P_1) / \partial z_1)(z_1, z_2) - (\partial P_1 / \partial z_2)(z_1, z_2) \leq C_{\alpha}(\partial P_1 / \partial z_2; 1/\sqrt{n}, 1/\sqrt{n}),
\]
\[
(\partial B_n(P_2) / \partial z_2)(z_1, z_2) - (\partial P_2 / \partial z_1)(z_1, z_2) \leq C_{\alpha}(\partial P_2 / \partial z_1; 1/\sqrt{n}, 1/\sqrt{n}),
\]
taking into account the Lemma 3.2 too, we easily get the estimate
\[
(\partial \overline{B}_n(f) / \partial \overline{z})(z, \overline{z}) - (\partial f / \partial \overline{z})(z) \leq C_{\alpha}(\partial f / \partial \overline{z}; 1/\sqrt{n}), \quad z \in R^*(0,1), n \in \mathbb{N}.
\]

Thus, Remark 3 together with Theorem 3.6 represent a simple constructive solution (containing even quantitative estimates) in the hyperbolic complex analysis, of a similar result in the classical complex analysis [22].

### 4. INTERPOLATION

Firstly, we deal with the interpolation of pseudoholomorphic functions.

Let \( g: [a, b] \to \mathbb{R} \) be and \( a = x_1 < \ldots < x_n = b \). It is known that the Lagrange interpolatory polynomials \( L_n(g)(x) \) of degree \( n-1 \) which satisfies \( L_n(g)(x_k) = g(x_k), k = 1, n, \) is given by
\[
L_n(g)(x) = \sum_{k=1}^{n} g(x_k) \cdot l_{k,n}(x)
\]
where \( l_{k,n}(x) \) are given by
\[
l_{k,n}(x) = (x-x_1) \ldots (x-x_{k-1})(x-x_{k+1}) \ldots (x-x_n)/(x_k-x_1) \ldots (x_k-x_{k-1})(x_k-x_{k+1}) \ldots (x_k-x_n).
\]

Now, let \( f: R^*(a,b) \to C_{q_1,q_2}, q^2 = +1 \) be pseudoholomorphic on \( R^*(a,b) \), i.e. \( f(z) = f_1(z_1)q_1 + f_2(z_2)q_2, z = z_1 q_1 + z_2 q_2 \), with \( a, b \in \mathbb{R}, a < b \), and
\[
f_1, f_2: [a,b] \to \mathbb{R}, \text{ continuous on } [a,b].
\]

Let us consider the complex Lagrange interpolatory polynomial
\[
L_n(f)(z) = \sum_{k=1}^{n} f(x_k) \cdot l_{k,n}(z).
\]

Obviously \( L_n(f)(x_k) = f(x_k), k = 1, n, \) and by simple calculus
\[
L_n(f)(z) = \left[ \sum_{k=1}^{n} f_1(x_k) l_{k,n}(z_1) \right] q_1 + \left[ \sum_{k=1}^{n} f_2(x_k) l_{k,n}(z_2) \right] q_2 = L_n(f_1)(z_1)q_1 + L_n(f_2)(z_2)q_2,
\]
\( \forall z = z_1 q_1 + z_2 q_2. \)

This implies
\[
f(z) - L_n(f)(z) = [f_1(z_1) - L_n(f_1)(z_1)] q_1 + [f_2(z_2) - L_n(f_2)(z_2)] q_2.
\]
and taking into account that for \( z_k - x_i \neq 0, k = \overline{1,2}, i = \overline{1,n} \)

\[
f_k(z_k) - L_n(f_k)(z_k) = (z_k - x_1) \cdots (z_k - x_i)[z_{i+1}, \ldots, x_n; f], \quad k = \overline{1,2},
\]

by simple calculus we obtain the following

**THEOREM 4.1.** If we denote \( R_n(f)(z) = f(z) - L_n(f)(z) \), then

\[
R_n(f)(z) = (z-x_1) \cdots (z-x_i)[z_{i+1}, \ldots, x_n; f] \quad \forall z \in R'(a,b) \text{ with } N_i(z-x_i) \neq 0, \quad i = \overline{1,n},
\]

where \( [z, x_1, \ldots, x_n; f] \) denotes the divided difference of \( f \) at the points \( z, x_1, \ldots, x_n \in R'(a,b) \), (where if \( a_k \in C_q, k = 1, m \) are such that \( N_i(a_k - a_j) \neq 0 \) for \( k \neq i \), then by definition

\[
[a_1, \ldots, a_m; f] = \sum_{k=1}^m f(a_k)/[(a_k - a_1)(a_k - a_{k-1}) \cdots (a_k - a_m)].
\]

Now, let \( L_k(x) \) be the Legendre's polynomials of degree \( n \) and \(-1 < x_1^* < \ldots < x_n^* < 1\), where \( x_i^* \) represent the zeros of \( L_k(x) \).

**THEOREM 4.2.** Let \( f: R'(-1,1) \to C_q \) be pseudoholomorphic on \( R'(-1,1) \),

\[
f(z) = f_1(z_1)q_1 + f_2(z_2)q_2, \quad z = z_1q_1 + z_2q_2 \in R'(-1,1).
\]

Let \( \gamma: [a,b] \to R'(-1,1), \gamma(a) = -1, \gamma(b) = 1, \) be a rectifiable path and let us denote

\[
L_n(f)(z) = \sum_{i=1}^n f(x_i^*) \cdot L_n(z), \quad n \in \mathbb{N}.
\]

Then, \( \lim_{n \to \infty} \int_{\gamma} L_n(f)(z)dz = \int_{\gamma} f(z)dz. \)

If moreover, the geometric image of \( \gamma \) is the interval \([-1,1]\), then

\[
\lim_{n \to \infty} \int_{\gamma} [f(z) - L_n(f)(z)]^2dz = 0.
\]

**PROOF.** By (4) and by \([7, \text{chapter IV}]\) we get

\[
\int_{\gamma} [f(z) - L_n(f)(z)]dz = \left\{ \int_{[1]} [f_1(z_1) - L_n(f_1)(z_1)]dz_1 \right\}q_1 + \left\{ \int_{[2]} [f_2(z_2) - L_n(f_2)(z_2)]dz_2 \right\}q_2.
\]

But by the classical theorem of Erdős and Turan \([9]\) we have

\[
\lim_{n \to \infty} \int_{[k]} [f_k(z_k) - L_n(f_k)(z_k)]^2dz_k = 0, \quad k = 1, \overline{2}.
\]

This obviously implies that

\[
\lim_{n \to \infty} \int_{[k]} [f_k(z_k) - L_n(f_k)(z_k)]dz_k = 0, \quad k = 1, \overline{2}
\]

and therefore \( \lim_{n \to \infty} \int_{\gamma} [f(z) - L_n(f)(z)]dz = 0. \)

Now, let \( \gamma \) be the path such that the geometric image of \( \gamma \) is the interval \([-1,1]\). By (4) we obtain

\[
[f(z) - L_n(f)(z)]^2 = [f_1(z_1) - L_n(f_1)(z_1)]^2 + [f_2(z_2) - L_n(f_2)(z_2)]^2/2 = F_n(z_1, z_2) =
\]

\[
F_n(z_1, z_2) = F_n(z_1, z_2)q_1 + F_n(z_1, z_2)q_2, \quad z = z_1q_1 + z_2q_2 \in R'(-1,1).
\]

Applying the definition of the integral for \( F_n(z_1, z_2) \) on
\( \gamma(t) = [-1,1], \ t \in [a,b], \) since \( \gamma(t) = \gamma(t)q_1 + \gamma(t)q_2, \) in [7, chapter IV] we immediately get

\[
\int_{[-1,1]} F_n(z_1, z_2) \, dz = (1/2) \left[ \int_a^b f_1(\gamma(t)) - L_n(f_1)(\gamma(t))^2 \, d\gamma(t) + \int_a^b f_2(\gamma(t)) - L_n(f_2)(\gamma(t))^2 \, d\gamma(t) \right] = (1/2) \left[ \int_{-1}^1 f_1(u) - L_n(f_1)(u)^2 \, du + \int_{-1}^1 f_2(u) - L_n(f_2)(u)^2 \, du \right] \xrightarrow{n \to \infty} 0, \text{ by (5)}.
\]

The theorem is proved.

REMARK. In the case when \( f:R^*(a,b) \to C_\mathbb{R}, \ a,b \in R, \ a < b, \) is of the form 
\( f(z) = P_1(z_1, z_2)q_1 + P_2(z_1, z_2)q_2, \ z \in R^*(a,b), \) with \( P_1 \) and \( P_2 \) continuous on \( R^*(a,b), \) as in Section 3 we can introduce the interpolatory areolar polynomials of Lagrange-type.

\[
L_n(f)(z, \overline{z}) = \sum_{k=1}^{n} \sum_{j=1}^{n} c_k c_j f(x_k, x_j) \cdot I_{k,n}(z) \cdot I_{j,n}(\overline{z}) = \\
\sum_{k=1}^{n} \sum_{j=1}^{n} \left[ P_1(x_k, x_j) q_1 + P_2(x_k, x_j) q_2 \right] \cdot I_{k,n}(z) \cdot I_{j,n}(\overline{z}) = \\
\left[ \sum_{k=1}^{n} \sum_{j=1}^{n} P_1(x_k, x_j) \cdot I_{k,n}(z) \cdot I_{j,n}(\overline{z}) \right] q_1 + \left[ \sum_{k=1}^{n} \sum_{j=1}^{n} P_2(x_k, x_j) \cdot I_{k,n}(z) \cdot I_{j,n}(\overline{z}) \right] q_2,
\]

where \( a \leq x_1 \leq ... \leq x_n \leq b \) and \( f(z) = P_1(z_1, z_2)q_1 + P_2(z_1, z_2)q_2, \ z = z_1q_1 + z_2q_2. \)

It is easy to check that 
\( L_n(f)(x_p, x_p) = f(x_p), \ \forall p = 1, n. \)

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