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Validity of the formula "$vp\frac{1}{x}\delta = -\frac{1}{2}\delta'$" from the point of view of Non-Standard Analysis

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Abstract

The objective of this work is to interpret the formula:

$$\text{"}vp\frac{1}{x}\delta = -\frac{1}{2}\delta'\text{"}$$

from the point of view of Nonstandard Analysis. The validity of this identity in the classic sense was established in [1]. For this purpose, we use Takeuchi's space G of nonstandard functions,([2]). This space is an algebra and it contains,in some sense, the classical distributions of Laurent Schwartz ([9]).

1 Preliminaries

We explain here the nonstandard basic concepts. According to Robinson's theory, the system of real numbers $I\!R$ may be view as a subfield of a more ample field, totally ordered called the *hiperreal system numbers* $I\!R^*$.
In order to make this we have included the basic definitions and properties of the theory of filters.

1.1 Filters and Ultrafilters

Definition 1.1 : *A non-empty set $\mathcal{F}$ of subset of a non-empty set X is called a* filter *if it has the following properties:*

(i) If $E \in \mathcal{F}$ and $E \subset F \Rightarrow F \in \mathcal{F}$.

(ii) If $E, F \in \mathcal{F} \Rightarrow E \cap F \in \mathcal{F}$.

(iii) $\emptyset \notin \mathcal{F}$.

Moreover, a filter $\mathcal{F}$ is called ultrafilter *iff*

(iv) If $E \subset X$ then $E \in \mathcal{F}$ or $X - E \in \mathcal{F}$,(but not both, by (ii) and (iii).

Example 1.2 : *If X is an infinite set, the set:*

$$\mathcal{F} = \{A \subseteq X \backslash X - A \; is \; finite\}$$

is a filter called filter of Frèchet *on X.*

Definition 1.3 : *A filter $\mathcal{F}$ on X is called* free *if $\bigcap_{E \in \mathcal{F}} E = \emptyset$.*

Remark 1.4 : *If X is an infinite set, the Frèchet filter on X is free.*

The following result shows the existence of free ultrafilter:

Theorem 1.5 : *For every filter $\mathcal{F}$ on X exists an ultrafilter $\mathcal{U}$ on X which contains to $\mathcal{F}$.*

Corollary 1.6 : *If X is an infinite set then it exists a free ultrafilter on X.*

1.2 The system of hiperreal numbers

Let $I\!N$ be the set of positive integral numbers and let $I\!R^N$ be the set of every sequences of real numbers. Let $< r_1, r_2, \ldots >$ or simply $< r_i >$ denote the elements of $I\!R^N$. We define in $I\!R^N$ the operation of addition and multiplication in the following way:

If $r, s \in I\!R^N$ $\quad r =< r_i > \quad s =< s_i >$,
$r \oplus s =< r_i + s_i >$;
$r \odot s =< r_i . s_i >$.

Thus, $I\!R^N$ be a conmutative ring with an identity $< 1, 1, \ldots >$ and a zero $< 0, 0, \ldots >$.

We introduce in $I\!R^N$ a equivalence relation " $\sim$ " which makes $I\!R^N/\!\sim$ a linearly ordered field. In fact, let $\mathcal{U}$ be a free ultrafilter on $I\!N$:

Definition 1.7 : *If $r = \langle r_i \rangle$ and $s = \langle s_i \rangle$ are in $I\!R^N$, then $r \sim s$ iff $\{i \in I\!N : s_i = r_i\} \in \mathcal{U}$. We then say that $\langle r_i \rangle = \langle s_i \rangle$ almost everywhere (a.e).*

Remark 1.8 : *The relation $\sim$ is an equivalence relation of $I\!R^N$.*

Definition 1.9 : *Let $I\!R^*$ denote the set of all the equivalence classes of $I\!R^N$ induced by " $\sim$ ". The equivalence class containing a particular sequence $s = \langle s_i \rangle$ is denoted by $\langle s \rangle$.*

Elements of $I\!R^*$ are called nonstandard or hyperreal numbers.

Definition 1.10 : *Let $r, s \in I\!R^*$; $r = [\langle r_i \rangle]$; $s = [\langle s_i \rangle]$. Then:*

(i) $r + s = [\langle r_i + s_i \rangle]$

(ii) $r.s = [\langle r_i.s_i \rangle]$

(iii) $r < s$ *iff* $\{i \in I\!N : r_i < s_i\} \in \mathcal{U}$, *and* $r \leq s$ *iff* $r < s$ *or* $r = s$.

Theorem 1.11 : *$I\!R^*$ with the operations defined in 1.10 is a linearly ordered field.*

We define now a mapping $* : I\!R \to I\!R^*$ as follows:

Definition 1.12 : *If $r \in I\!R$, we define $*(r) = *r$ where $*r = [< r, r, \ldots >]$*

Thus, $I\!R^*$ contains a isomorphic copy to $I\!R$ because $* : I\!R \to I\!R^*$ is an order-preserving isomorphism. If $(I\!R)_* = \{*r : r \in I\!R\}$ then $(I\!R)_*$ is the set of standard numbers of $I\!R^*$ and we will identify with $I\!R$. $I\!R^*$ contains numbers other than standard numbers, for example $w = [< 1, 2, 3, \ldots >]$ and $\frac{1}{w}$.

Definition 1.13 : *If $s \in I\!R^*$, we define the* absolute value *of s as follows:*

$$| s | = \begin{cases} s & if \;\; s \geq 0, \\ -s & if \;\; s < 0. \end{cases}$$

Definition 1.14 :

(i) A number $s \in I\!R^*$ will be called infinite number *if $| s | > n$ for all $n \in I\!N$.*

(ii) A number $s \in I\!R^*$ will be called finite number *if $| s | < n$ for any $n \in I\!N$.*

(iii) A number $s \in I\!R^*$ will be called infinitesimal number *if $| s | < \frac{1}{n}$ for all $n \in I\!N$.*

Definition 1.15 : *Let $x, y \in I\!R^*$, we say that x and y are* infinitely close *and we denote $x \approx y$ if $x - y$ is an infinitesimal number.*

Remark 1.16 : *By definition 1.15 we conclude that if $s \in I\!R^*$ is infinitesimal then $s \approx 0$.*

Theorem 1.17 : *If $x \in I\!R^*$ is finite, there is a unique standard number $r \in I\!R$ with the property $x \approx r$.*

Definition 1.18 : *If $x \in I\!R^*$ is finite, the unique standard number $r \in I\!R$ with $x \approx r$ we call standard part of x and we denote $st(x) = r$.*

Definition 1.19 :

(i) We call $G(0)$ *the subset of* $\mathbb{R}^*$ *of all finite numbers.*

(ii) We call standard part map, the map:

$$\begin{array}{rcl} st : G(0) & \rightarrow & \mathbb{R} \\ r & \rightarrow & st(r) \end{array}$$

Theorem 1.20 : *The map st is an order-preserving homomorphism of* $G(0)$ *onto* $\mathbb{R}$*, i.e:*

(i) $st(x \pm y) = st(x) \pm st(y)$,

(ii) $st(x.y) = st(x).st(y)$,

(iii) $st(\frac{x}{y}) = \frac{st(x)}{st(y)}$, *if* $st(y) \neq 0$,

(iv) $st(x) < st(y)$ *if* $x < y$.

The following lemma shows that there is a reasonable relationship between the asymptotic behavior of $\{a_n\}$ and the value of $a = [< a_n >]$,
$a \in \mathbb{R}^*$.

Lemma 1.21 : *If a sequence of real numbers* $\{s_n\}$ *has limit* L *then* $L \approx s = [< s_n >]$.

1.3 The algebra G of Generalized Functions

Definition 1.22 :

(i) A sequence $\{A_n\}$ *of subsets of* $\mathbb{R}$ *defines a subset* (A_n) *of* $\mathbb{R}^*$ *by:*

$$[< x_n >] \in (A_n) \qquad \text{iff} \qquad \{n : x_n \in A_n\} \in \mathcal{U}.$$

The subset of $\mathbb{R}^*$ *which can be obtained in this way is called* internal.

(ii) A sequence of functions $\{f_n\}$, $f_n : \mathbb{R} \rightarrow \mathbb{R}$, *defines a function* $(f_n) : \mathbb{R}^* \rightarrow \mathbb{R}^*$ *in the following way:*

$$(f_n)([< x_n >]) = [< f_n(x_n) >],$$

and any function on $\mathbb{R}^*$ *which can be obtained in this way is called* internal.

A function $f : \mathbb{R}^* \rightarrow \mathbb{R}^*$ is called *nonstandard function*.

Our interest will be to study the nonstandard functions obtained from a sequence $\{f_n\}$ of real functions, i.e, the *nonstandard internal functions* defined in 1.22.
According to Yu Takeuchi in [2], we define:

Definition 1.23 : *We called G the set of all nonstandard internal functions.*

Thus, if $g \in G$ then there exists a sequence $\{g_n\}$ of real functions, $g_n : I\!R \to I\!R$, so that $g = (g_n)$.

Theorem 1.24 : *Let $f, g \in G$, $f = (f_n)$ and $g = (g_n)$. Then, the following statements are equivalent:*

(i) $f = g$,

(ii) $\{n : f_n = g_n\} \in \mathcal{U}$,

(iii) There are sequences of real functions $\{\hat{f}_n\}$ and $\{\hat{g}_n\}$ so that $f = (\hat{f}_n)$ and $g = (\hat{g}_n)$ and $\hat{f}_n = \hat{g}_n$ for all $n \in I\!N$.

We define addition, product and product by a number in $I\!R^*$ in G, in the following way:

Let $f = (f_n)$, $g = (g_n)$, $\gamma \in I\!R^*$, then:

- $f + g = (f_n + g_n)$,
- $f.g = (f_n.g_n)$,
- $\gamma.f = (c_n.f_n)$ if $\gamma = [\langle c_n \rangle]$.

Moreover, if we consider a sequence of functions $\{f_n\}$ where for each n, $f_n(x) = 1$ for all $x \in I\!R$, then $I = (f_n)$ is the unit in G. The function $0 \in G$ generated by the sequence of null functions is the neutral element of the addition. In this way G is a commutative algebra with a unit and zero.

Example 1.25 : *Let $f : I\!R \to I\!R$. The map $f^* \in G$ generated by the sequence $\langle f, f, \ldots \rangle$ will be called "canonical extention of f".*

Example 1.26 : *The functions $g : I\!R^* \to I\!R^*$ defined by $g(x) = \gamma$, $\gamma \in I\!R^*$, where γ is fixed, belong to G. In fact if $\gamma = [\langle c_n \rangle]$ then $g = (g_n)$ with $g_n = c_n \, \forall n \in I\!N$, $\forall x \in I\!R$.*

Example 1.27 : *Let $\delta : I\!R^* \to I\!R^*$ defined by:*

$$\delta(\tau) = \begin{cases} \frac{1}{2\varepsilon} & \text{if } -\varepsilon < \tau < \varepsilon; \\ 0 & \text{if } |\tau| \geq \varepsilon, \end{cases}$$

where $\varepsilon = [\langle e_n \rangle]$ is a positive infinitesimal number (≥ 0 in $I\!R^*$). Then $\delta = (\delta_n)$ where $\delta_n : I\!R \to I\!R$ are defined by:

$$\delta_n(x) = \begin{cases} \frac{1}{2e_n} & \text{if } -e_n < x < e_n; \\ 0 & \text{if } |x| \geq e_n. \end{cases}$$

The function thus defined is by no means the canonical extention for some real function.

Remark 1.28 : *There exist nonstandard functions which are not in G (cf.[2], Theorem 1, p.124).*

Theorem 1.29 : *Let $f \in G, f = (f_n)$. If f_n converges to f_0 uniformly in $[a, b]$ then $f(\tau) \approx f_0(\tau)$ for any $\tau \in I\!R^*$, $a \leq \tau \leq b$, (a and b are not necessarily finite).*

1.4 Continuity of functions in G

Let $f : I\!R \to I\!R$ be a real function. In Nonstandard Analysis the continuity of a real function is equivalent to the following fact:

$$f^*(a + \varepsilon) = f^*(a) \qquad \forall \varepsilon \approx 0$$

where f^* is the canonical extension of the real function f.

Definition 1.30 : *Let $f : I\!R^* \to I\!R^*$ a function in G. Then f is continuous in $\alpha \in I\!R^*$ if:*

$$f(\alpha + \epsilon) \approx f(\alpha) \qquad \forall \epsilon \approx 0.$$

Remark 1.31 :

- *The canonical extension of a continuous real function is continuous.*
- *The nonstandard function generated by a sequence of equicontinuous real functions, is continuous.*

1.5 Differentiation of functions in G

Definition 1.32 : *Let $f = (f_n)$, then f is differentiable if*

$$\{n : f_n \text{ is piecewise smooth}\} \in \mathcal{U}.$$

We assume that a function g is piecewice smooth if it is continuous and differentiable with continuous differential except for a finite number of point, in which we consider $g'(c) = \lim_{x \to 0^+} g'(x)$.

Definition 1.33 : *If $f = (f_n)$ is differentiable, the function $f' \in G$ defined by:*

$$f' = (f'_n)$$

is called the differential of f.

Example 1.34 : *Let:*

$$H_n(x) = \begin{cases} 1 & \text{if} \quad \frac{1}{n} \leq x, \\ nx & \text{if} \quad 0 \leq x < \frac{1}{n}, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Then $H = (H_n) \in G$ *is given by:*

$$H(\tau) = \begin{cases} 1 & if \quad \frac{1}{\lambda} \leq \tau, \\ \lambda\tau & if \quad 0 \leq \tau < \frac{1}{\lambda}, \\ 0 & if \quad \tau < 0, \end{cases}$$

where $\lambda = [\langle n \rangle]$. *Then, it is valid:*

$$H'(\tau) = \begin{cases} \lambda & if \quad 0 \leq \tau < \frac{1}{\lambda}, \\ 0 & if \quad \tau < 0 \ and \ \tau \geq \frac{1}{\lambda} \end{cases}$$

We observe that H' *is not a continuous function because* $H'(0) = \lambda$ *and* $H'(-\frac{1}{\lambda}) = 0$ *and* $-\frac{1}{\lambda} \approx 0$.

1.6 Integration of functions in G

Definition 1.35 : *Let* $f \in G, f = (f_n)$. *If*

$$\{n : f_n \text{ is integrable in any interval}\} \in \mathcal{U},$$

we shall say that the nonstandard function f *is* integrable.

Definition 1.36 : *Let* $f = (f_n) \in G$ *be an integrable function. For* $\alpha = [\langle \alpha_n \rangle]$ *and* $\beta = [\langle \beta_n \rangle]$ *with* $\alpha < \beta$, *we define the* definite integral *of* f *in the following way:*

$$\int_{\alpha}^{\beta} f(\tau)d\tau = [\langle \int_{\alpha_n}^{\beta_n} f_n(x)dx \rangle].$$

Example 1.37 : *Let*

$$\delta(\tau) = \begin{cases} \lambda & if \quad 0 < \tau < \lambda, \\ 0 & in \ any \ other \ case, \end{cases}$$

where λ *is an infinite positive number. Then:*

$$\int_{\alpha}^{\beta} \delta(\tau)d\tau = 1 \qquad and \qquad \int_{\alpha}^{\beta} \delta^2(\tau)d\tau = \lambda,$$

for any $\alpha < 0, \beta > 0$ *non infinitesimals.*

1.7 Primitive of a function in G

Definition 1.38 :

- *A differentiable function* $F \in G$ *is a primitive of the function* $f \in G$ *if:*

$$F' = f,$$

- *A function* F *is called primitive of order* k, $k \geq 0$, *of the function* f *if:*

$$F^{(k)} = f.$$

1.8 NonStandard Model of the Dirac's delta function

Definition 1.39 *: A function $\delta \in G$ is a* NonStandard Model of the Dirac's delta function *if it has the following properties:*

(i) $\delta(\tau) \geq 0, \quad \forall \tau \in I\!R^*$ *and* $\delta(\tau) \approx 0 \quad \forall \tau$ *non infinitesimal.*

(ii) $\int_\alpha^\beta \delta(\tau)d\tau \approx 1$ *for* $\alpha, \beta \in I\!R^*, \alpha < 0$ *and* $\beta > 0$ *non infinitesimals.*

Theorem 1.40 *: If $f \in G$ is of finite value (i.e $f(x) \in G(0), \forall x \in I\!R^*$) and continuous in 0 then:*

$$\int_c^d f(\tau)\delta(\tau)d\tau \approx f(0)$$

where $st(c) < 0, st(d) > 0$.

1.9 NonStandard representation of Distributions

The following theorem deals with the sufficient conditions which must be performed by a nonstandard function in G in order to determine a distribution:

Theorem 1.41 *: (Takeuchi,[2], Teor.9,pg.144) Let $h \in G$ be a nonstandard function so that given a finite interval exists some primitive bounded in it (of order $k, k \geq 0$). Then the function h determine a distribution by:*

$$\langle h, f\rangle = st \int_{-\infty}^{\infty} h(\tau)\phi^*(\tau)d\tau \tag{1}$$

where ϕ^* is the canonical extension of $\phi \in \mathcal{D}$.

Theorem 1.42 *: Let $h \in G$ be a C^∞ nonstandard function which satisfies the conditions of Theorem 1.41, thus h determines a distribution by formula (1). Then the differential of order i of the distribution h, $h^{(i)} \in \mathcal{D}'$ is equal to the distribution determined by the nonstandard function $\mathcal{D}^i h \in G$, that is:*

$$\langle h^{(i)}, f\rangle = st \int_{-\infty}^{\infty} D^i h(\tau) f^*(\tau)d\tau,$$

where f^* is the canonical extension of f in $\mathcal{D}$.

Remark 1.43 *: There exist nonstandard functions $h \in G$ which are not distributions. For example, if $\delta \in G$ is a nonstandard model of the Dirac's delta, then $\delta^2, \delta^3, \ldots..$ are functions in G which are not distributions.*

The following theorem asserts that all classic distributions can be represented by a function in G.

Theorem 1.44 *: Let $T \in \mathcal{D}'$, then it exists a nonstandard function $h \in G$ such that:*

$$\langle T, f\rangle = st \int_{-\infty}^{\infty} h(\tau) f^*(\tau)d\tau$$

for all $f \in \mathcal{D}$, where f^* is the canonical extension of f.

2 Validity of the formula "$vp\frac{1}{x}\delta = -\frac{1}{2}\delta'$" in the nonstandard space G

The classic formula is due to A. González Domínguez and R. Scarffiello [1]. They have considered singular kernels $\{g_n\}$ which satisfy:

(i) $\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dx = 1$,

(ii) $\int_{-\infty}^{\infty} \mid g_n(x) \mid dx < M$,

(iii) $\lim_{n\to\infty} \int_I \mid g_n(x) \mid dx = 0$ for all interval I so that $0 \notin I$.

Moreover if

(iv) $h_n = g_n(x) * vp\frac{1}{x} = \int_{-\infty}^{\infty} \frac{g_n(y)}{x-y} dy$,

we order that $\mid xh_n(x) \mid < M$ and that g_n has a bounded derivative for every n.

To turn the formula "$vp\frac{1}{x}\delta = -\frac{1}{2}\delta'$" into nonstandard language we have to see under what conditions a sequence that verifies the conditions (i), (ii) and (iii), generates a nonstandard model for Dirac's delta, according with the model proposed by Yu Takeuchi ([2], §6, pg. 139).

Proposition 2.1 : *The conditions* (i) *and* (iii) *involves that* $\int_\alpha^\beta \delta(\tau) \approx 1$ *where* $\delta \in G, \delta = (g_n), \alpha < 0$ *non infinitesimal and* $\beta > 0$ *non infinitesimal.*

Proof. Let $\alpha, \beta \in I\!R^*$ of the form $\alpha = [-n], \beta = [n]$ where $n \in I\!N$. We shall show that:

$$st \int_\alpha^\beta \delta(\tau)d\tau = 1. \tag{2}$$

By definition:

$$\int_\alpha^\beta \delta(\tau)d\tau = [\langle \int_{-n}^n g_n(\tau)d\tau \rangle]. \tag{3}$$

For each n, we have

$$\int_{-n}^n g_n(x)dx = \int_{-\infty}^{\infty} g_n(x)dx - \int_{|x|>n} g_n(x)dx, \tag{4}$$

then

$$\mid \int_{|x|>n} g_n(x)dx \mid \leq \int_{|x|>n} \mid g_n(x) \mid dx = \int_{-\infty}^{-n} \mid g_n(x) \mid dx + \int_n^{\infty} \mid g_n(x) \mid dx. \tag{5}$$

We choose $a \in I\!R, 0 < a < 1$, then $a < n, \forall n \in I\!N$. Taking into account $(-\infty, -n) \subset (-\infty, -a)$ and $(n, \infty) \subset (a, \infty)$, then:

$$\int_{-\infty}^{-n} \mid g_n(x) \mid dx + \int_{n}^{\infty} \mid g_n(x) \mid dx \leq \int_{-\infty}^{-a} \mid g_n(x) \mid dx + \int_{a}^{\infty} \mid g_n(x) \mid dx. \qquad (6)$$

The inequality (6) holds for every $n \in I\!N$ so, that passing to the limits in both sides, we obtain that the left hand side tends to zero (from (iii)), and we deduce that

$$\lim_{n \to \infty} \int_{|x|>n} g_n(x) dx = 0 \qquad (7)$$

Then, we obtain from (4) and (i):

$$\lim_{n \to \infty} \int_{-n}^{n} g_n(x) dx = 1 \qquad (8)$$

and so (2) holds.†

We shall prove now the other condition required in order to the nonstandard function $\delta = (g_n)$ be a nonstandard model of Dirac's delta, i.e:

$$\delta(\tau) \approx 0, \; \forall \tau, \; \tau \not\approx 0, \; \tau \in I\!R^*. \qquad (9)$$

For it, the sequence $\{g_n\}_{n \in I\!N}$ must converges uniformly to the null function on each interval I that $0 \notin I$. This last fact involves the validity of condition (iii), i.e:

$$\lim_{n \to \infty} \int_I \mid g_n(x) \mid dx = 0, \text{ for all interval I so that } 0 \notin I.$$

So, we asume the uniform convergence of the sequence $\{g_n\}$ for all interval I so that $0 \notin I$. Let be $\tau \in I\!R^*, \tau \not\approx 0$. Then $\mid \tau \mid > c, c \in I\!R^+$. Given the uniform convergence of $\{g_n\}$ in $[c, \infty)$, by theorem 1.29 we obtain:

$$\delta(\tau) \approx 0, \; \forall \tau \in I\!R^*, \; \tau \geq c.$$

The same happens if $\tau < -c$. So, (9) is valid. Moreover, if the singular kernel $\{g_n\}$ satisfies the condition $g_n(x) \geq 0, \; \forall x \in I\!R$ then $\delta(\tau) \geq 0, \; \forall \tau \in I\!R^*$.

To conclude, if the singular kernel $\{g_n\}$ satisfies the additional conditions:

- $g_n(x) \geq 0, \; \forall x \in I\!R$
- $g_n \stackrel{\cdot}{\longrightarrow} 0, \; \forall I$ so that $0 \notin I$

the funtion $\delta \in G, \delta = (g_n)$ is a nonstandard model of Dirac's delta, according to the definition (1.39). We consider now the sequence $\{h_n\}$ (iv) which converges weakly to $vp\frac{1}{x}$, i.e:

$$\lim_{n \to \infty} \int_{-a}^{a} h_n(x) \phi(x) dx = (vp\frac{1}{x}, \phi); \; supp\phi \subset (-a, a).$$

This means that the nonstandard function $Vp = (h_n) \in G$ is a nonstandard model to the distribution $vp\frac{1}{x}$ in the sense that:

$$st \int_{-\infty}^{\infty} Vp(\tau)\phi^*(\tau)d\tau = (vp\frac{1}{x}, \phi)$$

where ϕ^* is the canonical extension of $\phi \in \mathcal{D}$. According to the definition of product in G, we obtain:

$$\delta.Vp = (g_n)(h_n) = (g_n h_n),$$

the sequence $k_n = g_n.h_n$ converges weakly to $-\frac{1}{2}\delta'$,(cf.,[1]). Then the nonstandard function $K \in G, K = (k_n)$ is a nonstandard model of the distribution $-\frac{1}{2}\delta'$, and is valid:

$$\delta.Vp = K$$

References

[1] A. González Domínguez y Roque Scarfiello: Nota sobre la fórmula "$vp\frac{1}{x}\delta = -\frac{1}{2}\delta'$". Revista de la Unión Matemática Argentina. Vol. 17, pg. 53-66, 1955.

[2] Yu Takeuchi: Funciones No-Estándar y Teoría de Distribuciones. Revista Colombiana de Matemáticas. Vol. XVII, pg. 117-151, 1983.

[3] A. Robinson: Nonstandard Analysis. North-Holland Publ., Amsterdam, 1966.

[4] José I. Téllez: Un modelo No-Estándar de Funciones Generalizadas. Thesis work to the Magister Scientiae in the Facultad de Ciencias, Departamento de Matemáticas y Estadística, Universidad Nacional de Colombia , directed by Yu Takeuchi, Bogotá, 1982.

[5] Gaisi Takeuti: Dirac Space. Proc. Japan Acad.38, pg. 414-418, 1962.

[6] A.E.Hurd; P.A.Loeb: An Introduction to Non-Standard Real Analysis. Academic Press, 1985.

[7] W.A.J.Luxemburg: Non-Standard Analysis. Mathematics Department. California Institute of Technology. Pasadena, California, 1973.

[8] S. Albeverio; R. Hoegh-Krohn; Jens Erik Fenstad; Tom Lindstrom: Nonstandard Methods in Stochastic Analysis and Mathematical Physics, Academic Press, 1986.

[9] Laurent Schwartz: Théorie des Distributions. Ed. Hermann, Paris, 1966

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