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Uniqueness of limit cycles for a class of Liénard systems

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Abstract

We shall give three criteria for the uniqueness of limit cycles of systems of Liénard type $x'=\alpha(y)-\beta(y)F(x)$, $y'=-g(x)$, examples are provided to illustrate our results.

1. Preliminars.

The main goal of this work is to study uniqueness of limit cycles of system:

$$\begin{aligned} x' &= \alpha(y) - \beta(y)F(x), \\ y' &= -g(x), \end{aligned} \tag{1}$$

where the functions in (1) are assumed to be continuous and such that uniqueness of solutions for initial value problems is guaranteed.

If we define as usual $G(x)=\int_0^x g(s)ds$, $A(y)=\int_0^y \alpha(r)dr$, then we assume that the following conditions hold:

i) $\alpha(0)=0$, $\alpha(y)$ is strictly increasing and $\alpha(\pm\infty)=\pm\infty$;

ii) $xg(x)>0$ when $x\neq 0$ and $G(\pm\infty)=\infty$;

iii) $\beta(y)>0$ for $y\in\mathbf{R}$, is a nonincreasing function;

iv) there exist constant x_1 , x_2 with $x_1 < 0 < x_2$ such that $F(x_1)=F(0)=F(x_2)=0$ and $xF(x)<0$ for $x\in(x_1,x_2)\setminus\{0\}$;

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v) there exist constants $M>0$, k, k_0 with $k>k_0$, such that $F(x)>k$ for $x\geq M$ and $F(x)<k_0$ for $x\leq -M$;

vi) one of the following:

1. $G(x_1)=G(x_2)$, or
2. $G(-x)\geq G(x)$ for $x>0$.

Furthermore, we assume (see [22]) that first equation of system (1) defines implicitly, a function $y=h(x)$ such that, $h:(-m,m)\to\mathbf{R}$ and

- $m>0$,
- $h(0)=0$,
- $\alpha(h(x))-\beta(h(x))F(x)=0$, $x\in(-m,m)$,
- $\operatorname{sgn} h(x)=\operatorname{sgn} F(x)$ when $x\neq 0$.

Also, in that paper, we proved the following result:

Lemma A. If there exist some positive constants N and M such that:

$$|F(x)|\leq N,\ \forall x\in\mathbf{R} \text{ and } \beta(y)\leq M,\ \forall y\in\mathbf{R},$$

then $h(x)$ is bounded and $m=+\infty$.

Considering $W(x)=\int_0^{h(x)}\alpha(y)dy$, where h is the above function, we have:

1. if $x_2\leq|x_1|$ then $\max\limits_{0\leq x\leq x_2}\{G(x)+W(x)\}\geq G(x_1)$,
2. if $0<|x_1|<x_2$ then $\max\limits_{x_1\leq x\leq 0}\{G(x)+W(x)\}\geq G(x_2)$.

We remark that system (1) is the classical Liénard differential equation $x''+f(x)x'+x=0$ when $\alpha(y)=y$, $\beta(y)\equiv 1$, $F'(x)=f(x)$ and $g(x)=x$. The following facts are know:

a) Conditions i)-iv) imply that system (1) has a unique singularity, which will be an unstable focus or node (see [23]).

b) If v) holds then there exists a closed curve Γ such that every trajectory intersecting it crosses it in the exterior - to - interior direction, hence implying the existence of at least one stable limit cycle, by the Poincaré-Bendixon theorem, see for instance [4].

c) Condition vi) assures that all closed trajectories of system (1) have intersect both $x=x_1$ and $x=x_2$ (use [11] and a comparison result, see for example of [25]).

d) In [14] we proved that under conditions i)-iv) all solutions of (1) are continuable to the future.

Some attempts [1-3,5-13,18,20,21,23,25-28] have been made to find sufficient conditions for existence and uniqueness of limit cycles of some particular cases of system (1), under the condition $F(\pm\infty)=\pm\infty$. In this paper we obtain sufficient conditions for uniqueness of limit cycles of (1) without make use of above condition. These criteria are refinements of early results of the author (see [15-18,23]), so we consider that the following condition is added:

$$F(x) \text{ is nondecreasing for } x\in(-\infty,x_1)\cup(x_2,\infty). \tag{2}$$

If i)-vi) and (2) do hold we will give a short proof that system (1) has exactly one limit cycle, not by using a comparison method but by estimating the divergence of system (1) integrated along a limit cycle. By this we can show that the limit cycle is hyperbolic. A limit cycle is hyperbolic, or simple, if for any arbitrarily small analytic perturbation of the system there is not other limit cycle in a sufficiently small neighborhood of the limit cycle.

Let X be a vectorial field plane and γ a closed trajectory of X with period T. The number

$$c(\gamma) = \int_0^T \operatorname{div}X(\gamma(t))dt,$$

is called "characteristic exponent of γ".

The next proposition is a classical result; for a proof see [4].

Lemma B. Let γ be a periodic orbit of a vector field X in $\mathbf{R}^2$. Then γ is a stable limit cycle if $c(\gamma)<0$ and unstable if $c(\gamma)>0$.

Next we will state an additional condition to guarantee the uniqueness of the limit cycle in case (2) is violated. If the functions in (1) are all odd then system (1) exhibits symmetric with respect to the origin and the conditions of our theorem can be weakened.

Finally we provide some examples that illustrate our results.

2. Three uniqueness criteria for system (1).

We will first state a theorem in case both i)-vi) and (2) hold.

Theorem 1. If conditions i)-vi) and (2) hold, then system (1) has exactly one closed orbit, a hyperbolic stable limit.

This theorem will be proved by showing that if γ is a closed orbit then its characteristic exponent $c(\gamma) = -\int_\gamma \beta(y)f(x)dt$ satisfies $c(\gamma)<0$, where $f(x)=F'(x)$. This shows that γ is hyperbolic and stable. Because two adjacent limit cycles cannot both be stable, the uniqueness of γ follows. In order to estimate the characteristic exponent the following lemma will appear to be useful.

Lemma 1. Let γ be an arc of an orbit of the system (1), described by $y(x)$, $a\le x\le b$. Then

$$-\int_{\gamma}\beta(y)f(x)dt=\operatorname{sgn}(\alpha(y(a))-\beta(y(a))F(a))\left[\ln\left|\frac{\beta(y(b))F(b)-\alpha(y(a))}{\beta(y(a))F(a)-\alpha(y(a))}\right|+\right.$$

$$\left.+\int_a^b\frac{(\beta(y(b))F(b)-\beta(y(x)F(x))g(x)\frac{d\alpha}{dy}}{(\beta(y(b))F(b)-\alpha(y(x))(\beta(y(x)F(x)-\alpha(y(x)))^2}dx\right].$$

The proof of this lemma is based on ideas of proof presented in [26]. In that paper is considered the system:

$$\begin{aligned}x' &= h(y)-F(x),\\ y' &= -g(x).\end{aligned}\qquad(3)$$

To prove Lemma 1 we consider, instead of the above system, the system (1). Modifying ideas of [26] we obtain the expected result.

Proof of Theorem 1. It was shown in [18] that it follows from the conditions i)-vi) that system (1) has at least one limit cycle Γ and it intersects both $x=x_1$ and $x=x_2$.
Denote the intersection point of Γ with the positive y-axis by A. Let B and C be the intersection points of Γ with $x=x_2$ In the first and fourth quadrant, respectively. If we denote the arc of Γ between A and B by γ_1, then applying Lemma 1 with $a=0$ and $b=x_2$, yields:

$$-\int_{\gamma_1}\beta(y)f(x)dt=\int_0^{x_2}\frac{\beta(y(x))F(x)g(x)\frac{d\alpha}{dy}}{\alpha(y(x))(\beta(y(x))F(x)-\alpha(y(x)))^2}dx.$$

This integral is negative because the integrand is negative by virtue of i)-iv). Thus have we proved:

$$-\int_{\gamma_1}\beta(y)f(x)dt<0.$$

For γ_2, the arc of Γ between B and C, we obtain by (2) and $f(x)=F'(x)$:

$$-\int_{\gamma_2}\beta(y)f(x)dt<0.$$

In an analogous way, the inequality $-\int_{\Gamma}\beta(y)f(x)dt<0$ is obtained. This completes the proof.

If the monotonicity of F(x) is only assumed on the intervals $(\overline{x_1},x_1)$ and $(x_2,\overline{x_2})$ then we can obtain the following:

Corollary 1. If conditions i)-vi) hold and F(x) is nondecreasing on $(\overline{x_1},x_1)$ and $(x_2,\overline{x_2})$ then in the strip $\overline{x_1}\le x\le\overline{x_2}$ system (1) has at most one closed orbit, a hyperbolic stable limit cycle.

Proof. If system (1) has a closed orbit then its uniqueness can be proved as in Theorem 1. However, in the strip $\overline{x_1}\le x\le\overline{x_2}$ the existence of a closed orbits is no longer guaranteed.

Next we present the Liénard equation of degree five on the plane, studied by Billeke, Burgos and Wallace (see [2-3]) which shows that if the conditions i)-vi) hold but (2) does not, then system (1) can have more than one limit cycle.
Consider the following system:

$$\begin{aligned} x' &= y-\varepsilon(a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5),\\ y' &= -x, \end{aligned} \tag{4}$$

with $0\le\varepsilon<<1$. For $\varepsilon=0$ all trajectories of (1) are closed and satisfy $H(x,y)=x^2+y^2=r^2$ (linear center).
To find the closed orbits for $0\le\varepsilon<<1$ we have to study:

$$\int_{x^2+y^2=r^2} dH = -2\varepsilon\int_0^{2\pi} r\cos t F(r\cos t)dt + O(\varepsilon^2),$$

whose zeros correspond with limit cycles for system (2), see [2,3]. An elementary calculation reveals that:

$$I(r)=\int_0^{2\pi} r\cos tF(r\cos t)dt = \pi r^2\left(a_1+\frac{3a_3}{4}r^2+\frac{15a_5}{24}r^4\right). \tag{5}$$

Thus we have that system (4) has two limit cycles if a_1 , a_3 , a_5 have alternated signs, $0<|a_1|<<|a_3|<<|a_5|$ and $|a_2|$, $|a_4|$ are sufficiently small (see[2,3]).
It is easy to check that the conditions i)-vi) hold but (2) is not satisfied, as can be seen by studying the graph of F(x).
If (2) is violated then we need an additional condition to guarantee the uniqueness of the limit cycle. In order to formulate this condition we will use the following lemma, which is easily obtained from Theorem 7.9, Chapter 4 of [28].

Lemma 2. Let $F_1(x)=F(x)$ and $F_2(x)=F(-x)$, both for $0\le x\le d$, where either $d\in\mathbf{R}_+$ or $d=+\infty$. Suppose the conditions i)-iv) hold and in addition, assume the following assumptions are fulfilled:

I) g is odd and nondecreasing function;

II) $y=F_1(x)$ intersects $y=F_2(x)$ at two points, (0,0) and (a,b) with $0<a\le d$;

III) $F_2(x)\ge F_1(x)$ for $x\in(0,a)$;

IV) For j=1,2 there exist τ_j, $\varepsilon_j\in[a,d]$ with $\tau_j\le\varepsilon_j$ such that:

a) $(-1)^j F_j(x)\le 0$ for $x\in[\tau_j, r]\subset[a,d]$, where $r=\max\limits_{j=1,2}\{\tau_j+\varepsilon_j\}$,

b) $(-1)^j F_j(x)+(-1)^{3-j}F_{3-j}(x+\varepsilon_j)\le 0$, not identically zero, for $x\in[0,\tau_j]$;

c) $F_1(x)>0$ and $F_2(x)<0$ for $x\in[r,d]$.

Then for all $x_0\in[r,d]$ the backward and forward orbits passing through $(x_0,h(x_0))$ cross the y-axis in A and B, respectively. Similarly, the forward and backward orbits passing through $(-x_0,h(-x_0))$ cross the y-axis in C and D, respectively, where $y_A>y_C$ and $y_B>y_D$.

As an application of this theorem we obtain the next result.

Corollary 2. Under conditions of above lemma, for all $x_0\in[r,d]$ system (1) has no closed orbits in the strip $|x|\le d$ which cross $x=x_0$ or $x=-x_0$.

Proof. Suppose that there exists a closed orbit Γ_1 intersecting $y=h(x)$ in $S(x_S,h(x_S))$ and $T(x_T,h(x_T))$, with $x_S>x_0$ and $x_T<-x_0$.
First consider $x_S>-x_T$. Let R denote the intersection of Γ_1 with the positive y-axis. Then by Lemma 2 the forward orbit γ passing through $(-x_S,h(-x_S))$ will cross the positive y-axis, say in U, such that $y_U<y_R$. This is impossible because obviously γ cannot intersect Γ_1. The case $x_S\le -x_T$ can be proved in a similar way.
An oscillatory orbit intersecting $x=-x_0$ but not $x=x_0$ has to cross the y-axis from A to C. But then, because $y_B>y_D$, this trajectory cannot intersect $x=-x_0$ again so it cannot be closed. The same argument holds for trajectories crossing $x=x_0$ but not $x=-x_0$. This exclude the possibility of a closed orbit intersecting only $x=-x_0$ or $x=x_0$. This completes the proof.

Remark 1. When g(x) does not satisfy condition I) of Lemma 2, we can define functions $F^*(u)$ and $\phi(x)$ on $\mathbf{R}$ by expressions:

$$F^*(u)=\begin{cases} F\left(G_1^{-1}\left(\dfrac{u^2}{2}\right)\right), u \geq 0,\\ F\left(G_1^{-1}\left(-\dfrac{u^2}{2}\right)\right), u < 0,\end{cases}$$

$$\phi(x)=\begin{cases}(2G_1(x))^{1/2}, x \geq 0,\\ -(-2G_1(x))^{1/2}, x<0,\end{cases}$$

with $G_1(x)=\int_0^x |g(s)|ds$, and the mapping $\Phi:(x,y)\to(u,v)$ by $\Phi(x,y)=(\phi(x),y)$. Then $F^*(u)$, $\phi(x)$ and $\Phi(x,y)$ are continuous. Consider the system (see [16]):

$$\begin{aligned} u' &= \alpha(v)-\beta(v)F^*(u),\\ v' &= -u. \end{aligned} \tag{6}$$

Now (6) satisfies condition I) of Lemma 2, because g(u)=u, but in general it will be quite cumbersome to check the other conditions.

Theorem 2. Suppose that system (1) satisfies the conditions i)-iv), I)-IV) and in addition assume that:

$$F'(x)\geq 0 \text{ for } x\in(-r,x_1)\cup(x_2,r). \tag{7}$$

Then in the strip $|x|\leq d$ system (1) has exactly one closed orbit, a hyperbolic stable limit cycle.

Proof. Consider the backward and forward trajectories passing through B_0 (r,h(r)) and suppose they cross the y-axis in A_0 and C_0, respectively. Similarly, suppose the forward and backward trajectories passing through E_0 (-r,h(-r)) cross the y-axis in F_0 and D_0, respectively. Then by Lemma 2 all trajectory of (1) intersecting the curve $\overline{A_0B_0C_0D_0E_0F_0A_0}$ crosses is in the exterior-to-interior direction, because $y_{A_0} > y_{F_0}$ and $y_{C_0} > y_{D_0}$.
Because O(0,0) is an unstable antisaddle it follows from Poincaré-Bendixon theorem that system (1) has al least one limit cycle in the strip $|x|<r$. By condition vi) follows from Corollary 1 that the limit cycle is hyperbolic and stable and hence unique. It follows from applying Corollary 2 with $x_0=r$, that there are no limit cycles in the strip $|x|\leq d$ that cross x=-r or x=r. This completes the proof.

If the functions α, g and F are odd and β is even then system (1) is symmetric with respect to the origin. This means that the conditions of Theorem 2 can be weakened. For this case we will not use Lemma 2 but the following:

Lemma 3. Under conditions i)-iii) suppose in addition that the following assumptions are fulfilled:

a) $\alpha(-y)=\alpha(y)$, $g(-x)=g(x)$, $\beta(-y)=\beta(y)$ and $F(-x)=F(x)$;
b) there exists $x_2>0$ such that $F(0)=F(x_2)=0$ and $F(x)<0$ for $x\in(0,x_2)$;
c) let $I=[0,x_2]$ and $J=[x_2,d]$ with $x_2<d$, $\varphi:I\rightarrow J$ is weakly increasing, continuous and $g(\varphi(x))\,\varphi'(x)\geq g(x)$ for $x\in I$;
d) with φ as above we have $F(\varphi(x))\geq -F(x)$ for $x\in I$;
e) $F(x)>0$ for all $x_0\in[x_2,d]$.

Then for all $x_0\in[\varphi(x_2),d]$ the backward and forward orbits passing through $(x_0,h(x_0))$ cross the y-axis, in A and B respectively and $y_A>-y_B$.
The method of proof of this lemma is exactly the same as in Alsholm [1], Corollary 3.

Remark 2. If the functions in system (1) are as in Lemma 3, then Lemma 2 is a special case of this with $\varphi(x)=x+\varepsilon$, and $\varepsilon=\varepsilon_1=\varepsilon_2$ by symmetry.

Theorem 3. Suppose system (1) satisfies the conditions i), ii), iii), a)-e). Furthermore assume that:

$$F'(x)\geq 0 \text{ for } x\in(x_2,\varphi(x_2)). \tag{8}$$

Then in the strip $|x|\leq d$ system (1) has exactly one closed orbit, a hyperbolic stable limit cycle.

The proof of Theorem 3 is basically the same as that Theorem 2 and we leave it to the reader. Note that we have dropped condition vi) because all symmetric curve, respect to the origin, of a trajectory of system (1) is also a trajectory, i.e., if $(x(t),y(t))$ is a trajectory of (1) then $(-x(t),-y(t))$ is also a trajectory.

1. Examples and related results.

We present here, some illustrative examples of our results.

Example 1. In [1] the existence of limit cycle of the equation:

$$x''+(x^4+7x^3+2x^2+x)x'+5x^3+x^2+x=0,$$

is considered, and Guidorizzi (see [6]) proved that the origin is globally asymptotically stable and, for all nontrivial solution $x=x(t)$ the trajectories $\gamma(t)=(x(t),x'(t))$ approaches the origin, in spiral, as $t\rightarrow+\infty$. Is easily check that condition iv) is not fulfilled.
Lins, de Melo and Pugh [13] proved that if $F(x)$ is a polynomial and the condition:

$$F(x)\neq F(-x) \text{ for all } x>0,$$

Holds, then there exist no nontrivial periodic solutions of (3) with $\alpha(y)=y$ and $g(x)=x$

(see also [7]). This example show the necessity of condition iv) or any other on F(x).

Example 2. Consider the system:

$$\begin{aligned} x' &= y - (a_3x^3 + a_2x^2 + a_1x), \\ y' &= -x, \end{aligned} \tag{9}$$

this example was discussed by Coll, Gasull and Llibre [5], equation (2). They proved the following:

Corollary 8. The polynomial Liénard equation (9) has at most one limit cycle which if it exists, is hyperbolic and stable, when $|a_2|$ is sufficiently small.
Is clear that if $a_1a_3<0$ the system (9) satisfies all conditions of Theorem 1.

Example 3. Consider again the system (4), and $I\in\mathbf{R}$ as in (5), thus we have the following facts (see [2,3]):

The system (4) has exactly one limit cycle if:
i) $a_1a_5<0$ and $a_3\geq 0$ or
ii) $a_1a_5<0$, $a_3<0$ and $-N<a_4<M$ for $N,M>0$.
iii) $a_5=0$ and $a_1a_5<0$.
iv) $A_1=0$ and $a_3a_5<0$ and $-N<a_4<M$ for $N,M>0$.
The conditions of Theorem 1 are fulfilled, this analysis is easy and we leave this as an exercise to the reader.

Example 4. Consider the system:

$$\begin{aligned} x' &= \alpha(y) - kF(x), \\ y' &= -g(x), \end{aligned}$$

with $k>0$, $F(x)=x(x^2-1)(x^{20}-140x+247)/20$ and where $\alpha(y)$ and $g(x)$ satisfy i), ii), iii). This example was analyzed in [12] and they showed that it satisfies all conditions of Theorem 2. If we take $F(x)=4x(x^2-1)/(4+3x^4)$, $\alpha(y)$ satisfies i) and $\alpha(-y)=-\alpha(y)$ we can check that all conditions of Theorem 3 hold (see [12] and [20]).

Remark 3. Our results are consistent with those of [1-3,5-8,10-11,13,18,20,26-27,29] related to existence, uniqueness and stability of limit cycle for Liénard equation:

$$x''+f(x)x'+g(x)=0, \tag{10}$$

and with the nonexistence of periodic solutions (see [8,16,19,24]).

Remark 4. The above remark still valid if we consider the results of [9,12,21,25,28] refer to system (1).

Remark 5. Our results contains, in particular, those of [12] and [21], refer to the system (3).

Remark 6. Consider the equation (10) with $f(x)=(2x-1)\exp(x^2+99x-100)$ and $g(x)=x$. In [6] Guidorizzi proved that, in this case, the equation admits at least one non trivial periodic solution.

Now the following question arises.

¿Under which additional assumptions on F(x), we can obtain a similar uniqueness result for system (1), if F(x) has a unique root?.

This is not trivial question, its resolution implies to extend the results of [6,7,21,24] to system (1) (and consequently to (3)).

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