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JONES INDEX THEORY FOR TYPE II_∞ VON NEUMANN ALGEBRAS

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Abstract

R. Longo's definition of index ([L2]) is extended to the case where the involved algebras are not factors, assuming they are of type II_∞. Main tools are generalizations of technics used by R. Longo. It is shown that our definition agrees with that of Baillet, Denizeau and Havet for von Neumann algebras, and also that it is equivalent to the one given in [AS] by E. Andruchow y D. Stojanoff. We obtain some properties about the tower and the tunnel of the inclusion. Also the techniques involved allow us to prove some known results, generally straightforwardly. Results obtained are applied to inclusion of type III von Neumann algebras with separable predual.

1 INTRODUCTION

In his work about Jones' index theory for inclusions of factors ([L2, L3]), R. Longo has developed a new definition of index and also techniques used to prove many results, all based strongly on the fact that the factors involved are properly infinite. In this paper we extend those techniques and the Index definition to the case where the inclusion is not any more of factors, but of arbitrary type II_∞ von Neumann algebras with separable predual. This type of algebras are a natural place where to make this extension, as they are properly infinite and they have a faithful normal semifinite trace, both of this facts being essential assumptions for the mathematics involved in the proofs.
The generalization made forces the index to be no more a scalar but a positive invertible operator of the center of the algebra, as in the work of Y. Watatani ([Wat]) and Baillet, Denizeau y Havet ([BDH]).
Main techniques used are the canonical endomorphism of R. Longo ([L2, L3]) and the existence of a joint cyclic and separating vector for each of the algebras of

the inclusion. This last condition is guaranteed by the property of the algebras of being infinite. These techniques allow us to get new proofs of known results from [BDH, Popa], and let us get more information about the basic construction, the tower and the tunnel, in particular the relation stablished for each pair of indices of consecutive inclusions of the tower (Theorems 4.3 and 4.4).

Let $\mathcal{B} \subseteq \mathcal{A}$ be von Neumann algebras of type II_∞, with separable predual. Being $\mathcal{A}$ is of type II_∞, it is well known that there exists faithful normal semifinite trace τ on $\mathcal{A}$.

Suppose moreover that we have a faithful and normal conditional expectation $E : \mathcal{A} \to \mathcal{B}$, such that $\tau \cdot E = \tau$. Theorem 2.1 assures that τ is also semifinite in $\mathcal{B}$.

As $\mathcal{A}$ and $\mathcal{B}$ are properly infinite, it is known by the classical standard representation theory that we can consider $\mathcal{A}$ and $\mathcal{B}$ acting over a Hilbert space $\mathcal{H}$ where there is a joint cyclic and separating vector $\Omega \in \mathcal{H}$ for $\mathcal{A}$ and $\mathcal{B}$. We call

$$\varphi = < \cdot\, \Omega, \Omega > \in \mathcal{A}_*, \tag{1.1}$$

and

$$\phi = \varphi \cdot E \in \mathcal{A}_*^+. \tag{1.2}$$

We call the *standard cone of* $\mathcal{A}$ the set (see [DL]):

$$P_\Omega(\mathcal{A}) = \{J_\mathcal{A} a J_\mathcal{A} a \Omega, a \in \mathcal{A}\}^-$$

By [DL] A.3, there is a positive vector representing ϕ, that we will note

$$\xi \in P_\Omega(\mathcal{A}). \tag{1.3}$$

This vector ξ gives rise to the Jones projection in this context: we will define the Jones projection to be the projection with range equal to $[\mathcal{B}\xi]$.

In [L1, L2], R. Longo introduces what he calls the "canonical endomorphism" of $\mathcal{A}$, noted γ, the following way:

$$\gamma(x) = J_\mathcal{B}\, J_\mathcal{A}\, x\, J_\mathcal{A}\, J_\mathcal{B} \qquad x \in \mathcal{A}, \tag{1.4}$$

where $J_\mathcal{A}$ and $J_\mathcal{B}$ are modular conjugations of $\mathcal{A}$ and $\mathcal{B}$ respectively with respect to Ω.

The Index will be defined to be the Radon-Nikodym derivative (see [PT]) of the tracial weight $\tau \cdot \gamma$ with respect to τ, in the sense of Pedersen and Takesaki (Theorem 2.2). This will give us an operator with the required properties, that is to be central and invertible.

In Section 2 we state some results needed in many of the proofs.

In Section 3 we define the Index and show that is has the usual properties

In Section 4 we show some special properties of the Index, specially the ones concerning the tower and tunnel of the inclusion.

In section 5 we consider the relation between the newly defined Index and the other definitions (specially [BDH]), showing that it extends the previous considered cases. Main Theorem is 5.6, where equivalence between the finite index notion is stated for the case considered.

Finally, in section 6 we apply Theorem 5.6 to obtain some information both in the type II_1 and type III case.

2 PRELIMINARIES

In this section we state well known technical results that will be used troughout this paper.
We start with Takesaki's theorem stating the existence of a conditional expectation ([Tak 2]).

Theorem 2.1 *(M. Takesaki) Let N be a von Neumann subalgebra of the algebra M and φ a normal faithful semifinite weight on M. Let σ_t^φ be the modular group of M with respect to φ. The following conditions are equivalent:*

(i) the faithful normal weight $\varphi|N^+$ is semifinite and $\sigma_t^\varphi(N) \doteq N$ for every $t \in I\!R$;

(ii) there exists a faithful normal conditional expectation $E : M \to N$ such that

$$\varphi(x) = \varphi(E(x)) \quad (x \in M^+).$$

Condition (i) determines uniquely the faithful normal conditional expectation $E : M \to N$.

Another result that will be useful to us is the Radon-Nikodym theorem of Pedersen and Takesaki ([PT]).
Before stating the theorem we recall that given a normal weight ρ and a positive selfadjoint operator k (possibly not bounded), we define

$$\rho(kx) = \lim_{\delta \to 0} \rho(k_\delta^{1/2} x k_\delta^{1/2}),$$

where $k_\delta = k(1+\delta k)^{-1}$, and it is known that the limit exists because the net is increasing.
We call

$$M^{\sigma^\varphi} = \{x \in M : \sigma^\varphi(x) = x\}.$$

Theorem 2.2 *(G. Pedersen and M. Takesaki) Let φ be a normal faithful semifinite weight on a von Neumann algebra M. If ψ is a σ^φ-invariant normal semifinite weight on M then there is a unique selfadjoint positive operator h affiliated with M^{σ^φ} such that $\psi = \varphi(h\cdot)$.*

When φ is a trace, the set M^φ is all M and the operator h is affiliated with the center $\mathcal{Z}(M)$ of M.

3 DEFINITION OF INDEX

Let $\mathcal{B} \subseteq \mathcal{A}$ an inclusion of type II_∞ von Neumann algebras. Consider an expectation $E : \mathcal{A} \to \mathcal{B}$ and a faithful normal semifinite trace τ on $\mathcal{A}$ such that E commutes with τ.
We recall that we are considering $\mathcal{A}$ and $\mathcal{B}$ acting over a Hilbert space $\mathcal{H}$ where there is a joint cyclic and separating vector Ω for $\mathcal{A}$ and $\mathcal{B}$. This is the main advantage obtained in restricting ourselves to the infinite case.

Let $J_{\mathcal{A}}$ be the modular conjugation of $\mathcal{A}$ and $J_{\mathcal{B}}$ the modular conjugation of $\mathcal{B}$. From now on we will also suppose that $\tau \cdot \gamma$ is semifinite. If this condition does not occur, we will say that the index is infinite.
As τ y $\tau \cdot \gamma$ are normal weights and τ is also tracial and semifinite, we can apply theorem 2.2 to obtain a unique invertible operator, selfadjoint and positive $h_{\mathcal{A}}$, affiliatted to center of $\mathcal{A}$, such that

$$\tau \cdot \gamma(x) = \tau(h_{\mathcal{A}} x). \tag{3.1}$$

It is shown in Lemma 2.1 of [L2] that $h_{\mathcal{A}}$ does not depend on the bicyclic vector chosen. This operator $h_{\mathcal{A}}$ is our candidate to be the Index.

Definition 3.1 *If the operator $h_{\mathcal{A}}$ is bounded, we will say that the expectation E has finite index, and we will call the operator $Ind(E) = h_{\mathcal{A}}$ the Index of the expectation E. If $h_{\mathcal{A}}$ is not bounded, we will say that E has infinite Index.*

Let us define a projection

$$p = [\mathcal{B}\xi] \in \mathcal{B}', \tag{3.2}$$

with ξ as in 1.3. We will state as a Lemma, without proof, the following result, that appears in the proof of 2.1 (see, for instance, [St], 10.2):

Lemma 3.2 *With the above notations, $J_{\mathcal{A}} p J_{\mathcal{A}} = p$.*

As the state ϕ defined in 1.2 is in $\mathcal{B}_*$ and it is faithful, the vector ξ is separating for $\mathcal{B}$, and then the homomorphism

$$\begin{aligned} \Phi : \mathcal{B} &\to \mathcal{B}p \\ x &\mapsto xp \end{aligned} \tag{3.3}$$

is an isomorphism.

Definition 3.3 *We say that the projection p of equation 3.2 is the Jones projection associated to the expectation E.*

Remark 3.4 *The projection $p \in L(\mathcal{H})$ satisfies*

$$p(x\xi) = E(x)\xi. \tag{3.4}$$

Definition 3.5 *The extension of $\mathcal{A}$ by E is the algebra $\mathcal{M} =< \mathcal{A}, p >$.*

Now we can extend τ to a trace $\tilde{\tau}$ of $\mathcal{M}$ in the following way: as $J_{\mathcal{M}} = J_{\mathcal{A}} J_{\mathcal{B}} J_{\mathcal{A}}$, the same γ of $\mathcal{A}$ is also the canonical endomorphism of $\mathcal{M}$. As $h_{\mathcal{A}}$ is invertible, we can define, for $x \in \mathcal{M}$,

$$\tilde{\tau}(x) = \tau(\gamma(h_{\mathcal{A}}^{-1} x)). \tag{3.5}$$

Proposition 3.6 *$\tilde{\tau}$ is a semifinite trace in $\mathcal{M}$ extending τ, and $\tilde{\tau} \cdot \gamma$ is also a semifinite trace.*

Proof. Consider two elements $x, y \in \mathcal{M}$. Then we have

$$\tilde{\tau}(xy) = \tau(\gamma(h_{\mathcal{A}}^{-1}xy) = \tau(\gamma(h_{\mathcal{A}}^{-1})\gamma(x)\gamma(y)).$$

As $h_{\mathcal{A}}^{-1}$ is central in $\mathcal{A}$ and τ is a trace, we have

$$\tilde{\tau}(xy) = \tau(\gamma(h_{\mathcal{A}}^{-1})\gamma(y)\gamma(x)) = \tilde{\tau}(yx).$$

Given $x \in \mathcal{A}$,

$$\begin{aligned}\tilde{\tau}(x) &= \tau(\gamma(h_{\mathcal{A}}^{-1}x)) = \lim_{\delta\to 0}\tau(\gamma((h_{\mathcal{A}}^{-1})_\delta x) = \\ &= \lim_{\delta\to 0}\tau(h_{\mathcal{A}}(h_{\mathcal{A}}^{-1})_\delta x) = \tau(x),\end{aligned}$$

so that $\tilde{\tau}$ extends τ.
If K is the σ-WOT dense subspace of $\mathcal{A}$ where τ is finite, it is easily verified that $K + \sum KpK$ is dense in $\mathcal{M}_0$, and then it will be enough to see that $\tilde{\tau}|_{KpK}$ is finite. Indeed, we must prove that if we have an element apb with $a, b \in \mathcal{A}$, there is a net $\{a_i p b_i\}$ in KpK that converges σ-WOT to apb. By density of K in $\mathcal{A}$ and Kaplansky's density theorem, we can assume there are bounded nets $\{a_i\}$ and $\{b_i\}$ with

$$a_i \to a, \;\; b_i \to b$$

and

$$\|a_i\| \leq c \;\text{ for every } i$$

in the σ-WOT topology. So, if μ is a vector in the underlying Hilbert space $\mathcal{H}$, it suffices to show convergence in the SOT topology, as all the nets and operators involved are bounded. Then

$$\begin{aligned}\|(apb - a_i p b_i)\mu\| &= \|(apb - a_i pb + a_i pb - a_i p b_i)\mu\| \\ &\leq \|(a - a_i)pb\mu\| + \|a_i p(b - b_i)\mu\| \\ &\leq \|(a - a_i)pb\mu\| + c\|(b - b_i)\mu\| \to 0.\end{aligned}$$

Now let us see that $\tilde{\tau}|_{KpK}$ is finite. If $a, b \in K$, we use the polar identity for the bilineal map $(a, b) \mapsto apb$, $apb = \frac{1}{2}(a + b^*)p(a + b^*)^* + \frac{1}{2}(a - ib^*)p(a - ib^*)^*$, so that $apb \leq \frac{1}{2}(a + b^*)(a + b^*)^* + \frac{1}{2}(a - ib^*)(a - ib^*)^*$. In each one of the summands, one of the factors is in K by linearity, and if $x \in K$, $x^*x \in K$, so that all the right member is in K, and so $\tilde{\tau}(apb) < \infty$. For $\tilde{\tau}\cdot\gamma$ the dense subspace where it is finite is $\gamma^{-1}(\mathcal{B}_0)$, where $\mathcal{B}_0$ is the σ-WOT dense subspace of $\mathcal{B}$ where τ is finite. $\square$

As $\tilde{\tau}$ is a faithful normal semifinite trace of $\mathcal{M}$, there exists, by Theorem 2.1 a new conditional expectation

$$E_{\mathcal{A}} : \mathcal{M} \to \mathcal{A} \tag{3.6}$$

commuting with $\tilde{\tau}$, and a positive invertible operador $h_{\mathcal{M}}$ of the center of $\mathcal{M}$ in a similar manner as $h_{\mathcal{A}}$ before.

4 Properties of Ind(E)

We start with a Lemma where we stablish general properties similar to those appearing in classical Jones' Index Theory.

Lemma 4.1

1. $\mathcal{M} = J_{\mathcal{A}}\mathcal{B}'J_{\mathcal{A}}$.
2. $\mathcal{B} = \gamma(\mathcal{M})$.
3. $\mathcal{M}_0 = \mathcal{A} + \sum \mathcal{A}p\mathcal{A}$ is WOT dense in $\mathcal{M}_0$.

Proof. As $J_{\mathcal{A}}\ p\ J_{\mathcal{A}} = p$ (Lemma 3.2), we can reproduce the proof in Proposition 3.1.5 of [J] to obtain 1 and 3. To see 2, simply note that

$$\gamma(\mathcal{M}) = J_{\mathcal{B}}J_{\mathcal{A}}J_{\mathcal{A}}\mathcal{B}'J_{\mathcal{A}}J_{\mathcal{A}}J_{\mathcal{B}} = J_{\mathcal{B}}\mathcal{B}'J_{\mathcal{B}} = \mathcal{B}.$$

□

As this Lemma does not depend on the existence of a trace, it will remain true for general inclusion of infinite algebras.

Considering $\mathcal{B} \subseteq L(\mathcal{H})$, $\mathcal{B}p \subseteq L(p\mathcal{H})$, it is possible to apply the unitary implementation theorem (see [KR], Theorem 7.2.9) to the isomorphism $\Phi(x) = xp$, to obtain a unitary operator $V : \mathcal{H} \to p\mathcal{H}$ such that

$$\Phi(x) = VxV^*. \tag{4.1}$$

This operator V can be seen as a parcial isometry of $\mathcal{H}$ satisfying

$$VxV^* = xp,\ \ x \in \mathcal{B}, \tag{4.2}$$

and $V^*V = 1$. Moreover,

$$Vx = VxV^*V = xpV = xV, \tag{4.3}$$

so that $V \in \mathcal{B}'$.
If $x \in \mathcal{A}$,

$$E(x) = V^*VE(x) = V^*E(x)V = V^*E(x)pV = V^*pxpV = V^*xV. \tag{4.4}$$

Remark 4.2 *The triple $(id, V, p\mathcal{H})$ is the Stinespring dilation of E.*

We also have (see the proof of Proposition 5.1 of [L2],) that

$$E(x) = V^*xV = J_{\mathcal{B}}\,V^*\,J_{\mathcal{B}}\,\gamma(x)\,J_{\mathcal{B}}\,V\,J_{\mathcal{B}} \tag{4.5}$$

If $(\tau\cdot\gamma)|_{\mathcal{A}}$ is semifinite, we also have that $\tau|_{\gamma(\mathcal{A})}$ is semifinite, and by theorem 2.1 there is a faithful normal conditional expectation

$$E_0 : \mathcal{B} \to \gamma(\mathcal{A}) \tag{4.6}$$

commuting with τ; in particular we can repeat the construction made before to obtain a projection

$$e \in \mathcal{A}, \text{ with } \mathcal{A} = < \mathcal{B}, e >, \; exe = E_0(x)e, x \in \mathcal{B}. \tag{4.7}$$

Note that from the argument above, the canonical endomorphism γ allows us to construct in a canonical way a tunnel for the inclusion, as the properties from the original inclusion are preserved.
The following Theorem stablishes a relation between different indexes from inclusions in the tower. Briefly, it is proved that the expectation E "moves down" the index one level. In particular, it shows that the downward construction inherits the finitness of the index.

Theorem 4.3 *If E has finite Index , then the expectation E_0 defined in equation 4.6 has finite index and $Ind(E_0) = E(Ind(E))$.*

Proof. It is enough to see that if $x \in \mathcal{B}$,

$$\tau(E(h_{\mathcal{A}})x) = \tau(E(h_{\mathcal{A}}x)) = \tau(h_{\mathcal{A}}x) = \tau(\gamma(x)),$$

and that E takes elements from the center of $\mathcal{A}$ in elements of the center of $\mathcal{B}$. By [PT] $h_{\mathcal{B}}$ is unique, so that $E(h_{\mathcal{A}}) = h_{\mathcal{B}}$. □

The following result justifies that it is possible to construct the Jones' tower of the inclusion preserving the main properties: it is shown that finite index in one level implies finite index in every level of the tower. In 1.2.2 of [Popa] one can find a somewhat analogous version of these results related to the weak index.

Theorem 4.4 *If E has finite Index, then the expectation $E_{\mathcal{A}}$ defined in equation 3.6 has finite Index, and moreover $||Ind(E_{\mathcal{A}})|| = ||Ind(E)||$.*

Proof. We will show that $\tilde{\tau} \cdot \gamma \leq ||h_{\mathcal{A}}||\tilde{\tau}$, this implies that $h_{\mathcal{M}}$ is bounded with norm lower that $||h_{\mathcal{A}}||$ by the first part of the proof of Theorem 5.12 of [PT].
Given $x \in \mathcal{M}$,

$$\begin{aligned}
(\tilde{\tau} \cdot \gamma)(x) &= \tau(\gamma(x)) \\
&= \lim_{\epsilon \to 0} \tau(\gamma(h_{\mathcal{A}}(h_{\mathcal{A}}^{-1})_{\epsilon}x)) \leq \\
&\leq ||h_{\mathcal{A}}|| \lim_{\epsilon \to 0} \tau(\gamma((h_{\mathcal{A}}^{-1})_{\epsilon}x)) \\
&= ||h_{\mathcal{A}}||\tau(\gamma(h_{\mathcal{A}}^{-1}x)) \\
&= ||h_{\mathcal{A}}||\tilde{\tau}(x).
\end{aligned}$$

We have shown then that $||h_{\mathcal{M}}|| \leq ||h_{\mathcal{A}}||$, but we also have, by the previous theorem, that

$$||h_{\mathcal{A}}|| = ||E_{\mathcal{A}}(h_{\mathcal{M}})|| \leq ||h_{\mathcal{M}}||,$$

thus proving the equality. □

In the usual Jones' Index Theory for inclusions of factors, it is satisfied the relation

$$E_{\mathcal{A}}(p) = \mathrm{Ind}(E)^{-1},$$

where Ind(E) is in that context a real number greater than 1. The following proposition is the generalization of that result. The limitation appearing with respect to the factors' case is the possibility of Ind($E_{\mathcal{A}}$) and Ind(E) not to be equal. Note that when they are equal, that is when Ind(E) belongs to the center of $\mathcal{B}$, Proposition 4.5 expresses exactly the mentioned result.

Proposition 4.5 *If E has finite Index, then*

$$E_{\mathcal{A}}(Ind(E_{\mathcal{A}})p) = 1$$

and

$$Ind(E)E_{\mathcal{A}}(p) = 1.$$

Proof. As $V \in \mathcal{B}'$, $J_{\mathcal{B}}VJ_{\mathcal{B}} \in \mathcal{B}$, so that if $x \in \mathcal{A}$,

$$\begin{aligned} \tau(E(x)) &= \tau(J_{\mathcal{B}}V^*J_{\mathcal{B}}\gamma(x)J_{\mathcal{B}}VJ_{\mathcal{B}}) = \\ &= \tau(\gamma(x)J_{\mathcal{B}}VJ_{\mathcal{B}}J_{\mathcal{B}}V^*J_{\mathcal{B}}) = \tau(\gamma(xp)), \end{aligned} \tag{4.8}$$

because $p = J_{\mathcal{A}}pJ_{\mathcal{A}}$. Now, $\gamma(xp) \in \mathcal{B}$, as xp is in $\mathcal{M}$, so we can write, using that $\tau(x) = \tau(E(x))$,

$$\tau(x) = \tilde{\tau}(\gamma(px)) = \tilde{\tau}(h_{\mathcal{M}}px) = \tilde{\tau}(E_{\mathcal{A}}(h_{\mathcal{M}}px)) = \tau(E_{\mathcal{A}}(h_{\mathcal{M}}p)x).$$

We have then, as $E_{\mathcal{A}}(h_{\mathcal{M}}p) \in \mathcal{A}$ and $\tau((1 - E_{\mathcal{A}}(h_{\mathcal{M}}p))x) = 0$ for every $x \in \mathcal{A}$, that $E_{\mathcal{A}}(h_{\mathcal{M}}p) = 1$.
To see the second assertion, let $x \in \mathcal{A}$. Then

$$\tau(E_{\mathcal{A}}(p)x) = \tilde{\tau}(px) = \tau(\gamma(h_{\mathcal{A}}^{-1}px)) = \tau(\gamma(pxh_{\mathcal{A}}^{-1})) = \tau(E(xh_{\mathcal{A}}^{-1}) = \tau(h_{\mathcal{A}}^{-1}x),$$

thus proving that $E_{\mathcal{A}}(p) = h_{\mathcal{A}}^{-1}$. □

Lemma 4.6 *If E has finite Index, then $\mathcal{M}p = \mathcal{A}p$.*

Proof. The set $K = \{a_0 + \sum_i a_i p b_i : a_i, b_i \in \mathcal{A}\}$ is WOT-dense in $\mathcal{M}$. Given $x \in \mathcal{M}$, we have a net $\{x_\alpha\}_\alpha \subseteq \mathcal{M}_0$ such that $x_\alpha \to x$ with the weak operator topology. It is trivial then that $x_\alpha p \to xp$. For each element x_α there exists an $a_\alpha \in \mathcal{A}$ with $a_\alpha p = x_\alpha p$ (using that $pyp = E(y)p$ for every y of $\mathcal{A}$). By Kaplansky's density theorem, we can assume that the net $\{x_\alpha\}_\alpha$ is norm bounded, that is, $\|x_\alpha\| \leq c$, $c \in I\!R$, for every α. Then we can write

$$\begin{aligned} a_\alpha p &= x_\alpha p \\ E_{\mathcal{A}}(a_\alpha p) &= E_{\mathcal{A}}(x_\alpha p) \\ a_\alpha E_{\mathcal{A}}(p) &= E_{\mathcal{A}}(x_\alpha p) \\ a_\alpha &= E_{\mathcal{A}}(p)^{-1}E_{\mathcal{A}}(x_\alpha p). \end{aligned}$$

So,

$$\|a_\alpha\| \leq \|E_{\mathcal{A}}(p)^{-1}\| \, \|E_{\mathcal{A}}(x_\alpha p)\| \leq c\|E_{\mathcal{A}}(p)^{-1}\|.$$

Choosing a subnet of $\{a_\alpha\}$ that converges to an $a \in \mathcal{A}$, we have clearly that $ap = xp$.
□

Remark 4.7 *The preceding proposition is stablished in [AS] for the case where the inclusion has finite weak index. In our approach, this is not useful to us, because we want to deduce this fact from our own (Longo's, strictly speaking) definition of index.*

5 RELATION BETWEEN IND(E) AND OTHER DEFINITIONS OF INDEX

We are interested in comparing the following numbers:

$$\begin{aligned}
\lambda_1 &= \max\{\lambda : E(x) \geq \lambda x, \ x \in \mathcal{A}_+\} \\
\lambda_2 &= \max\{\lambda : \|E(x)\| \geq \lambda\|x\|, \ x \in \mathcal{A}_+\} \\
\lambda_3 &= \max\{\lambda : E(x) - \lambda x \text{ es completamente positivo}, \ x \in \mathcal{A}_+\} \\
\lambda_4 &= \inf\{\|E(q)\| : \ q \in \mathcal{P}(\mathcal{A}),\}
\end{aligned}$$

that in the finite factor case are equivalent definitions of Jones' index ([PiPo-1], section 2). In the case considered in [AS] they use λ_1^{-1} as definition of index. This number λ_1 coincides with the *weak index* of E of [BDH],

$$\operatorname{Ind}_w(E) = \lambda_1. \tag{5.1}$$

It is well known ([BDH]) that when the index is a scalar and the algebra $\mathcal{B}$ if properly infinite, it happens that

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4.$$

The following lemma, also from Baillet, Demizeau y Havet, proves that $\lambda_1 = \lambda_2$.

Lemma 5.1 *([BDH]) For every positive $x \in \mathcal{M}$ and $\lambda > 0$, $\|E(x)\| \geq \lambda\|x\|$ if and only if $E(x) \geq \lambda x$.*

Proof. The reverse implication is clear. To see the other, let $y \in \mathcal{M}^+$, $\lambda > 0$ and suppose that

$$\|E(x)\| \geq \lambda\|x\|.$$

For every $n \in I\!N$, let $b_n = (E(y * y) + 1/n)^{-1/2}$. Then

$$\begin{aligned}
b_n y^* y b_n &\leq \|b_n y^* y b_n\| \leq \|y^* y\| \|b_n^{-2}\| \leq \\
&\leq \lambda^{-1} \|E(y^* y)\| \|(E(y^* y) + 1/n)^{-1}\| \leq \lambda,
\end{aligned}$$

and we can write

$$y^* y \leq \lambda^{-1}(E(y^* y) + 1/n) \text{ for every } n \in I\!N$$

and

$$\lambda y^* y \leq E(y^* y)$$

□

As a general observation, it is seen easily that $\lambda_1 \leq \lambda_4$.

Lemma 5.2 *If $\lambda_4 > 0$, or if E has finite Index, then the set $\mathcal{M} = \{apb : \ a, b \in \mathcal{A}\}$.*

Proof. Let us call

$$\mathcal{M}_0 = \{apb : \; a, b \in \mathcal{A}\}.$$

It is trivial that $\mathcal{M}_0$ is a *-subálgebra. By Lemma 4.6, it suffices then to show that the identity can be reached, and in the context we are working it is easily seen that $1 \in \mathcal{M}_0$. To see this note that $V \in \mathcal{B}'$, so that $V_0 = J_{\mathcal{A}}\, V\, J_{\mathcal{A}}$ is an isometry of $\mathcal{M}$ satisfying:

$$V_0^* V_0 = J_{\mathcal{A}}\, V^*\, J_{\mathcal{A}}\, J_{\mathcal{A}}\, V\, J_{\mathcal{A}} = J_{\mathcal{A}}\, V^* V\, J_{\mathcal{A}} = 1 \tag{5.2}$$

$$V_0 V_0^* = J_{\mathcal{A}}\, V\, J_{\mathcal{A}}\, J_{\mathcal{A}}\, V^*\, J_{\mathcal{A}} = J_{\mathcal{A}}\, p\, J_{\mathcal{A}} = p. \tag{5.3}$$

So, $1 = V_0^*\, p\, V_0$, and it is shown in [AS] that if $\lambda_4 > 0$ then $\mathcal{M}p = \mathcal{A}p$ (with the hypotesis of $h_{\mathcal{A}}$ being bounded we have the same result by Proposition 4.6), so there exists $m \in \mathcal{A}$ with $V_0{}^* p = mp$, and $1 = mpm^* \in \mathcal{M}_0$. □

Proposition 5.3 *If E has finite Index, then $E(x) \geq ||Ind(E)||^{-1} x$ for $x \in \mathcal{A}_+$.*

Proof.
By the preceding Lemma, $1 = m_0 e m_0^*$, $m_0 \in \mathcal{B}$, and by 4.6 there is an $a \in \mathcal{B}$ with $x^{1/2} m_0 e = ae$. Then,

$$\begin{aligned} x = x^{1/2} 1 x^{1/2} &= x^{1/2} m_0 e m_0^* x^{1/2} = aea^* \\ &\leq aa^* = aE(h_{\mathcal{A}} e) a^* \\ &\leq ||h_{\mathcal{A}}|| aE(e) a^* = ||h_{\mathcal{A}}|| E(aea^*) = ||h_{\mathcal{A}}|| E(x). \end{aligned}$$

As E is normal, the inequality is stated for every x in $\mathcal{A}_+$.
We have then that $x \leq ||h_{\mathcal{A}}|| E(x)$. □

Proposition 5.4 *If E has finite Index, then $Ind(E)$ coincides with the index of Baillet, Denizeau y Havet. Reciprocally, if the expectation E has finite index in the sense of [BDH], then E has finite Index.*

Proof. Taking m from lemma 5.2, it is easily seen that (m, m^*) is a quasi-base for $\mathcal{A}$, because if $x \in \mathcal{A}_+$,

$$\begin{aligned} xp &= mpm^* xp = mE(m^* x) p \\ px &= pxmpm^* = pE(xm) m^* \end{aligned}$$

As E is faithful, we can deduce that $x = mE(m^* x) = E(xm) m^*$ for every $x \in \mathcal{A}_+$ and, by linearity, for every $x \in \mathcal{A}$.
On the other side,

$$mm^* = mE_{\mathcal{A}}(h_{\mathcal{M}} p) m^* = E_{\mathcal{A}}(h_{\mathcal{M}} mpm^*) = E_{\mathcal{A}}(h_{\mathcal{M}}) = h_{\mathcal{A}},$$

so we have that $h_{\mathcal{A}}$ is the index of E.

To see the reverse implication, let $\{m_i\}_i$ be a quasi base for $\mathcal{A}$. We will show that $h_{\mathcal{A}} = \sum m_i m_i^*$. As in the proof of 4.5, we have, for $x \in \mathcal{A}$,

$$\begin{aligned}
\tau(\sum m_i m_i^* x) &= \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon p) \sum m_i m_i^* x) = \\
&= \sum \lim_{\epsilon\to 0} \tau(m_i E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon p) m_i^* x) = \\
&= \sum \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon m_i p m_i^* x)) = \\
&= \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon (\sum m_i p m_i^*) x)) = \\
&= \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon x)) = \\
&= \lim_{\epsilon\to 0} \tau((h_{\mathcal{M}})_\epsilon x) = \\
&= \tau(h_{\mathcal{M}} x) = \\
&= \tau(\gamma(x)).
\end{aligned}$$

and the assertion is proved by unicity of Theorem 2.2. □

Proposition 5.5 *If $\lambda_4 > 0$, then E has finite Index.*

Proof. We take again m from the proof of lemma 5.2. We know that mm^* belongs to the center of $\mathcal{A}$ for being (m, m^*) a quasi-base. Proceeding again as in the proof of 4.5, we have, for $x \in \mathcal{A}$,

$$\begin{aligned}
\tau(mm^* x) &= \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon p) mm^* x) = \\
&= \lim_{\epsilon\to 0} \tau(m E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon p) m^* x) = \\
&= \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon m p m^* x)) = \\
&= \lim_{\epsilon\to 0} \tau(E_{\mathcal{A}}((h_{\mathcal{M}})_\epsilon x)) = \\
&= \lim_{\epsilon\to 0} \tau((h_{\mathcal{M}})_\epsilon x) = \\
&= \tau(h_{\mathcal{M}} x) = \\
&= \tau(\gamma(x)).
\end{aligned}$$

Once again, by unicity in Theorem 2.2, it has to be $h_{\mathcal{A}} = mm^*$, so that $h_{\mathcal{A}}$ is bounded. □

Next theorem, conclusion of this section, stablishes that in the context we are working we can take any of the numbers λ_i as an "scalar definition" of index, and the notion of finite index agrees with any choice of the definition, and also with our extension of Longo's defintion.

Theorem 5.6 *Let $\mathcal{B} \subseteq \mathcal{A}$ be type II_∞ von Neumann algebras, $E : \mathcal{A} \to \mathcal{B}$ a faithful normal conditional expectation commuting with a trace τ in $\mathcal{A}$. Then they are equivalent:*

(i) E has finite weak index.

(ii) $\lambda_2 > 0$.

(iii) $\lambda_3 > 0$.

(iv) $\lambda_4 > 0$.

(v) E has finite Index.

(vi) E has strong finite index in the sense of Baillet, Denizeau y Havet.

Proof. It only remains to prove that $\lambda_4 > 0$ implies that E has finite weak index, but if $\lambda_4 > 0$, then $h_{\mathcal{A}}$ is bounded by Proposition 5.5, and so E has obviously finite weak index. □

6 An application: generalization to inclusions of type II_1 and type III algebras

6.1 The type II_1 case

First of all we consider an inclusion $\mathcal{B} \subseteq \mathcal{A}$ of type II_1 von Neumann algebras with an expectation E commuting with the canonical trace of $\mathcal{A}$. In order to apply the ideas used troughout this paper, we can use the known trick of making tensor products with some type I_∞ factor, obtaing thus an inclusion of II_∞ algebras. This construction preserves the Index, in the sense that if the inclusion has finite index in the sense of [BDH], then we have $\mathrm{Ind}(E \otimes id) = (\mathrm{Ind}_{BDH}E) \otimes 1$ (see below). Moreover, the results stablished in 4.3, 4.4 and 4.5 are satisfied.
Let F be a type I_∞ factor and define

$$\tilde{\mathcal{A}} = \mathcal{A} \otimes F \tag{6.1}$$

and

$$\tilde{\mathcal{B}} = \mathcal{B} \otimes F \tag{6.2}$$

with the expectation $E \otimes \mathrm{id}$. So we have an operator index $\mathrm{Ind}(E \otimes \mathrm{id}) \in \mathcal{Z}(\tilde{\mathcal{A}})$. Now observe that being $\tilde{\mathcal{A}}$ a tensor product, its center is also the tensor product of the centers of the algebras, so

$$\mathcal{Z}(\tilde{\mathcal{A}}) = \mathcal{Z}(\mathcal{A}) \otimes \mathbb{C}, \tag{6.3}$$

as F is a factor. So the operator $\mathrm{Ind}(E \otimes \mathrm{id})$ is only an amplification $h \otimes 1$ of an operator $h \in \mathcal{Z}(\mathcal{A})$. This provides us with a way to define the index of a II_1 inclusion even though there is no quasi base.

6.2 The type III case

We consider now an inclusion $\mathcal{B} \subseteq \mathcal{A}$ of type III von Neumann algebras with separable predual, and a faithful normal conditional expectation $E : \mathcal{A} \to \mathcal{B}$. In this context we will be able to stablish a result concerning equivalence of the "scalar" definitions of index. It is not obvious to state a definition of Index in the sense of Longo here because of the lack of a trace, which is essential in the definition and properties of $\mathrm{Ind}(E)$.

Following the construction in section 4 of [L2], we consider a faithful normal state $\phi_0 \in \mathcal{B}_*$. The state $\phi \in \mathcal{A}_*$ given by $\phi = \phi_0 \cdot E$ has modular group σ^ϕ that leaves $\mathcal{B}$ stable and we have an inclusion of semifinite von Neumann algebras $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} = \mathcal{B} \rtimes_{\sigma^{\phi_0}} I\!R$, $\tilde{\mathcal{A}} = \mathcal{A} \rtimes_{\sigma^\phi} I\!R$, are the crossed products of $\mathcal{B}$ y $\mathcal{A}$ respectively by their modular groups with respect to the states ϕ_0 and ϕ. R. Longo proves that the inclusion $\tilde{\mathcal{B}} \subseteq \tilde{\mathcal{A}}$ does not depend on ϕ_0 up to isomorphism, because if ψ_0 is another faithful state of $\mathcal{B}$ and $\psi = \psi_0 \cdot E$, then the Connes cocycle

$$(D\psi : D\phi)_t = (D\psi_0 : D\phi_0)_t$$

is in $\mathcal{B}$.
We call $\hat{\sigma}_t$ the dual action. In $\tilde{\mathcal{A}}$ we have a canonical trace τ, and there is a canonical construction of an expectation $\tilde{E}$ extending a E and commuting with τ, satisfying $\tilde{E}(\tilde{\mathcal{A}}) = \tilde{\mathcal{B}}$.
By isomorphism we can consider the inclusions

$$\mathcal{A} \subseteq \tilde{\mathcal{A}} \subseteq \tilde{\tilde{\mathcal{A}}}\,.$$

Takesaki's duality gives us an isomorphism between $\tilde{\tilde{\mathcal{A}}}$ and $\mathcal{A} \otimes F$, where F is a type I_∞ subfactor of $\mathcal{B}$, and this isomorphism takes $\tilde{\tilde{E}}$ in $E \otimes \mathrm{id}$. As $\mathcal{A}$ is properly infinite, it is possible to build and isomorphism between $\mathcal{A}$ and $\mathcal{A} \otimes F$ that maps E in $E \otimes \mathrm{id}$.

Theorem 6.1 *Let $\mathcal{B} \subseteq \mathcal{A}$ be type III von Neumann algebras with separable predual, $E : \mathcal{A} \to \mathcal{B}$ a faithful normal conditional expectation. Then they are equivalent:*

(i) E has finite weak index.

(ii) $\lambda_2 > 0$.

(iii) $\lambda_3 > 0$.

(iv) $\lambda_4 > 0$.

Proof. We consider

$$\tilde{\lambda}_1 = \max\{\lambda : \tilde{E}(x) \geq \lambda x,\ x \in \tilde{\mathcal{A}}_+\}$$
$$\tilde{\tilde{\lambda}}_1 = \max\{\lambda : \tilde{\tilde{E}}(x) \geq \lambda x,\ x \in \tilde{\tilde{\mathcal{A}}}_+\}$$
$$\bar{\lambda}_1 = \max\{\lambda : E \otimes \mathrm{id}(x) \geq \lambda x,\ x \in (\mathcal{A} \otimes F)_+\}$$

One has obviously that

$$\lambda_1 \geq \tilde{\lambda}_1 \geq \tilde{\tilde{\lambda}}_1 = \bar{\lambda}_1.$$

Moreover, the isomorphism between $\mathcal{A}$ and $\mathcal{A} \otimes F$ shows that

$$\lambda_1 = \bar{\lambda}_1,$$

and so

$$\lambda_1 = \tilde{\lambda}_1.$$

We can proceed in the same way with $\tilde{\lambda}_4$, so the theorem is proved by reduction to Theorem 5.6. □

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