DUAL SPACES
FOR ONE-SIDED WEIGHTED HARDY SPACES

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ABSTRACT. Let $H^p(w)$ be the Hardy spaces introduced in [3] defined for
one-sided weights $w$, see [4], and a suitable one-sided maximal function for
distributions on the real line. The purpose of this paper is to give a character-
ization of the dual spaces of $H^p(w)$ in terms of certain classes of weighted
BMO of Lipschitz spaces. The method used here is similar to that of J.
García-Cuerva in [1] for $H^p(w)$ spaces, where $w$ belongs to $A_q$ classes of B.
Muckenhoupt. For the case of $w(x) > 0$ almost everywhere, the characteriza-
tion obtained generalizes the one given in [1], see Theorem (2.4).

1. NOTATIONS, DEFINITIONS AND PREREQUISITES

Given a Lebesgue measurable set $E \subset \mathbb{R}$, we denote its Lebesgue measure by $|E|$ and the characteristic function of $E$ by $\chi_E$.

Let $f$ be a measurable function defined on $\mathbb{R}$. The one-sided Hardy-Littlewood maximal functions $M^-f$ and $M^+f$ are given by

$$M^-f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^{x} |f(t)| dt$$

and

$$M^+f(x) = \sup_{h > 0} \frac{1}{h} \int_{x}^{x+h} |f(t)| dt.$$

As usual, a weight $w$ is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a measurable set, we denote its $w$-measure by $w(E) = \int_E w(t) dt$.

A weight $w$ belongs to the class $A^+_q$, $1 \leq q < \infty$, if there exists a constant $c$ such that

$$\sup_{h > 0} \left( \frac{1}{h} \int_{x-h}^{x} w(t) dt \right) \left( \frac{1}{h} \int_{x}^{x+h} w(t)^{-\frac{1}{q-1}} dt \right)^{q-1} \leq c,$$

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for all real number \( x \). We observe that \( w \) belongs to \( A^+_1 \) if and only if \( M^- w(x) \leq c w(x) \) holds for almost every \( x \).

Given \( w \) belonging \( A^+_q \), \( 1 \leq q < \infty \), we can define \( x_{-\infty} \geq -\infty \) and \( x_{\infty} \leq +\infty \), such that

(i) \( w(x) = 0 \) a.e. in \((-\infty, x_{-\infty})\),

(ii) \( w(x) = \infty \) a.e. in \((x_{\infty}, \infty)\) and,

(iii) \( 0 < w(x) < \infty \) for almost every \( x \in (x_{-\infty}, x_{\infty}) \).

We always have \( x_{-\infty} \leq x_{\infty} \). In order to avoid the non-interesting case of \( x_{-\infty} = x_{\infty} \), it is necessary and sufficient that there exists a measurable set \( E \) satisfying \( 0 < w(E) < \infty \).

Let \( f \) be a measurable function with support contained in an interval \( I \) (\( I \) not necessarily bounded). We shall say that \( f \) belongs to \( L'(I, w) \), \( 0 < r \leq \infty \), if \( \|f\|_{L'^r(I, w)} = \left( \int |f(x)|^r w(x) \, dx \right)^{1/r} \) is finite. If \( I = \mathbb{R} \) or \( w \equiv 1 \) we simply write \( L'(w) \) or \( L'(I) \) respectively, and \( L'(\mathbb{R}) \) shall be denoted by \( L' \). Given a positive integer \( \gamma \), we say that a function \( f \) belongs to \( L'_\gamma(I, w) \) if \( f \in L'(I, w) \) and, if \( |I| < \text{dist}(x_{-\infty}, I) \), then we require \( f \) to have null moments up to the order \( \gamma - 1 \), i.e., \( \int f(x)x^k \, dx = 0 \) holds for every integer \( k \), \( 0 \leq k \leq \gamma - 1 \).

The following lemma contains the basic results for \( A^+_q \) weights and one-sided maximal functions that we shall need in this paper.

**Lemma 1.2.**

(i) Let \( 1 \leq q_1 < q_2 < \infty \). If the weight \( w \) belongs to the class \( A^+_{q_1} \), then it also belongs to \( A^+_{q_2} \).

(ii) Let \( 1 < q < \infty \). The one-sided Hardy-Littlewood maximal \( M^+ \) is bounded on \( L'(w) \) if and only if \( w \) belongs to \( A^+_q \).

(iii) Given \( w \in A^+_q \), \( 1 \leq q < \infty \) for every \( a \in \mathbb{R} \), the \( w \)-measure of the interval \((a, \infty)\) is equal to infinite.

(iv) Let \( w \in A^+_q \), \( 1 \leq q < \infty \). Let \( \alpha < \beta \) be the end points of the bounded interval \( I \). Then, the interval \( \tilde{I} \) with end points \( \alpha - |I| \) and \( \alpha \), satisfies

\[
w(\tilde{I}) \leq c_w w(I)
\]

where the constant \( c_w \) does not depend on \( I \).

A proof of (ii) may be found in [4] or in [2]. As for parts (i) and (iii) the proofs are easy. Part (iv) is an immediate consequence of (ii).
Let $w$ belong to $A_q^+$, $1 \leq q < \infty$, and let $x_{-\infty}$ be defined as in (1.1) for the weight $w$. As usual, $C_0^\infty(\mathbb{R})$ denotes the set of all functions with compact support having derivatives of all orders. We shall denote by $\mathcal{D}(x_{-\infty}, \infty)$ the space of all functions in $C_0^\infty(\mathbb{R})$ with support contained in $(x_{-\infty}, \infty)$ equipped with the usual topology and by $\mathcal{D}'(x_{-\infty}, \infty)$ the space of distributions on $(x_{-\infty}, \infty)$.

Given a positive integer $\gamma$ and $x \in \mathbb{R}$, we shall say that a function $\psi$ in $C_0^\infty(\mathbb{R})$, belongs to the class $\Phi_\gamma(x)$ if there exists a bounded interval $I_x = [\alpha, \beta]$ containing the support of $\psi$ such that $D^\gamma \psi$ satisfies

$$|I_x|^{\gamma+1} \|D^\gamma \psi\|_\infty \leq 1.$$ 

Let $F$ be a distribution in $\mathcal{D}'(x_{-\infty}, \infty)$. We define as in [3] the one-sided maximal function $F^*_{+\gamma}$, as

$$(1.3) \quad F^*_{+\gamma}(x) = \sup \{| f, \psi | : \psi \in \Phi_\gamma(x)\},$$

for every $x > x_{-\infty}$.

Fixed $w$ belonging to $A_q^+$ ($1 \leq q < \infty$), a positive integer $\gamma$ and, $0 < p \leq 1$ such that $(\gamma + 1)p \geq q > 1$ or $(\gamma + 1)p > q$ if $q \neq 1$, the distribution $F$ in $\mathcal{D}'(x_{-\infty}, \infty)$ belongs to $H^p_{+\gamma}(w)$ if the "$p$-norm"

$$\|F\|_{H^p_{+\gamma}(w)} = \left( \int_{x_{-\infty}}^\infty F^*_{+\gamma}(x)^p w(x) dx \right)^{1/p},$$

is finite.

In the sequel we shall suppose that $w$ belongs to $A_q^+$, $\gamma$ is a positive integer, $0 < p \leq 1$ and, that they satisfy $(\gamma + 1)p \geq q > 1$ or $(\gamma + 1)p > q$ if $q = 1$.

**Lemma 1.4.** Let $I \subset (x_{-\infty}, \infty)$ be an interval and let $f$ belong to $L^\infty(I)$. Then for any $x > x_{-\infty}$, we have

$$f^+_{+\gamma}(x) \leq c_\gamma \|f\|_{\infty} [M^+ x f(x)]^{\gamma+1}.$$ 

Moreover,

$$\|f\|_{H^p_{+\gamma}(w)} \leq c_{\gamma, w} \|f\|_{\infty} w(I)^{1/p}.$$ 

The constants $c_\gamma$ and $c_{\gamma, w}$ do not depend on $f$.

This lemma can be found in [3] as Lemma (3.2). Thus, as in [3] we have the following definition of $p$-atom with respect to a weight $w$. 
A function $a(x)$ defined on $\mathbb{R}$ is called a $p$-atom with respect to $w$ if there exists an interval $I$ containing the support of $a(x)$, such that

(i) $I$ is contained in $(x_{-\infty}, \infty)$ and $w(I) < \infty$,

(ii) $a(x) \in L^\infty(I)$ and,

(iii) $\|a\|_\infty \leq w(I)^{-1/p}$.

We shall say that $I$ is the interval associated to the atom $a(x)$.

The following theorems are of fundamental importance in the theory of the $H^p_{\gamma, \gamma}(w)$ spaces. Their proofs can be found in section 5 of [3].

**Theorem 1.5.** (Decomposition into atoms). If $F$ belongs to $H^p_{\gamma, \gamma}(w)$, then there exists a sequence $\{a_k\}$ of $p$-atoms with respect to $w$ and a sequence $\{\lambda_k\}$ of real numbers such that

$$F = \sum \lambda_k a_k \quad \text{in} \quad D'(x_{-\infty}, \infty)$$

and,

$$c_p' \|F\|_{H^p_{\gamma, \gamma}(w)} \leq \sum |\lambda_k|^p \leq c_p \|F\|_{H^p_{\gamma, \gamma}(w)}$$

holds.

**Remark 1.6.** By Lemma (1.4) and Theorem (1.5) we have that the set $D$ of all functions $f$ such that there exists an interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$ and $f \in L^\infty(I)$, is dense in $H^p_{\gamma, \gamma}(w)$.

**Theorem 1.7.** Under the hypotheses of Theorem (1.5) and if, in addition, we assume that $x_{-\infty} = -\infty$, then the $p$-atoms $\{a_k\}$ in the decomposition can be taken in such a way that the corresponding associated intervals are bounded and therefore all the $p$-atoms in the decomposition have null moments up to the order $\gamma - 1$.

**Remark 1.8.** If $x_{-\infty} = -\infty$, by Lemma (1.4) and Theorem (1.7) we have that the set $D_1$ of all functions $f$ such that there exists a bounded interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$ and $f \in L^\infty(I)$, is dense in $H^p_{\gamma, \gamma}(w)$.

We shall denote $[H^p_{\gamma, \gamma}(w)]^*$ the dual space of $H^p_{\gamma, \gamma}(w)$ formed by all the real valued continuous linear functionals $L$ with the norm

$$\|L\| = \sup\{|L(F)| : \|F\|_{H^p_{\gamma, \gamma}(w)} \leq 1\}.$$
Let $\gamma$ be a positive integer and let $P_\gamma$ be the linear space of all real polynomials of degree less than $\gamma$. For any bounded interval $I$, we define the inner product on $P_\gamma$ by the formula

$$(P, Q)_I = \int_I P(x) Q(x) \, dx.$$ 

Let $\{e_k\}_{k=0}^{\gamma-1}$ be an orthonormal basis of $P_\gamma$ for the case when $I = [0, 1]$. It is easy to verify that for any $I = [a, b]$, the polynomials

$$(1.9) \quad e_{k,I}(x) = |I|^{-1/2} e_k((x - a)/|I|), \quad 0 \leq k \leq \gamma - 1$$

form an orthonormal basis of $P_\gamma$ with the inner product $(\cdot, \cdot)_I$. Given a function $f$ such that $f \chi_I \in L^1$, we define its orthogonal projection on $P_\gamma$, as

$$(1.10) \quad P_I(f)(x) = \sum_{k=0}^{\gamma-1} \left( \int_a^b f(y) \, e_{k,I}(y) \, dy \right) e_{k,I}(x).$$

We observe that, by (1.9),

$$(1.11) \quad \sup_{x \in I} |e_{k,I}(x)| = |I|^{-1/2} \sup_{x \in [0,1]} |e_k(x)| \leq c_\gamma |I|^{-1/2},$$

holds for every integer $k$, $0 \leq k \leq \gamma - 1$. Then, if $f \chi_I \in L^\infty$, by (1.10) and (1.11), we have that

$$(1.12) \quad |P_I(f)(x)| \leq c_\gamma \|f \chi_I\|_\infty,$$

holds for every $x \in I$, with a constant $c_\gamma$ depending on $\gamma$ only.

We shall need a result that allows us to compare $P_I(f)$ and $P_J(f)$. To be more precise we state the following lemma.

**Lemma 1.13.** Let $I \subset J$ be two bounded intervals such that $|J| \leq 5|I|$. Then, if $f \chi_J \in L^1$, we have that

$$|P_I(f)(x) - P_J(f)(x)| \leq c_\gamma \frac{1}{|I|} \int_I |f - P_J(f)| \, dx,$$

holds for every $x$ belonging to $J$.

**Proof.** Let $\{e_k\}_{k=0}^{\gamma-1}$ be the orthonormal basis of the subspace $P_\gamma$ defined above and let $\{e_{k,I}\}_{k=0}^{\gamma-1}$ be the orthonormal basis given in (1.9). Thus

$$P_I(f)(x) - P_J(f)(x) = P_I[f - P_J(f)](x)$$

$$= \sum_{k=0}^{\gamma-1} \left( \int_I [f - P_J(f)](s) e_{k,I}(s) \, ds \right) e_{k,I}(x).$$
Consequently, if \( x \) belongs to \( J \) we get

\[
|P_I(f)(x) - P_J(f)(x)| \leq \sum_{k=0}^{\gamma-1} \int_I |[f - P_J(f)](s)| \, ds \|e_k \chi_{J\setminus I}\|_{\infty} \|e_k \chi_{J}\|_{\infty}.
\]

By (1.11), we have \( \|e_k \chi_{J\setminus I}\|_{\infty} \leq c_\gamma |J|^{-1/2} \). Moreover, since \( I \subset J \) and \( |J| \leq 5|I| \), it follows that if \( x \in J \) then \( |x-a|/|I| \leq 5 \), which implies that \( \|e_k \chi_{J}\|_{\infty} \leq |I|^{-1/2} \sup_{y \in I} |e_k(y)| \leq c_\gamma |J|^{-1/2} \). Therefore, for every \( x \in J \) we obtain

\[
|P_I(f)(x) - P_J(f)(x)| \leq \gamma c_\gamma c_\gamma' |J|^{-1} \int_I |[f - P_J(f)](s)| \, ds,
\]

as we wanted to show.

We shall say that a function \( \ell \), defined on \( (x_{-\infty}, x_{\infty}) \), belongs to \( BMO_+(p, \gamma, w) \) if for every interval \( I \subset (x_{-\infty}, x_{\infty}) \) with \( w(I) < \infty \), we have

(i) \( \ell \chi_I \) belongs to \( L^1 \),
(ii) if \( |I| \geq \text{dist}(x_{-\infty}, I) \) then \( \int_I |\ell(x)| \, dx \leq c \, w(I)^{1/p} \) and,
(iii) if \( |I| < \text{dist}(x_{-\infty}, I) \) then the orthogonal projection \( P_I(\ell) \) is well defined and

\[
\int_I |\ell(x) - P_I(\ell)(x)| \, dx \leq c \, w(I)^{1/p}.
\]

holds.

The constant \( c \) does not depend on the intervals \( I \) and the least constant \( c \) for which (ii) and (iii) hold, shall be denoted by \( \|\ell\|_{BMO_+(p, \gamma, w)} \).

**Remark 1.14.** Let \( \ell \) belong to \( BMO_+(p, \gamma, w) \) and let \( A \) belong to \( L^\infty_\gamma(I) \), where \( I \subset (x_{-\infty}, x_{\infty}) \) is an interval with \( w(I) < \infty \). If \( |I| \geq \text{dist}(x_{-\infty}, I) \), by the definition of \( BMO_+(p, \gamma, w) \), we have that

\[
\int |A(x) \ell(x) \, dx| \leq \|A\|_{\infty} \int |\ell(x)| \, dx \leq \|A\|_{\infty} \|\ell\|_{BMO_+(p, \gamma, w)} w(I)^{1/p}.
\]

In the case that \( |I| < \text{dist}(x_{-\infty}, I) \), since, by definition of \( L^\infty_\gamma(I) \), the function \( A \) has null moments up to the order \( \gamma - 1 \), we get

\[
\int |A(x) \ell(x) \, dx| = \left| \int A(x) \left[ \ell(x) - P_I(\ell)(x) \right] \, dx \right| \leq \|A\|_{\infty} \int |\ell(x) - P_I(\ell)(x)| \, dx \leq \|A\|_{\infty} \|\ell\|_{BMO_+(p, \gamma, w)} w(I)^{1/p}.
\]
Remarks.

(a) If there exists $\beta > x_\infty$ such that $w((x_\infty, \beta)) < \infty$, then $(BMO_+(p, \gamma, w), \| \cdot \|_{BMO_+(p, \gamma, w)})$ is a normed space.

(b) If we have that $w((x_\infty, \beta)) = \infty$ holds for every $\beta > x_\infty$ then $\| \cdot \|_{BMO_+(p, \gamma, w)}$ is a seminorm. Indeed, $\| \ell \|_{BMO_+(p, \gamma, w)}$ is equal to zero if and only if $\ell$ belongs to $\mathcal{P}_\gamma$, the set of all polynomials of degree less than $\gamma$. Therefore defining, as usual, for $\ell$ belonging to $BMO_+(p, \gamma, w)/\mathcal{P}_\gamma$ the application

$$\| \ell \|_{BMO_+(p, \gamma, w)/\mathcal{P}_\gamma} = \| \ell' \|_{BMO_+(p, \gamma, w)},$$

where $\ell - \ell' \in \mathcal{P}_\gamma$, we obtain the normed space $(BMO_+(p, \gamma, w)/\mathcal{P}_\gamma, \| \cdot \|_{BMO_+(p, \gamma, w)/\mathcal{P}_\gamma})$.

We shall say that a function $\ell$ defined on $(x_\infty, x_\infty)$, belongs to $BMOF_+(p, \gamma, w)$ if for every bounded interval $I \subset (x_\infty, \infty)$ with $w(I) < \infty$, we have

(i) $\ell x_I$ belongs to $L^1$ and,

(ii) $\int_I |\ell(x) - P_I(\ell)(x)| dx \leq c w(I)^{1/p}$ holds with a constant $c$ not depending on the intervals $I$.

The least constant $c$ for which (ii) holds shall be denoted by $\| \ell \|_{BMOF_+(p, \gamma, w)}$.

Remarks.

(a) The application $\| \cdot \|_{BMOF_+(p, \gamma, w)}$ is a seminorm and, as usual, it induces a norm $\| \cdot \|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma}$ in the quotient space $BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma$.

(b) If we have that $w((x_\infty, \beta)) = \infty$ holds for every $\beta > x_\infty$, then the space $BMOF_+(p, \gamma, w)$ coincides with $BMO_+(p, \gamma, w)$.

2. STATEMENT OF THE RESULTS

In this paragraph we state the results that characterize the dual space of $H^p_{+\gamma}(w)$, which is the purpose of the paper.

**Theorem 2.1.** Let $w \in A^+_r$, $r > q$, $\gamma$ a positive integer and $0 < p \leq 1$ such that $(\gamma + 1)p \geq q$ if $q > 1$ or $(\gamma + 1)p > 1$ if $q = 1$. If $L$ belongs to $[H^p_{+\gamma}(w)]^*$ we have that

(i) if there exists $\beta > x_\infty$ such that $w((x_\infty, \beta)) < \infty$, then there exists a unique $\ell$ belonging to $BMO_+(p, \gamma, w)$ such that

$$L(f) = \int \ell(x) f(x) \, dx$$
holds for every \( f \in L^r(I,w) \) where \( I \subset (x_-,\infty) \) is any interval with \( w(I) < \infty \). Moreover,

\[
\|\ell\|_{BMO^+(p,\gamma,w)} \leq c_{\gamma,r,p,w}\|L\|.
\]

(ii) if we have that \( w(\langle x_-,\beta \rangle) = \infty \) holds for every \( \beta > x_- \), then there exists a unique class \( \tilde{\ell} \) belonging to \( BMO^+(p,\gamma,w)/P_\gamma \) such that for any \( \ell' \) belonging to \( \tilde{\ell} \), we have that

\[
L(f) = \int \ell'(x) f(x) \, dx
\]

holds for every \( f \in L^r(I,w) \), where \( I \subset (x_-,\infty) \) is any interval with \( w(I) < \infty \). Moreover

\[
\|\ell\|_{BMO^+(p,\gamma,w)/P_\gamma} \leq c_{\gamma,r,p,w}\|L\|.
\]

**Theorem 2.2.** Let \( w \in A^+_\gamma \), \( \gamma \) a positive integer and \( 0 < p \leq 1 \) such that \((\gamma + 1)p \geq q \) if \( q > 1 \) or \((\gamma + 1)p > 1 \) if \( q = 1 \). Then, we have

(i) if there exists \( \beta > x_- \) such that \( w(\langle x_-,\beta \rangle) < \infty \), given \( \ell \) belonging to \( BMO^+(p,\gamma,w) \), the functional

\[
L(f) = \int \ell(x) f(x) \, dx
\]

is well defined on the dense set \( D \) (see Remark (1.6)) and,

\[
\|L\| \leq c_{p,\gamma,w}\|\ell\|_{BMO^+(p,\gamma,w)}.
\]

(ii) if we have that \( w(\langle x_-,\beta \rangle) = \infty \) holds for every \( \beta > x_- \), given \( \tilde{\ell} \) belonging to \( BMO^+(p,\gamma,w)/P_\gamma \) and \( \ell' \) in the class \( \tilde{\ell} \), the functional

\[
L(f) = \int \ell'(x) f(x) \, dx
\]

is well defined on the dense set \( D \), \( L \) is independent of \( \ell' \in \tilde{\ell} \) and

\[
\|L\| \leq c_{p,\gamma,w}\|\tilde{\ell}\|_{BMO^+(p,\gamma,w)/P_\gamma}.
\]

**Theorem 2.3.** Let \( w \in A^+_\gamma \), \( \gamma \) a positive integer and \( 0 < p \leq 1 \) such that \((\gamma + 1)p \geq q \) if \( q > 1 \) or \((\gamma + 1)p > 1 \) if \( q = 1 \). Then, we have
(i) if there exists $\beta > x_{-\infty}$ satisfying $w((x_{-\infty}, \beta)) < \infty$, then there exists a bijective linear application $i$ from $[H_{+}^{p, \gamma}(w)]^{*}$ into $BMO_{+}(p, \gamma, w)$ such that if $i(L) = \ell$, then

$$L(f) = \int \ell(x) f(x) \, dx$$

holds for every $f \in D$. Moreover,

$$c_{1} \|L\| \leq \|\ell\|_{BMO_{+}(p, \gamma, w)} \leq c_{2} \|L\|.$$ 

(ii) if we have $w((x_{-\infty}, \beta)) = \infty$ holds for every $\beta > x_{-\infty}$, then there exists a bijective linear application $i$ from $[H_{+}^{p, \gamma}(w)]^{*}$ into $BMO_{+}(p, \gamma, w)/P_{\gamma}$ such that if $i(L) = \ell$ and $\ell'$ belongs to $\ell$, then

$$L(f) = \int \ell'(x) f(x) \, dx$$

holds for every $f \in D$. Moreover,

$$c_{1} \|L\| \leq \|\ell\|_{BMO_{+}(p, \gamma, w)/P_{\gamma}} \leq c_{2} \|L\|.$$ 

**Theorem 2.4.** Let $w \in A_{+}^{q, r}$, $\gamma$ a positive integer and $0 < p \leq 1$ such that $(\gamma + 1)p \geq q$ if $q > 1$ or $(\gamma + 1)p > 1$ if $q = 1$. If $x_{-\infty} = -\infty$ then the conclusions of part (ii) of Theorem (2.3) hold for every $f$ belonging to the dense set $D_{1}$ (see Remark (1.8)) even if there exists $\beta$ such that $w((-\infty, \beta)) < \infty$.

3. PROOFS OF THE RESULTS

**Lemma 3.1.** Let $w \in A_{+}^{q, r}$, $\gamma \geq 1$ an integer and, $0 < p \leq 1$ such that $(\gamma + 1)p \geq q > 1$ or $(\gamma + 1)p > 1$ and $r \geq q > 1$ or $r > q = 1$. Let $I \subset (x_{-\infty}, \infty)$ be an interval with $w(I) < \infty$ and let $f$ belong to $L_{r}^{r}(I, w)$. Then $f \in H_{+}^{p, \gamma}(w)$ and

$$\|f\|_{H_{+}^{p, \gamma}(w)} \leq c_{\gamma, r, p, w} \|f\|_{L_{r}^{r}(I, w)} w(I)^{\frac{1}{r} - \frac{1}{r'}}.$$ 

**Proof.** Let $\alpha < \beta$ be the end points of $I$. If $\max(x_{-\infty}, \alpha - |I|) \leq x$, by definition (1.3), we have $f_{+}^{\gamma}(x) \leq M_{+} f(x)$. Then, by Hölder's inequality and applying Lemma (1.2), we obtain

$$\int_{\max(x_{-\infty}, \alpha - |I|)}^{\infty} f_{+}^{\gamma}(x)^{p} w(x) \, dx \leq \left( \int_{-\infty}^{\beta} M_{+} f(x)^{r} w(x) \, dx \right)^{\frac{p}{r}} w(\tilde{I} \cup I)^{1 - \frac{p}{r}}$$

$$\leq c_{\gamma, r, p, w} \|f\|_{L_{r}^{r}(I, w)} w(I)^{1 - \frac{1}{r'}}.$$ (3.2)
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If there exists \( x \) such that \( x - \infty < x < \alpha - |I| \), then \( f \) has null moments up to the order \( \gamma - 1 \) and the interval \( I \) is bounded. Let \( \psi \) belong to the class \( \Phi_\gamma(x) \) and \( I_\psi \) the interval associated with \( \psi \) in this class. We have

\[
< f, \psi > = \int_I f(t) \left[ \psi(t) - \sum_{s=0}^{\gamma-1} \frac{D^s \psi(\alpha)}{s!} (t - \alpha)^s \right] dt .
\]

We may assume that \( I \cap I_\psi \neq \emptyset \), then \( \alpha - x \leq |I_\psi| \) and we get

\[
| < f, \psi > | \leq \frac{\|D^\gamma \psi\|_\infty}{\gamma!} |I|^{\gamma} \int_I |f(t)| dt \leq c_\gamma \left( \frac{|I|}{\alpha - x} \right)^{\gamma+1} \frac{1}{|I|} \int_I |f(t)| dt .
\]

Since for every \( x \) such that \( x - \infty < x < \alpha - |I| \), the one-sided maximal function \( M^+ \chi(x) \) satisfies:

\[
\frac{|I|}{\alpha - x} \leq M^+ \chi(x) ,
\]

it follows that

\[
f_{+\gamma}(x) \leq c_\gamma [M^+ \chi(x)]^{\gamma+1} \frac{1}{|I|} \int_I |f(t)| dt .
\]

Now, by Hölder’s inequality and taking into account that \( w \in A_+^r \), we have

\[
\frac{1}{|I|} \int_I |f(t)| dt \leq \|f\|_{L^r(I,w)} \frac{1}{|I|} \left( \int_I w(t)^{-r'/r} dt \right)^{1/r'} \leq c_{r,w} \|f\|_{L^r(I,w)} w(I)^{-1/r} ,
\]

which implies that

\[
f_{+\gamma}(x) \leq c_{\gamma,r,w} \|f\|_{L^r(I,w)} w(I)^{-1/r} [M^+ \chi(x)]^{\gamma+1} .
\]

Then, by Lemma (1.2), we get

\[
(3.3) \quad \int_{x - \infty}^{x - |I|} f_{+\gamma}(x)^p w(x) dx \leq c_{\gamma,r,p,w} \|f\|_{L^r(I,w)}^p w(I)^{1-\frac{p}{r}} .
\]

By (3.2) and (3.3), this lemma is proved. ■

Remark. The estimation for the \( p \)-norm \( \|f\|_{H^\gamma_{+,\gamma}(w)} \) in Lemma (1.4) also follows from Lemma (3.1).

Lemma 3.4. Let \( w \geq 0 \) and \( r > 1 \). Let \( I \) be an interval with \( w(I) < \infty \). Then, if \( g \chi_I \in L^{r'}(I,w) \) we have that \( g \chi_I \in L^1(I,w) \). In particular, the orthogonal projection \( P_I(gw) \) is well defined.
The proof is an immediate consequence of H"older's inequality.

**Lemma 3.5.** Let $w \in A_q^+$ and $r \geq q > 1$ or $r > q = 1$. We assume that $I \subset (x_{-\infty}, \infty)$ is an interval satisfying the condition $|I| < \text{dist}(x_{-\infty}, I)$. Then, if $f \in L^r(I, w)$ we have that $f \in L^1(I)$. In particular, the orthogonal projection $P_l(f)$ is well defined.

**Proof.** Let us observe the condition $|I| < \text{dist}(x_{-\infty}, I)$ implies that $I$ is a bounded interval and if we define $\tilde{I}$ as in Lemma (1.2) it follows that $w(\tilde{I}) > 0$. By H"older's inequality and the $A_q^+$ condition, $r > 1$, we get

$$
\int_I |f(x)| dx \leq \left( \int_I |f(x)|^r w(x) dx \right)^{1/r} \left( \int_I w(x)^{-r'/r} dx \right)^{1/r'}
$$

$$
\leq c_{r,w} |I| \|f\|_{L^r(I,w)} w(\tilde{I})^{-1/r'} < \infty,
$$

as we wanted to show.

**Proof of Theorem (2.1).**

Part (i). We consider a sequence $\{\beta_k\}_{k \geq 1} \uparrow x_\infty$, such that for every $k \geq 1$, the interval $I_k = (x_{-\infty}, \beta_k)$ satisfies $w(I_k) < \infty$. In the case of $w((x_{-\infty}, x_\infty)) < \infty$, we take $\beta_k = x_\infty$, $k \geq 1$. Given $f \in L^r(I_k, w)$, by Lemma (3.1), we have

$$
|L(f)| \leq \|f\|_{H^{p,\gamma}_+(w)}
$$

$$
\leq c_{\gamma,r,p,w} \|f\|_{L^r(I_k,w)} w(I_k)^{\frac{1}{p} - \frac{1}{r}}.
$$

Therefore, $L$ induces a continuous linear functional on $L^r(I_k, w)$. Then, by Riesz's Representation Theorem, there exists a unique $g_k \in L^r(I_k, w)$ such that

$$
L(f) = \int f(x) g_k(x) w(x) dx
$$

holds for every $f \in L^r(I_k, w)$. The uniqueness of $g_k$, implies that the restriction $g_{k+1}|_{I_k}$ is equal to $g_k$ almost everywhere in $I_k$; then, there exists a unique function $g$ defined on $(x_{-\infty}, x_\infty)$ such that for every interval $I \subset (x_{-\infty}, \infty)$ with $w(I) < \infty$, we have

(3.6) $\int_I |g(x)|^r w(x) dx < \infty$ and

(3.7) $L(f) = \int f(x) g(x) w(x) dx$, for every $f \in L^r(I, w)$.

Let us prove that $\ell = gw$ belongs to $BMO^+(p, \gamma, w)$. Let $I \subset (x_{-\infty}, \infty)$ be an interval with $w(I) < \infty$ and $\text{dist}(x_{-\infty}, I) \leq |I|$. The function $f = sg(\ell) X_I$ belongs to $L^r(I, w)$. Besides, by (3.7), Lemma (3.1) and taking into account that $\|f\|_{L^r(I, w)} \leq \|f\|_{L^\infty w(I)^{1/r'}}$, we have

(3.8) $\int_I |\ell| dx = \int \ell f dx = L(f) \leq c_{\gamma,r,p,w}|L| w(I)^{1/p}.$
Now, we assume that $|I| < \text{dist} (x_\infty, I)$. By (3.6) and Lemma (3.4), the orthogonal projection $P_I(\ell)$ is well defined. The function $f = s g [\ell - P_I(\ell)] \chi_I$ belongs to $L^r(I, w)$ and by Lemma (3.5) we get

$$\int_I |\ell - P_I(\ell)| \, dx = \int_I [\ell - P_I(\ell)] f \, dx$$

$$= \int_I [\ell - P_I(\ell)] [f - P_I(f)] \, dx$$

$$= \int_I \ell [f - P_I(f)] \, dx.$$  

Applying (3.7), Lemma (3.1) and (1.12), we obtain

$$\int_I |\ell - P_I(\ell)| \, dx = L([f - P_I(f)] \chi_I)$$

$$\leq \|L\| \, c_{\gamma, r, p, w} \, \|f - P_I(f)\chi_I\|_{L\infty (I, w)}^{1/p}$$

$$\leq c_{\gamma, r, p, w} \|f\|_{L^r(I, w)}^{1/p}.$$  

From (3.8) and (3.9) it follows that $\ell \in BMO_+(p, \gamma, w)$.

Part (ii). Now, for every $\beta > x_\infty$, $w((x_\infty, \beta))$ is infinite. This condition implies that $x_\infty = -\infty$. Let $\{\alpha_k\}_{k \geq 1} \downarrow -\infty$ and $\{\beta_k\}_{k \geq 1} \uparrow x_\infty$ be two sequences such that for every $k \geq 1$, the interval $I_k = (\alpha_k, \beta_k)$ satisfies $w(I_k) < \infty$. If there exists $\alpha$ satisfying $w((\alpha, x_\infty)) < \infty$, we take $\beta_k = x_\infty$, $k \geq 1$. By Lemma (3.1), $L$ induces a continuous linear functional on $L^r_\gamma(I_k, w)$, which, by Hahn-Banach, can be extended to $L^r(I_k, w)$. By Riesz Representation Theorem, the extension is represented by a function $g_k$ belonging to $L^r(I_k, w)$. Suppose there exist functions $g_k$ and $g'_k$ in $L^r(I_k, w)$ such that

$$\int f(x) g_k(x) w(x) \, dx = \int f(x) g'_k(x) w(x) \, dx,$$

holds for every $f \in L^r(I_k, w)$. We want to show that $g = g_k - g'_k$ is equal to $P w^{-1}$ almost everywhere in $I_k$, where $P$ is a polynomial of degree less than $\gamma$.

In fact, given $f \in L^r(I_k, w)$, the function $[f - P_{I_k}(f)] \chi_{I_k}$ belongs to $L^r_\gamma(I_k, w)$; then, using Lemma (3.4), we have

$$0 = \int_{I_k} [f - P_{I_k}(f)] g w \, dx$$

$$= \int_{I_k} [f - P_{I_k}(f)] \left[ g - \frac{P_{I_k}(gw)}{w} \right] w \, dx$$

$$= \int_{I_k} f \left[ g - \frac{P_{I_k}(gw)}{w} \right] w \, dx.$$
Thus, since $I_k \subset (x_{-\infty}, x_{\infty})$ it follows that $g = \frac{P_{I_k}(gw)}{w}$ a.e. in $I_k$.

Taking into account that $I_k = (\alpha_k, \beta_k) \uparrow (x_{-\infty}, x_{\infty})$, we can define a function $g$ on $(x_{-\infty}, x_{\infty})$ such that for every $I \subset (x_{-\infty}, x_{\infty})$ with $w(I) < \infty$, the properties (3.6) and

$$L(f) = \int f(x) \, g(x) \, w(x) \, dx, \quad \text{for every } f \in L^2_x(I, w)$$

also holds.

In this part (ii), if we have an interval $I$ with $w(I) < \infty$, then $|I| < \text{dist}(x_{-\infty}, I) = \infty$ and arguing as in (3.9), it follows that $\ell = gw \in BMO_+(p, \gamma, w)$. ■

Let $f$ be a locally integrable function on $(x_{-\infty}, \infty)$ belonging to $H^p_{\gamma}(w)$. For every integer $n$, we define the open set

$$\Omega_n = \{x : x > x_{-\infty}, f_+^n(x) > 2^n\}$$

and we denote its component intervals by $I_{n,i}$, $i \geq 1$, where $I_{n,1}$ is, if there exists, the connected component that starts at $x_{-\infty}$, and $I_{n,1} = \emptyset$ otherwise. In addition, for every $i > 1$ and $j \geq 1$, we define functions $\eta_{n,i,j}(x) \geq 0$ belonging to $C^\infty(\mathbb{R})$ such that

$$(3.10) \quad \left( \sum_{j \geq 1} \eta_{n,i,j}(x) \right) \chi_{I_{n,i}}(x) = \chi_{I_{n,i}}(x), \quad i > 1;$$

and polynomials $P_{n,i,j}(f)$ of degree less than $\gamma$, explicitly given by the formula

$$P_{n,i,j}(f)(x) = \sum_{k=0}^{\gamma-1} \left( \int f(s) \epsilon^{n,i,j}_k(s) \eta_{n,i,j}(s) \chi_{I_{n,i}}(s) \, ds \right) \epsilon^{n,i,j}_k(x),$$

where $\{\epsilon^{n,i,j}_k\}_{k=0}^{\gamma-1}$ is an orthonormal basis of the subspace of $L^2(\eta_{n,i,j} \chi_{I_{n,i}})$ generated by $1, x, \ldots, x^{\gamma-1}$. From their definition, it follows that the polynomials $P_{n,i,j}(f)$ satisfy

$$\int f(x) \, x^s \, \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) \, dx =$$

$$(3.11) \quad = \int P_{n,i,j}(f)(x) x^s \, \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) \, dx, \quad 0 \leq s < \gamma.$$

For an explicit definition of the functions $\eta_{n,i,j}$ see section 5 in [3].
We recall that in the proof of Theorem (2.2) in [3], (see (5.1)), for every \( i > 1 \) and \( j \geq 1 \), we obtained the estimate

\[
(3.12) \quad \sup_{x \in \text{support}(\eta_{n,i,j})} |P_{n,i,j}(f)(x)| \leq c \, 2^n ,
\]

where the constant \( c \) is independent of \( n \) and \( f \).

Taking into account the notations introduced above, for each integer \( n \), we consider the function \( g_n(x) \) defined as

\[
(3.13) \quad g_n(x) = f(x) \chi_{c_0}(x) + \sum_{i>1} \sum_{j \geq 1} P_{n,i,j}(f)(x) \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) ,
\]

which satisfies

\[
(3.14) \quad |g_n(x)| \leq c \, 2^n \quad \text{a.e. in } (x-\infty, \infty) ,
\]

where the constant \( c \) is independent of \( n \) and \( f \).

**Proof of Theorem (2.2).** Let \( \ell \) belong to \( BMO_+(p, \gamma, w) \). For every bounded function \( f \) supported in an interval \( I = (\alpha, \beta) \subset (x-\infty, \infty) \) with \( w(I) < \infty \), we have that

\[
\int |\ell(x)| \, |f(x)| \, dx \leq \|f\|_{\infty} \int |\ell(x)| \, dx < \infty .
\]

Then, the linear functional \( L(f) = \int \ell(x)f(x) \, dx \) is well defined on the dense set \( D \) (see Remark (1.6)). We want to show that \( L \) is a bounded functional and therefore that it can be extended to \( H^{p,\gamma,w}_w \). Since \( f \in L^\infty \), if \( M \) is large enough, then the set \( \Omega_M \) is empty and by (3.13), we have \( g_M = f \). Thus,

\[
(3.15) \quad f(x) = \sum_{n=N}^{M-1} (g_{n+1}(x) - g_n(x)) + g_N(x) .
\]

From the definition of \( g_n \), it follows that its support is contained in the union \( I \cup \Omega_n \subset (x-\infty, \beta) \).

If \( \ell \in BMO_+(p, \gamma, w) \) and \( w((x-\infty, \beta)) < \infty \), then \( \ell \) is integrable on \( (x-\infty, \beta) \) and taking into account (3.14), we get

\[
\int f \ell \, dx = \sum_{n=N}^{M-1} \int_{x-\infty}^{\beta} (g_{n+1} - g_n) \ell \, dx + \int_{x-\infty}^{\beta} g_N \ell \, dx .
\]

For the last integral on the right hand side, we have

\[
\left| \int g_N \ell \, dx \right| \leq c \, 2^N \int_{x-\infty}^{\beta} |\ell| \, dx \leq c \, 2^N \|\ell\|_{BMO_+(p, \gamma, w) w((x-\infty, \beta))^{1/p}} ,
\]
which goes zero for $N$ tending to $-\infty$.

Now, let us suppose $\omega((x_{-\infty}, \beta)) = \infty$. Then, the hypothesis $\omega(I) = \omega((\alpha, \beta)) < \infty$ implies that $x_{-\infty} = -\infty < \alpha < \beta < +\infty$. By Lemma (1.4) if $f$ belongs to $L^\infty(I)$, then we have

$$f^*_x(x) \leq c_\gamma \|f\|_\infty [M^+ \chi_I(x)]^{\gamma+1}.$$ 

On the other hand, it is easy to see that for $x < \beta$, the following inequalities hold. Thus,

$$\Omega_n = \{x : 2^n < f^*_x(x)\} \subset \{x : 2^n < c_\gamma \|f\|_\infty [M^+ \chi_I(x)]^{\gamma+1}\}$$

$$\subset \left\{ x : x < \beta, 2^n < c_\gamma \|f\|_\infty 4^{\gamma+1} \left[\frac{|I|}{\alpha - x + 2|I|}\right]^{\gamma+1} \right\} = J_n.$$ 

It can be verified without difficulty that $J_n$ is either the empty set or an interval with finite end points, where the upper end point is equal to $\beta$. Since $I = (\alpha, \beta)$, then $I \cup J_n = K_n$ is a bounded interval. Besides, if $n$ is negative enough then $J_n \supset I$ and therefore $K_n = J_n$. In conclusion, $g_n$ is supported in a bounded interval $K_n = (\delta, \beta)$ with $w(K_n) < \infty$. We shall estimate the $w$-measure of $K_n$ for very negative values of $n$, i.e., when $K_n = J_n$. In virtue of the first inequality of (3.16) we have

$$J_n \subset \{x : 2^n < c_\gamma \|f\|_\infty 8^{\gamma+1}[M^+ \chi_I(x)]^{\gamma+1}\}.$$ 

By Chebyshev's inequality, if $s > 0$ then

$$w(J_n) \leq c_\gamma \|f\|_\infty^s 2^{-ns} \int [M^+ \chi_I(x)]^{(\gamma+1)s} w(x) \, dx.$$ 

Since the weight $w$ satisfies the hypotheses we can assume that $w \in A^s_I$, with $(\gamma + 1)p > r > 1$. Let $s$ be a real number such that $0 < s < p$ and $(\gamma + 1)s = r > 1$. Then

$$\int [M^+ \chi_I(x)]^{(\gamma+1)s} w(x) \, dx \leq c_{\gamma,s,w} w(I),$$ 

and thus, we obtain

$$w(J_n) \leq c_{\gamma,s,w} \|f\|_\infty^s 2^{-ns} w(I).$$ 

(3.17)
It is easy to verify directly that $\int g_N(x) x^s \, dx = 0$ for $0 \leq s < \gamma$. In fact, adding in (3.11) for $j \geq 1$ and $i > 1$, we have

$$\sum_{i>1} \sum_{j \geq 1} \int f(x) \, x^s \, \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) \, dx =$$

$$(3.18) \quad = \sum_{i>1} \sum_{j \geq 1} \int P_{n,i,j}(f)(x) \, x^s \, \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) \, dx, \quad 0 \leq s < \gamma.$$  

In virtue of (3.12), since $|P_{n,i,j}(f)(x)\eta_{n,i,j}(x)| \leq c \, 2^n$, $\bigcup_{i \geq 1} I_{n,i} = \Omega_n$ (in this case: $I_{n,1} = \emptyset$) and $\Omega_n \subset K_n = (\delta, \beta)$, where $\delta$ is finite, then by Lebesgue’s Dominated Convergence Theorem the right hand side of (3.18) is equal to

$$\int x^s \left[ \sum_{i>1} \sum_{j \geq 1} P_{n,i,j}(f)(x) \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) \right] \, dx.$$  

On the other hand, taking into account that $f$ belongs to $L^\infty$ and that its support is a bounded set, by (3.10) and the Lebesgue’s Dominated Convergence Theorem, the left hand side of (3.18) is equal to

$$\sum_{i>1} \sum_{j \geq 1} \int f(x) x^s \eta_{n,i,j}(x) \chi_{I_{n,i}}(x) \, dx = \sum_{i>1} \int f(x) x^s \chi_{I_{n,i}}(x) \, dx$$

$$= \int_{\Omega_n} f(x) x^s \, dx.$$  

Thus,

$$\int g_n(x) x^s \, dx = \int_{\Omega_n} f(x) x^s \, dx + \int_{\Omega_n} f(x) x^s \, dx = \int f(x) x^s \, dx = 0$$  

holds for $0 \leq s < \gamma$.

Going back to (3.15), we have that if $\ell$ belongs to $BMO_+(p, \gamma, \omega)$ then

$$\int f(x) \ell(x) \, dx = \int_{K_n} \left[ \sum_{n=N}^{M-1} [g_{n+1}(x) - g_n(x)] + g_N(x) \right] \ell(x) \, dx,$$

and, since $\ell$ is integrable on $K_N$, we get

$$\int f(x) \ell(x) \, dx = \sum_{n=N}^{M-1} \int_{K_N} [g_{n+1}(x) - g_n(x)] \ell(x) \, dx + \int_{K_N} g_N(x) \ell(x) \, dx.$$  

If $N$ is negative enough, then $K_N = J_N$ and, from the fact that $g_N$ has null moments up to the order $\gamma - 1$, for the last integral on the right hand side we have

$$\left| \int g_N \ell \right| = \int_{J_N} g_N \ell \, dx = \int_{J_N} g_N [\ell - P_{J_N}(\ell)] \, dx$$

$$\leq c \, 2^N \int_{J_N} |\ell(x) - P_{J_N}(\ell)(x)| \, dx \leq c \, 2^N \|\ell\|_{BMO_+(p, \gamma, \omega)} w(J_N)^{1/p}.$$
which, in virtue of (3.17) is bounded by
\[
c 2^{N(1-s/p)} \|f\|_{\infty}^{s/p} w(I)^{1/p}.
\]
Since \( s < p \) it follows that \( 1 - s/p > 0 \) and then the last expression goes to zero when \( N \) tends to \( -\infty \). This proves that \( \lim_{N \to -\infty} \int g_N \ell \, dx = 0 \), also in this case. Therefore, we always have
\[
\int f(x)\ell(x) \, dx = \sum_{n=1}^{N-1} \int [g_{n+1} - g_n] \ell \, dx.
\]
In the proof of Theorem (2.2), in section 5 of [3], it was shown that
\[
g_{n+1}(x) - g_n(x) = \sum_{i>1} \tilde{A}_{n,i}(x) + \tilde{A}_{n,1}(x),
\]
where the support of the function \( \tilde{A}_{n,i} \) are contained in the connected components \( I_{n,i} \) of \( \Omega_n \), \( \|\tilde{A}_{n,i}\|_{\infty} \leq c 2^n \) and, moreover, if \( i > 1 \) then \( \int \tilde{A}_{n,i}(x)x^s \, dx = 0 \) holds for \( 0 \leq s < \gamma \). Since \( \Omega_n \) is contained in an interval with finite \( w \)-measure \( (x_{-\infty}, \beta) \) or \( J_n \) and, by definition, \( \ell \) is integrable on these intervals, the Lebesgue’s Dominated Convergence Theorem and Remark (1.14) imply that
\[
\left| \int (g_{n+1} - g_n) \ell \, dx \right| = \left| \sum_{i \geq 1} \int \tilde{A}_{n,i}(x)\ell(x) \, dx \right| \\
\leq c 2^n \|\ell\|_{BMO_+(p,\gamma, w)} \sum_{i \geq 1} w(I_{n,i})^{1/p} \\
\leq c 2^n \|\ell\|_{BMO_+(p,\gamma, w)} w(\Omega_n)^{1/p}.
\]
Then
\[
\left| \int f \ell \, dx \right| \leq \sum_n \left| \int (g_{n+1} - g_n) \ell \, dx \right| \\
\leq c \|\ell\|_{BMO_+(p,\gamma, w)} \left[ \sum_{-\infty}^{\infty} 2^{np} w(\Omega_n) \right]^{1/p} \\
\leq c \|\ell\|_{BMO_+(p,\gamma, w)} \|f\|_{H_+^{p,\gamma}(w)},
\]
as we wanted to show. 

**Proof of Theorem (2.3).** We define the application \( i \) as \( i(L) = \ell \), where \( \ell \) is the function associated to \( L \) in part (i) of Theorem (2.1), or \( i(L) = \tilde{\ell} \), where \( \tilde{\ell} \) is the class associated to \( L \) in part (ii) of the same theorem. Since \( D \) is a dense set
in \([H^p_{\gamma, \gamma}(w)]^*\), it follows that \(i\) is an injective application. Taking into account Theorems (2.1) and (2.2) then Theorem (2.3) follows immediately. 

For every non-negative integer \(n\) and every real number \(a > -2^{n+1}\), we define the interval

\[ I_{n,a} = [-2^{n+1}, a] . \]

If \(a = -2^n\) then, we denote by \(I_n\) the interval \(I_{n,-2^n} = [-2^{n+1}, -2^n]\). Moreover, given a function \(\ell\) belonging locally to \(L^1(-\infty, a]\), we denote the orthogonal projections \(P_{I_{n,a}}(\ell)\) and \(P_n(\ell)\) by \(P_{n,a}\) and \(P_n\) respectively.

**Lemma 3.19.** Let \(w \in A^+_\gamma\), \(\gamma\) a positive integer and \(0 < p \leq 1\) such that \((\gamma+1)p \geq q\) if \(q > 1\) or \((\gamma+1)p > 1\) if \(q = 1\). Assume \(x_{-\infty} = -\infty\) and let \(a\) and \(n\) such that \(|a| \leq 2^n\) and \(w((-\infty, a)) < \infty\). If \(\ell\) belongs to \(BMOF^+(p, \gamma, w)\), then

(i) for every \(k > n\), we have

\[
\sup_{x \in I_{n,a}} |P_{k+1}(x) - P_k(x)| \leq c_\gamma \|\ell\|_{BMOF^+(p, \gamma, w)} 2^{-k} w(I_{k+1,a})^{1/p} \]

and,

(ii) \(\sup_{x \in I_{n,a}} |P_{n+1}(x) - P_{n,a}(x)| \leq c_\gamma \|\ell\|_{BMOF^+(p, \gamma, w)} 2^{-n} w(I_{n+1,a})^{1/p} . \)

**Proof.** For \(|a| \leq 2^n\) and \(k > n\), we have

\[ |I_{k+1,a}| \leq 5|I_k| \leq 5|I_{k+1}| . \]

Then, by Lemma (1.13) and since \(I_k \cup I_{k+1} \subset I_{k+1,a}\), we get

\[
\sup_{x \in I_{n,a}} |P_{k+1}(x) - P_k(x)| \leq \sup_{x \in I_{k+1,a}} |P_{k+1}(x) - P_{k+1,a}(x)| + \sup_{x \in I_{k+1,a}} |P_{k+1,a}(x) - P_k(x)| \\
\leq c_\gamma 2^{-k-1} \int_{I_{k+1,a}} |\ell - P_{k+1,a}| dx + c_\gamma 2^{-k} \int_{I_k} |\ell - P_{k+1,a}| dx \\
\leq \frac{3}{2} c_\gamma 2^{-k} \int_{I_{k+1,a}} |\ell - P_{k+1,a}| dx .
\]

Therefore

\[
\sup_{x \in I_{n,a}} |P_{k+1}(x) - P_k(x)| \leq \frac{3}{2} c_\gamma 2^{-k} \|\ell\|_{BMOF^+(p, \gamma, w)} w(I_{k+1,a})^{1/p} ,
\]

\[ \square \]
which is part (i) of the lemma. In order to prove part (ii), we observe that 
\[ |I_{n+1,a}| \leq (5/2)|I_{n+1}| \text{ and } |I_{n+1}| \leq 2|I_{n,a}|, \text{ thus } |I_{n+1,a}| \leq 5|I_{n+1}| \text{ and } |I_{n+1,a}| \leq 5|I_{n,a}|. \]
Then, by Lemma (1.13) and since \( I_{n+1} \cup I_{n,a} \subset I_{n+1,a} \),
\[
\sup_{x \in I_{n,a}} |P_{n+1}(x) - P_{n,a}(x)| \leq
\leq c_\gamma |I_{n+1}|^{-1} \int_{I_{n+1}} |\ell - P_{n+1,a}| dx + c_\gamma |I_{n,a}|^{-1} \int_{I_{n,a}} |\ell - P_{n+1,a}| dx
\leq \frac{3}{2} c_\gamma 2^{-n} \int_{I_{n+1,a}} |\ell - P_{n+1,a}| dx
\leq \frac{3}{2} c_\gamma \|\ell\|_{\text{BMO}_{+}(p,\gamma, w)} 2^{-n} w(I_{n+1,a})^{1/p},
\]
as we wanted to show. \( \Box \)

In order to prove Theorem (2.4) we need the following proposition.

**Proposition 3.20.** Let \( w \in A^1_q \), \( \gamma \) a positive integer and \( 0 < p \leq 1 \) such that 
\( (\gamma + 1)p \geq q \) if \( q > 1 \) or \( (\gamma + 1)p > 1 \) if \( q = 1 \). Assume \( x_{-\infty} = -\infty \) and that there exists \( \beta \) satisfying \( w((-\infty, \beta)) < \infty \). Then, given \( \ell \in \text{BMO}_{+}(p,\gamma, w) / \mathcal{P}_\gamma \) there exists a unique \( \ell' \in \ell \) belonging to \( \text{BMO}_{+}(p,\gamma, w) \) such that
\[
\|\ell'\|_{\text{BMO}_{+}(p,\gamma, w)} \leq c_{p,\gamma} \|\ell\|_{\text{BMO}_{+}(p,\gamma, w) / \mathcal{P}_\gamma}.
\]

**Proof.** Let \( \ell \in \text{BMO}_{+}(p,\gamma, w) \) and \( b \) such that \( w((-\infty, b)) < \infty \). Choose \( m \) such that \( |b| \leq 2^m \). By part (i) of Lemma (3.19), we have that for every \( k > m \)
\[
\sup_{x \in I_{m,b}} |P_{k+1}(x) - P_k(x)| \leq c_\gamma \|\ell\|_{\text{BMO}_{+}(p,\gamma, w)} 2^{-k} w(I_{k+1,b})^{1/p}.
\]
Then given \( i > j > m \)
\[
\sup_{x \in I_{m,b}} |P_i(x) - P_j(x)| \leq \sum_{k=j}^{i-1} \sup_{x \in I_{m,b}} |P_{k+1}(x) - P_k(x)|
\leq c_\gamma \|\ell\|_{\text{BMO}_{+}(p,\gamma, w)} \sum_{k=j}^{i-1} 2^{-k} w(I_{k+1,b})^{1/p}
\leq c_\gamma \|\ell\|_{\text{BMO}_{+}(p,\gamma, w)} w((-\infty, b))^1/p 2^{-j+1}.
\]
This implies that \( \{P_k\}_{k>m} \) is a Cauchy sequence in the Banach space of the continuous functions on \( I_{m,b} \). Therefore there exists a polynomial \( P \in \mathcal{P}_\gamma \) such that
\[
\lim_{k \to \infty} \sup_{x \in I_{m,b}} |P(x) - P_k(x)| = 0.
\]
Thus,
\[
\int_{-\infty}^{b} |\ell - P| dx = \sum_{n=m+1}^{\infty} \int_{I_n} |\ell - P| dx + \int_{-2^{m+1}}^{b} |\ell - P| dx
\]
\[
\leq \sum_{n=m+1}^{\infty} \int_{I_n} |\ell - P_n| dx + \sum_{n=m+1}^{\infty} \int_{I_n} |P_n - P| dx
\]
\[
+ \int_{I_{m,b}} |\ell - P_{m,b}| dx + \int_{I_{m,b}} |P_{m,b} - P| dx
\]
\[
= I + II + III + IV.
\]

Let us estimate $I + III$. By definition of $BMOF_+(p, \gamma, w)$ and recalling that $0 < p \leq 1$, we have
\[
I + III \leq \|\ell\|_{BMOF_+(p, \gamma, w)} \left( \sum_{n=m+1}^{\infty} w(I_n)^{1/p} + w(I_{m,b})^{1/p} \right)
\]
\[
\leq \|\ell\|_{BMOF_+(p, \gamma, w)} w((-\infty, b))^{1/p}.
\]

Next, we shall estimate $II$. We have
\[
II \leq \sum_{n=m+1}^{\infty} \sum_{k=n}^{\infty} \int_{I_n} |P_{k+1} - P_k| dx.
\]

Using part (i) and (ii) of Lemma (3.19) with $a = -2^n$, we get
\[
II \leq c_\gamma \|\ell\|_{BMOF_+(p, \gamma, w)} \sum_{n=m+1}^{\infty} 2^n \sum_{k=n}^{\infty} 2^{-k} w(I_{k+1} \cup \ldots \cup I_n)^{1/p}.
\]

Since $0 < p \leq 1$, the double series on the right hand side is bounded by
\[
\sum_{n=m+1}^{\infty} 2^{np} \sum_{k=n}^{\infty} 2^{-kp} \sum_{j=n}^{k+1} w(I_j)
\leq \sum_{n=m+1}^{\infty} 2^{np} \sum_{j=n}^{\infty} w(I_j) \sum_{k=j}^{\infty} 2^{-kp}
\leq c_p \sum_{n=m+1}^{\infty} 2^{np} \sum_{j=n}^{\infty} 2^{-jp} w(I_j)
\leq c_p \sum_{j=m+1}^{\infty} 2^{-jp} w(I_j) \sum_{n=m+1}^{\infty} 2^{np}
\leq c_p \sum_{j=m+1}^{\infty} w(I_j)^{1/p} \leq c'_p w((-\infty, b))^{1/p}.
\]
Thus,
\[ II \leq c_{p,\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} w((-\infty, b))^{1/p}. \]

Finally, let us estimate IV. We have
\[
\int_{I_{m,b}} |P_{m,b} - P| dx \leq \sum_{k=m+1}^{\infty} \int_{I_{m,b}} |P_{k+1} - P_k| dx + \int_{I_{m,b}} |P_{m+1} - P_{m,b}| dx \\
= A + B.
\]

Using part (i) of Lemma (3.19) with \( n = m \) and \( a = b \), we get
\[
A \leq c_{\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} |I_{m,b}| \sum_{k=m+1}^{\infty} 2^{-k} w(I_{k+1,b})^{1/p} \\
\leq c_{\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} w((-\infty, b))^{1/p},
\]
and using part (ii) of Lemma (3.19), we have
\[
B \leq c_{\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} |I_{m,b}| 2^{-m} w(I_{m,b})^{1/p} \\
\leq c_{\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} w((-\infty, b))^{1/p}.
\]

Let us consider two different values of \( b \), say \( b \) and \( b' \), and let \( P \) and \( P' \) the polynomials obtained above that satisfy
\[
\int_{-\infty}^{b} |\ell - P| dx \leq c w((-\infty, b))^{1/p} < \infty
\]
and
\[
\int_{-\infty}^{b'} |\ell - P'| dx \leq c w((-\infty, b'))^{1/p} < \infty.
\]
Then, if \( \beta = \min(b, b') \), we have
\[
\int_{-\infty}^{\beta} |P - P'| dx \leq \int_{-\infty}^{b} |\ell - P| dx + \int_{-\infty}^{b'} |\ell - P'| dx < \infty.
\]
Thus, \( P - P' \equiv 0 \) showing that there exists a unique \( P \in \mathcal{P}_\gamma \) satisfying
\[
\int_{-\infty}^{b} |\ell - P| dx \leq c_{p,\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} w((-\infty, b))^{1/p}.
\]
Taking \( \ell' = \ell - P \) we find that \( \ell' \in BMO^+(p,\gamma,w) \) and
\[
\|\ell'\|_{BMO^+(p,\gamma,w)} \leq c_{p,\gamma} \|\ell\|_{BMO^+(p,\gamma,w)} / P_\gamma = c_{p,\gamma} \|\ell\|_{BMO^+(p,\gamma,w)/\mathcal{P}_\gamma}.
\]
as we wanted to show.

**Proof of Theorem (2.4).** If we have that \( w((\infty, \beta)) = \infty \) holds for every \( \beta \) then, since \( D_1 = D \), Theorem (2.3) coincides with Theorem (2.4).

Now, let us assume that there exists \( \beta \) satisfying \( w((\infty, \beta)) < \infty \). If \( L \) belongs to \( [H_{p_0, \gamma}^0(w)]^* \), by part (i) of Theorem (2.3), we have that \( i(L) = \ell \in BMO_+(p, \gamma, w) \) and

\[
L(f) = \int \ell(x) f(x) \, dx
\]

holds for every \( f \in D \). If \( f \) belongs to the dense set \( D_1 \), then

\[
L(f) = \int \ell'(x) f(x) \, dx
\]

holds for every \( \ell' \in \tilde{\ell} \in BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma \), since \( \ell - \ell' = \ell \in \mathcal{P} \in \mathcal{P}_\gamma \) and \( \int f(x) P(x) \, dx = 0 \). Then, we can define \( \tilde{i}(L) = \tilde{\ell} \) and, in virtue of part (i) of Theorem (2.3) we obtain that

\[
\|\tilde{\ell}\|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma} = \|\ell\|_{BMOF_+(p, \gamma, w)} \leq \|\tilde{\ell}\|_{BMO_+(p, \gamma, w)} \leq \|L\|.
\]

Thus,

\[
(3.21)
\]

\[
\|\tilde{i}(L)\|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma} \leq \|L\|.
\]

By Proposition (3.20) given a class \( \tilde{\ell} \in BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma \), there exists a unique representative \( \ell' \) such that \( \ell' \in BMO_+(p, \gamma, w) \) and,

\[
(3.22)
\]

\[
\|\ell'\|_{BMO_+(p, \gamma, w)} \leq c_{p, \gamma} \|\tilde{\ell}\|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma}.
\]

Now, by part (i) of Theorem (2.2), the functional

\[
L(f) = \int \ell'(x) f(x) \, dx,
\]

is well defined on the dense set \( D \) and,

\[
(3.23)
\]

\[
\|L\| \leq c_{p, \gamma, w} \|\ell'\|_{BMO_+(p, \gamma, w)}.
\]

Therefore, \( i(L) = \ell' \) and, in consequence, \( \tilde{i}(L) = \tilde{\ell}' = \tilde{\ell} \) showing that \( \tilde{i} \) is a surjective application. Moreover, in virtue of (3.21), (3.23) and (3.22) we have that

\[
c_1 \|\tilde{i}(L)\|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma} \leq \|L\| \leq c_2 \|\tilde{\ell}'\|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma} = c_2 \|\tilde{i}(L)\|_{BMOF_+(p, \gamma, w)/\mathcal{P}_\gamma}.
\]
This finishes the proof. ■

REFERENCES


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