Counterexample to a conjecture of Mujica

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Abstract

Let $U$ be an open subset of a Banach space $E$. In [2] Mujica shows there is a unique Banach space $G^\infty(U)$ and a bounded holomorphic mapping $\delta_U$ from $U$ into $G^\infty(U)$ with the property that given any Banach space $F$ every bounded holomorphic function from $U$ into $F$ factors through $G^\infty(U)$ as a continuous linear mappings. The properties the Banach space $G^\infty(U)$ are similar to those of $E$. In [4] Mujica asks if $G^\infty(U)$ is weakly sequentially complete when $E$ is weakly sequentially complete. In this paper we provide a counterexample to this conjecture.

In [2] Mujica proves the following result:

**Theorem 1.** (Mujica) Let $U$ be an open subset of a Banach space $E$ then there is a Banach space $G^\infty(U)$ and $\delta_U \in H^\infty(U;G^\infty(U))$ such that the following universal property holds: Given any Banach space $F$ and any $f \in H^\infty(U;F)$, there is a unique continuous linear operator $T_f:G^\infty(U) \rightarrow F$ such that $f = T_f \circ \delta_U$.

The pair $G^\infty(U)$ and $\delta_U$ are characterized up to isometric isomorphism by this property. The Banach space $G^\infty(U)$ can be realised as the space of all linear functionals on $H^\infty(U)$ whose restriction to each multiple of the unit ball of $H^\infty(U)$ is continuous for the compact open topology. The holomorphic function $\delta_U$ is then defined by $\delta_U(x) = \delta_x$ where $\delta_x(f) = f(x)$. Furthermore $H^\infty(U)$ is isometrically isomorphic to $G^\infty(U)_0^*$, the strong dual of $G^\infty(U)$.

The vector space properties of $G^\infty(U)$ are closely related to those of $E$. Indeed in [2], Mujica shows that if $U$ is balanced open in $E$ then $G^\infty(U)$ has the approximation property if and only if $E$ has the approximation property, while if $B_E$ is the open unit ball of $E$ then $G^\infty(B_E)$ has the metric approximation property if and only if $E$ has the metric approximation property.

Continuing the study of $G^\infty(U)$ in [4] Mujica poses the following question:

**Problem.** Let $U$ be a bounded open subset of a weakly sequentially complete Banach space $E$. Is $G^\infty(U)$ weakly sequentially complete?

We will give an example to show that the answer to this question is no. We begin with the observation that $L^1(\partial \Delta)/H^1_0(\Delta)$ is the unique isometric predual of $H^\infty(\Delta)$ (see [1]) and that this space is weakly sequentially complete and has cotype 2. We shall need the following result of Pisier [5].
Theorem 2. (Pisier) There is a weakly sequentially complete Banach space $Z$ with cotype 2 such that $(L^1(\partial \Delta)/H^1_0(\Delta)) \bigotimes Z$ contains a copy of $c_0$.

In particular this will mean that $(L^1(\partial \Delta)/H^1_0(\Delta)) \bigotimes Z$ is not weakly sequentially complete and does not have cotype 2.

Clearly we have that $C \bigoplus \infty Z$ is weakly sequentially complete.

By Proposition 2.3 of [2] we see that $Z$ is isomorphic to a complemented subspace of $G^\infty(B_Z)$. By Theorem 6.1 of [3]

$$G^\infty(\Delta \times B_Z) \simeq G^\infty(\Delta) \bigotimes \pi G^\infty(B_Z)$$

which contains $(L^1(\partial \Delta)/H^1_0(\Delta)) \bigotimes Z$ as a complemented subspace. Therefore we see that $G^\infty(\Delta \times B_Z)$ cannot be weakly sequentially complete. The space $C \bigoplus \infty Z$ is also an example of a Banach space with cotype 2 with open unit ball $\Delta \times B_Z$ such that $G^\infty(\Delta \times B_Z)$ does not have cotype 2.


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