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Flag Spaces for Reductive Lie Groups

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Abstract

Let G_0 be a reductive Lie group and suppose K is the complexification of a maximal compact subgroup of G_0. In this study we define complex flag spaces for G_0 and characterize the isotropy groups for corresponding G_0 and K-actions.

1 Introduction

Suppose G_0 is a reductive Lie group of Harish-Chandra class [5, Section 3] and let G denote the corresponding complex adjoint group. Let K be the complexification of a maximal compact subgroup of G_0. By definition, *a complex flag space for* G_0 is a homogeneous, complex projective, algebraic G-space. *A parabolic subgroup of* G can be defined as any subgroup that contains a maximal connected solvable subgroup of G. From the work of Tits and Borel [6], one knows that the complex flag spaces Y for G_0 are the spaces of the form

$$Y = G/P$$

where $P \subseteq G$ is a parabolic subgroup. The groups G_0 and K act on Y.

The purpose of this paper is to prove a certain basic technical result characterizing the structure of the isotropy groups for the G_0 and K-actions in Y. This characterization is fundamental to the program of geometric construction of representations for G_0 in Y (consider for example [1, Proposition 1 and Lemma 1], where the result is used without proof, as well as [2] where the result is also used, but in a special, well-known context; we also mention [3] and [4] where the result of this paper is also needed). Although one finds an infinitesimal result in [9], it seems a general proof of the structure and decomposition for the isotropy groups does not appear in the existing literature. The aim of this article is to fill that void. In particular, our main result here is Theorem 4.4 in Section 4.

Actually, we open the scope a little wider in this paper. Specifically, we prove our result in the context of complex flag spaces for reductive Lie groups (not necessarily of Harish-Chandra class). Although quite simple, our definition and characterization of complex flag spaces for reductive Lie groups also does not seem to appear anywhere in the current literature. This enlarged context requires some simple generalizations as well as a consideration of flag spaces, parabolic subgroups, etc. for disconnected complex algebraic groups.

Our paper is organized as follows. The first section is the introduction. In the second section we usher in the class of reductive Lie groups [7, Definition 4.29], review some key structure theory, and define a corresponding complex adjoint group. In the third section we introduce the complex linear algebraic groups and prove a simple lemma about the fixed point sets of involutions. We then consider the theory of flag spaces, parabolic subgroups, etc. for disconnected complex linear algebraic groups. In the last section we define complex flag spaces for reductive Lie groups and prove our result about the structure of the isotropy groups.

We conclude this introduction with a few remarks about our terminology. The *nilradical* of a Lie algebra is defined to be the radical of the corresponding derived algebra. Thus the nilradical is the intersection of the kernels of all finite-dimensional irreducible representations [11, Theorem 3.8.1, Theorem 3.14.1 and Theorem 3.16.2]. With this definition, the unipotent radical of a parabolic subgroup of a complex reductive group has Lie algebra the nilradical of the corresponding parabolic subalgebra [12, Section 1.2.2, page 55]. When G_0 is a Lie group with Lie algebra $\mathfrak{g}_0$, then *the derived subgroup* of G_0 is the connected subgroup with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$. A finite-dimensional representation of G_0 always means a continuous representation in a finite-dimensional complex vector space. Suppose G is a complex Lie group with Lie algebra $\mathfrak{g}$. Then a *conjugation* of G is a continuous involutive automorphism whose derivative is conjugate linear on $\mathfrak{g}$. The conjugation is called *compact* if its fixed point set is a compact subset of G. A *real form* of G means a closed subgroup whose Lie algebra is a real form of $\mathfrak{g}$.

2 Reductive Lie Groups

In this section we begin by defining reductive Lie groups and reviewing the Cartan decomposition. Next, we review the well-known correspondence between complex reductive groups and compact Lie groups. We conclude the section by defining and considering what we call the complex adjoint group for G_0 (in the case of a Harish-Chandra class group, our definition coincides with the usual definition).

The Cartan Decomposition. Let G_0 be a Lie group with Lie algebra $\mathfrak{g}_0$. Then G_0 is called a *reductive Lie group* if:

(1) G_0 has finitely many connected components;
(2) The Lie algebra of G_0 is reductive;
(3) The derived subgroup of G_0 has finite center.

Suppose G_0 is a reductive Lie group and let G_0^{id} denote the identity component of G_0. By a classical result of G. Mostow [10, Section 3], the fact that G_0 has finitely many connected components implies that each maximal connected compact subgroup M_0 of G_0^{id} is contained in a maximal compact subgroup K_0 of G_0 such that

$$K_0 \cap G_0^{\mathrm{id}} = M_0 \text{ and } K_0 \cdot G_0^{\mathrm{id}} = G_0.$$

In addition, all maximal compact subgroups of G_0 are conjugate by an element of G_0^{id}.

Let K_0 be a maximal compact subgroup of G_0 and let $\mathfrak{K}_0$ denote the Lie algebra of K_0. We define a *Cartan complement* $\mathfrak{s}_0$ to $\mathfrak{K}_0$ in $\mathfrak{g}_0$ to be an $\mathrm{Ad}(K_0)$-invariant complement to $\mathfrak{K}_0$ in $\mathfrak{g}_0$ such that

$$[\mathfrak{s}_0, \mathfrak{s}_0] \subseteq \mathfrak{K}_0.$$

Cartan complements to $\mathfrak{K}_0$ in $\mathfrak{g}_0$ are known to exist and any two are conjugate by an automorphism of $\mathfrak{g}_0$ that pointwise fixes $\mathfrak{K}_0$ [5, Section 3]. In fact, the automorphism can be chosen as the exponential of a derivation of $\mathfrak{g}_0$.

Suppose $\mathfrak{s}_0$ is a Cartan complement to $\mathfrak{K}_0$ in $\mathfrak{g}_0$. Let $\xi \in \mathfrak{s}_0$ and let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$. Then it is not hard to show that adjoint map

$$\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$$

is semisimple. We also observe that the linear map

$$\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$$

defined to have eigenvalues +1 in $\mathfrak{K}_0$ and -1 in $\mathfrak{s}_0$ is an involutive automorphism of $\mathfrak{g}_0$. θ is called a *Cartan involution* of $\mathfrak{g}_0$ corresponding to K_0.

This analysis of $\mathfrak{g}_0$ descends to the group. In particular, one knows that $\exp(\mathfrak{s}_0)$ is a closed regular analytic submanifold of G_0 and that the exponential map

$$\exp : \mathfrak{s}_0 \to \exp(\mathfrak{s}_0)$$

is an isomorphism of analytic varieties. Letting $e \in G_0$ denote the identity, we have

$$K_0 \cap \exp(\mathfrak{s}_0) = \{e\} \text{ and } G_0 = K_0 \cdot \exp(\mathfrak{s}_0)$$

a so-called *Cartan decomposition of* G_0 with respect to K_0. Thus the Cartan involution of $\mathfrak{g}_0$ descends to an automorphism of the group, by defining

$$\theta(k \cdot \exp(\xi)) = k \cdot \exp(-\xi) \text{ for } k \in K_0 \text{ and } \xi \in \mathfrak{s}_0.$$

Complex Reductive Groups. A complex Lie group G is called a *complex reductive group* if G has finitely many connected components and if a maximal compact subgroup of G is a real form of G.

Observe that a complex reductive group G is a reductive Lie group. In particular, if M_0 is a maximal compact subgroup of G and $\mathfrak{m}_0$ is the Lie algebra of M_0 then

$$G = M_0 \cdot \exp(i\mathfrak{m}_0)$$

is a Cartan decomposition of G with respect to M_0.

From the Cartan decomposition it follows that each finite-dimensional representation of M_0 lifts to a holomorphic representation of G. But there is a finer point involved here: the group G carries a compatible linear algebraic structure that is uniquely determined by the condition that any holomorphic homomorphism of G into a complex linear algebraic group is in fact algebraic. Hence the holomorphic representations of G are algebraic and there is a natural equivalence between the finite-dimensional representation theory of M_0 and the finite-dimensional algebraic representation theory of G.

On the other hand let K_0 be a compact Lie group with Lie algebra $\mathfrak{k}_0$ and let K be the complexification of K_0. Since K_0 is compact, the canonical morphism

$$K_0 \to K$$

is injective and the image of K_0 is a real form of K. One knows that a complex Lie group is the complexification of a compact real form if and only if the compact real form is a maximal compact subgroup. Thus K is a complex reductive group with maximal compact subgroup K_0 and corresponding Cartan decomposition

$$K = K_0 \cdot \exp(i\mathfrak{k}_0).$$

In general, when G is a complex reductive group, we note that an open subgroup of the fixed point set of a conjugation in G is a reductive Lie group. More specifically, suppose

$$\tau : G \to G$$

is a conjugation of G. Then there exists a compact conjugation

$$\gamma : G \to G \text{ such that } \gamma\tau = \tau\gamma.$$

From this one can deduce the Cartan decomposition for the fixed point set of τ (and hence Property 1 from the definition of reductive Lie group).

The Complex Adjoint Group. Suppose G_0 is a reductive Lie group with Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{g}$ denote the complexification of $\mathfrak{g}_0$ and let $\mathrm{Aut}(\mathfrak{g})$ denote the complex automorphism group of $\mathfrak{g}$. Then the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ defines a morphism of complex Lie algebras

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})).$$

By definition, *the complex adjoint group of* $\mathfrak{g}$, denoted by $\mathrm{Int}(\mathfrak{g})$, is the connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$.

The adjoint action of G_0 on $\mathfrak{g}$ defines a morphism of Lie groups

$$\mathrm{Ad} : G_0 \to \mathrm{Aut}(\mathfrak{g}).$$

We define *the complex adjoint group G of G_0* to be the Zariski closure of $\mathrm{Ad}(G_0)$ in $\mathrm{Aut}(\mathfrak{g})$. Thus G has finitely many connected components and

$$G^{\mathrm{id}} = \mathrm{Int}(\mathfrak{g}).$$

In particular, G is either a finite group or a complex semisimple group with finitely many connected components.

Fix a maximal compact subgroup K_0 of G_0 and let θ be a corresponding Cartan involution of $\mathfrak{g}_0$. We use the same letter to indicate the unique extension of θ to $\mathrm{Aut}(\mathfrak{g})$. Let τ be the conjugation of $\mathfrak{g}$ determined by $\mathfrak{g}_0$. Then τ and θ commute and each normalizes the group $\mathrm{Int}(\mathfrak{g})$. In addition

$$\tau \circ \mathrm{Ad}(g) \circ \tau = \mathrm{Ad}(g) \ \text{ and } \ \theta \circ \mathrm{Ad}(g) \circ \theta = \mathrm{Ad}(\theta(g)) \ \text{ for each } g \in G_0.$$

Thus G is normalized by τ and θ, since $\mathrm{Ad}(G_0)$ meets each connected component of G. Let

$$\gamma : G \to G$$

be the product of θ and τ. Then γ is a compact conjugation of G. Indeed, since $\mathrm{Ad}(K_0)$ meets each connected component of G, it follows that γ is a Cartan involution of G. We will refer to γ as *the associated compact conjugation* of G.

3 Linear Algebraic Groups and Flag Spaces

In order to associate a family of complex flag spaces to a given reductive Lie group, we need to consider flag spaces for disconnected linear algebraic groups. Although our concepts are simple generalizations from the connected case, they are generalizations which we have not found in the literature. Since a fundamental tool in the theory of geometric realization of representations is the canonical projection from a full flag space to a general flag space, we will require that flag spaces be connected. This requirement leads naturally to our definition of parabolic subgroup and Proposition 3.2.

Complex Linear Algebraic Groups. Let G be a complex linear algebraic group. Then there exists a largest connected, simply connected, Zariski closed, nilpotent normal subgroup of G called *the unipotent radical U of G*. A *Levi factor L* of G is a complex subgroup such that

$$L \cdot U = G \ \text{ and } \ L \cap U = \{e\}.$$

Levi factors exist and any two are conjugate under U. Indeed, a Levi factor of G is a maximal complex reductive subgroup. In particular, Levi factors are Zariski closed and G is a complex reductive group if and only if $U = \{e\}$. The decomposition of G as the semidirect product of a Levi factor with the unipotent radical is called *a Levi decomposition of G*.

The following simple lemma will be utilized in the next section.

Lemma 3.1 *Let G be a complex linear algebraic group and suppose*

$$\theta : G \to G$$

is a continuous automorphism. Assume L is a Levi factor preserved under θ. Let G_θ denote the fixed point set of θ in G, let L_θ be the fixed point set of θ in L and let $\mathfrak{u}_0$ denote the fixed point set of θ in the Lie algebra $\mathfrak{u}$ of the unipotent radical U of G. Then we have the following.
(a) *$exp(\mathfrak{u}_0)$ is a closed nilpotent normal subgroup of G_0 and*

$$exp : \mathfrak{u}_0 \to exp(\mathfrak{u}_0)$$

is an equivalence of analytic varieties.
(b) *Let G_0 be an open subgroup of G_θ and put $L_0 = G_0 \cap L_\theta$. Then G_0 is a semidirect product of L_0 with $exp(\mathfrak{u}_0)$.*

Proof: First observe that θ stabilizes $\mathfrak{u}$. Since the unipotent radical U is a connected, simply connected nilpotent complex Lie group it follows that

$$\exp : \mathfrak{u} \to U$$

is an equivalence of complex algebraic varieties. Since $\mathfrak{u}_0$ is a subalgebra of $\mathfrak{u}$, it follows from standard facts about simply connected nilpotent Lie groups that $\exp(\mathfrak{u}_0)$ is a closed nilpotent subgroup of U. On the other hand, G_0 normalizes $\exp(\mathfrak{u}_0)$, since $G_0 \subseteq G_\theta$. Therefore (a) follows, since G_0 contains $\exp(\mathfrak{u}_0)$.
To establish (b) let U_θ be the fixed point set of θ in U. From part (a) of the lemma, together with the equation

$$\theta(\exp(\xi)) = \exp(\theta(\xi)) \text{ for } \xi \in \mathfrak{u}$$

it follows that $U_\theta = \exp(\mathfrak{u}_0)$. Thus, G_θ is a semidirect product of L_θ with $\exp(\mathfrak{u}_0)$. Therefore the result follows, since every open subgroup of G_θ contains $L_\theta^{\text{id}} \cdot \exp(\mathfrak{u}_0)$.
■

Flag Spaces. Fix a complex linear algebraic group G with Lie algebra $\mathfrak{g}$. We define *a flag space for* G to be a connected, homogeneous, complex projective, algebraic G-space.
By definition, a *Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$. More generally, *a parabolic subalgebra* $\mathfrak{p}$ *of* $\mathfrak{g}$ is a complex subalgebra that contains a Borel subalgebra of $\mathfrak{g}$. One has the following properties.

(1) Any two Borel subalgebras of $\mathfrak{g}$ are conjugate by an element of G^{id}.
(2) The normalizer in G^{id} of a parabolic subalgebra $\mathfrak{p}$ is the connected subgroup with Lie algebra $\mathfrak{p}$.
(3) The G-homogeneous variety X of Borel subalgebras of $\mathfrak{g}$ is projective.

The variety X from item (3) is called *the full flag space of* $\mathfrak{g}$ (or of G). We define a *Borel subgroup of* G to be the normalizer in G of a Borel subalgebra of $\mathfrak{g}$. Thus

$$X = G/B$$

where B is a Borel subgroup of G. Properties (1) and (2) imply that

$$B \cdot G^{\mathrm{id}} = G \text{ and } B \cap G^{\mathrm{id}} = B^{\mathrm{id}}.$$

Suppose $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. Then property (1) implies that any parabolic subalgebra of $\mathfrak{g}$ is conjugate, by an element of G^{id}, to a parabolic subalgebra containing $\mathfrak{b}$. On the other hand, let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and let P be the normalizer of $\mathfrak{p}$ in G. Applying property (1) to $\mathfrak{p}$ and P, it follows that any Borel subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ is conjugate to $\mathfrak{b}$ by an element of P^{id}. Since the normalizer in G^{id} of $\mathfrak{b}$ also normalizes $\mathfrak{p}$, one deduces that no two distinct parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$ can be conjugate under G^{id}. Thus we obtain the following additional property.

(4) Any parabolic subalgebra of $\mathfrak{g}$ is conjugate, under G^{id}, to a unique parabolic subalgebra containing $\mathfrak{b}$.

We define *a parabolic subgroup of* G to be any closed complex subgroup of G that contains a Borel subgroup of G. Observe that the Lie algebra of a parabolic subgroup is parabolic. However, it is important to realize that the correspondence between parabolic subalgebras and parabolic subgroups that occurs in the connected case, need not occur in the disconnected case [consider the example later in this section]. The following proposition shows that the connected, complex projective algebraic quotients of a complex linear algebraic group G are characterized by the parabolic subgroups of G. This generalizes the standard result ([6]) for the case when G is connected.

Proposition 3.2 *Let G be a complex linear algebraic group.*
(a) *Suppose P is a parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Then P is the normalizer of $\mathfrak{p}$ in G.*
(b) *Suppose $Y = G/Q$ is a complex algebraic quotient of G. Then Y is connected and projective if and only if Q is a parabolic subgroup of G.*

Proof: To establish (a) let B be a Borel subgroup of G contained in P and let $N_G(\mathfrak{p})$ denote the normalizer of $\mathfrak{p}$ in G. Suppose $g \in N_G(\mathfrak{p})$. Since $G = B \cdot G^{\mathrm{id}}$ we can write

$$g = h \cdot b \text{ for some } h \in G^{\mathrm{id}} \text{ and some } b \in B.$$

Thus h normalizes $\mathfrak{p}$. By property (4) $h \in P^{\mathrm{id}}$. Therefore $g \in P$.
To establish (b) we first suppose $Y = G/Q$ is a connected complex projective algebraic quotient of G. Then G^{id} acts transitively on Y. Thus $G = Q \cdot G^{\mathrm{id}}$ and it follows by the usual facts about connected linear algebraic groups and complete homogeneous spaces that $Q \cap G^{\mathrm{id}} = Q^{\mathrm{id}}$ is a parabolic subgroup of G^{id}. Thus the Lie algebra $\mathfrak{q}$ of Q is a parabolic subalgebra of $\mathfrak{g}$. To see that Q contains a Borel subgroup of G, let $\mathfrak{b}$ be a Borel subalgebra of contained in $\mathfrak{q}$ and let B be the corresponding Borel subgroup of G. Suppose $b \in B$. Then there exists $g \in G^{\mathrm{id}}$ such that $bQ = gQ$. But this implies

$$\mathrm{Ad}(b)\mathfrak{q} = \mathrm{Ad}(g)\mathfrak{q}.$$

Therefore property (4) implies g normalizes $\mathfrak{q}$. Arguing as in (a), one shows that Q is the normalizer in G of $\mathfrak{q}$. Thus $b \in Q$.

For the converse, let Q be a parabolic subgroup of G and let B be a Borel subgroup of G contained in Q. Then part (a) of the proposition implies Q is Zariski closed in G. Let $Y = G/Q$ be the quotient variety and let $X = G/B$ be the full flag space for G. Then we have a G-equivariant algebraic projection

$$\pi : X \to Y.$$

It follows immediately that Y is connected and complete. Thus the result is proved, since a complete G-homogeneous algebraic variety is necessarily projective. ■

Example. When G is a connected complex reductive group and $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ then there is a well-known $1-1$ correspondence between the G-equivariant equivalence classes of flag spaces for G and the parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$. However, this need not be true when G is disconnected.

For example, suppose G is the automorphism group of the complex semisimple Lie algebra $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ (it will follow from our analysis that G is not connected). Let $\mathfrak{c}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of the diagonal matrices in $\mathfrak{g}$ and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{c}$. If α is a root of $\mathfrak{c}$ in $\mathfrak{g}$, we let $\mathfrak{g}^{\alpha} \subseteq \mathfrak{g}$ denote the corresponding root subspace. Let α_1 and α_2 be the two simple roots of $\mathfrak{c}$ in $\mathfrak{b}$. Then there exist exactly four parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$, the two nontrivial cases being:

$$\mathfrak{p}_{\alpha_1} = \mathfrak{g}^{-\alpha_1} + \mathfrak{b} \quad \text{and} \quad \mathfrak{p}_{\alpha_2} = \mathfrak{g}^{-\alpha_2} + \mathfrak{b}\,.$$

Let θ be the complex automorphism of $\mathfrak{g}$ defined by

$$\theta(\xi) = -\xi^{\mathrm{t}} \quad \text{for} \quad \xi \in \mathfrak{sl}(3, \mathbb{C})$$

(ξ^{t} denotes the transpose of the matrix ξ). Then $\mathfrak{c}$ is θ-stable and θ acts as -1 on the roots of $\mathfrak{c}$ in $\mathfrak{g}$. Let g be an element from the normalizer of $\mathfrak{c}$ in G^{id} representing the longest element in the Weyl group of $\mathfrak{c}$ in $\mathfrak{g}$. Then a straightforward calculation shows that

$$g\theta(\mathfrak{p}_{\alpha_1}) = \mathfrak{p}_{\alpha_2}.$$

It follows that neither one of the normalizers $N_G(\mathfrak{p}_{\alpha_1})$ or $N_G(\mathfrak{p}_{\alpha_2})$ can contain a Borel subgroup of G. Therefore there are only two flag spaces for G: the full flag space X and the trivial one point space.

4 Complex Flag Spaces and the Isotropy Groups

In this section we associate a family of complex flag spaces to a reductive Lie group G_0 and characterize both the structure of the isotropy groups in G_0 as well as the structure of the isotropy groups for the natural action of the complexification of a maximal compact subgroup of G_0. The main result of our study is Theorem 4.4.

Throughout the remainaer we fix a reductive Lie group G_0 with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$. We let τ denote the complex conjugation of $\mathfrak{g}$ determined by $\mathfrak{g}_0$. We fix a maximal compact subgroup K_0 of G_0, as well as a Cartan involution θ determined by K_0. The group K denotes the complexification of K_0. The Lie algebras of K_0 and K will be denoted by $\mathfrak{k}_0$ and $\mathfrak{k}$, respectively.

Complex Flag Spaces for a Reductive Lie Group. Let G be the complex adjoint group of G_0 [Section 2]. We define *a complex flag space for* G_0 to be a flag space Y of G.

Suppose Y is a complex flag space of G_0. Thus Y is an Int($\mathfrak{g}$)-conjugacy classes of parabolic subalgebras of $\mathfrak{g}$ that remains invariant under the adjoint action of G_0. We want to make this correspondence between points in Y and parabolic subalgebras of $\mathfrak{g}$ explicit. In particular, if Lie(G) denotes the Lie algebra of G and if

$$T : \mathfrak{g} \to \mathfrak{g}$$

is a derivation of $\mathfrak{g}$ belonging to Lie(G), then there exists a unique $\xi \in [\mathfrak{g}, \mathfrak{g}]$ such that

$$T = \mathrm{ad}_{\xi}.$$

Thus we can (and will) identify Lie subalgebras of Lie(G) with Lie subalgebras of $\mathfrak{g}$ contained in $[\mathfrak{g}, \mathfrak{g}]$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}].$$

For $y \in Y$, we let P_y denote the parabolic subgroup of G that stabilizes y. Then Lie(P_y) is a parabolic subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ and

$$\mathfrak{p}_y = \mathfrak{z} \oplus \mathrm{Lie}(P_y)$$

is the unique parabolic subalgebra of $\mathfrak{g}$ containing Lie(P_y). With this identification, for $g \in G_0$, the point $g \cdot y$ corresponds to the parabolic subalgebra $\mathrm{Ad}(g)\mathfrak{p}_y$.

Observe that there is a natural algebraic K-action on Y. In particular, the adjoint action of K_0 on $\mathfrak{g}$ determines an algebraic action of K on $\mathfrak{g}$ (also referred to as the adjoint action) which, in turn, defines a morphism of algebraic groups

$$\mathrm{Ad} : K \to G.$$

We adopt the following convention for denoting the isotropy groups in G_0, K_0 and K. For $y \in Y$ we let $G_0[y]$, $K_0[y]$ and K_y denote the corresponding stabilizers. In particular, $G_0[y]$, $K_0[y]$ and K_y are the normalizers of $\mathfrak{p}_y$ in each of the respective groups. We reserve the notation $K[y]$ to indicate the Zariski closure of $K_0[y]$ in K_y. Thus $K[y]$ is the complexification of $K_0[y]$.

The Structure of the Isotropy Groups. Suppose $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ and let $\mathfrak{u}$ denote the nilradical of $\mathfrak{p}$. We define *a Levi factor* $\mathfrak{l}$ of $\mathfrak{p}$ to be a complex subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ such that

$$\mathfrak{l} \cap \mathfrak{u} = \{0\} \quad \text{and} \quad \mathfrak{l} + \mathfrak{u} = \mathfrak{p}.$$

We fix a complex flag space Y for G_0. For $y \in Y$ we let U_y denote the unipotent radical of the parabolic subgroup P_y and we put $\mathfrak{u}_y$ =Lie(U_y). Then $\mathfrak{u}_y$ is the nilradical of $\mathfrak{p}_y$ and there is a $1-1$ correspondence between Levi factors of P_y and Levi factors of $\mathfrak{p}_y$. Specifically, if L is a Levi factor of P_y then

$$\mathfrak{l} = \mathfrak{z} + \mathrm{Lie}(L)$$

is the corresponding Levi factor of $\mathfrak{p}_y$.

Let γ be the associated compact conjugation of G [Section 2]. We associate a specific Levi factor to each $y \in Y$ by way of the following lemma.

Lemma 4.1 *Suppose $y \in Y$ and define*

$$L = P_y \cap \gamma(P_y).$$

Then L is a Levi factor of P_y.

Proof: Since L is a Zariski closed subgroup of G invariant under γ (and thus a complex reductive group) it suffices to check that L contains a maximal compact subgroup of P_y. Let M_0 be a maximal compact subgroup of P_y and let F_0 be the fixed point set of γ in G. Since F_0 is a maximal compact subgroup of G there exists $g \in G$ such that $gM_0g^{-1} \subseteq F_0$. On the other hand, F_0 acts transitively on Y, so there exists $h \in F_0$ such that $hgP_yg^{-1}h^{-1} = P_y$. It follows that $hg \in P_y$. Therefore

$$hgM_0g^{-1}h^{-1} = P_y \cap F_0$$

is a maximal compact subgroup of P_y contained in L. ■

The Levi factor L of the previous lemma will be called *the γ-stable Levi factor of* P_y. Define

$$\mathfrak{l} = \mathfrak{z} + \mathrm{Lie}(L)$$

the corresponding γ-stable Levi factor of $\mathfrak{p}_y$. In fact, using the same letter γ to denote the conjugation of $\mathfrak{g}$ defined by

$$\gamma = \tau\theta$$

it follows that

$$\mathfrak{l} = \mathfrak{p}_y \cap \gamma(\mathfrak{p}_y).$$

Since it is known that the centralizer of $\mathfrak{l}$ in U_y is trivial, it follows from the Levi decomposition for P_y that L is the normalizer of $\mathfrak{l}$ in P_y.

A complex subalgebra $\mathfrak{r} \subseteq \mathfrak{g}$ will be called *bistable* if

$$\tau(\mathfrak{r}) = \mathfrak{r} \text{ and } \theta(\mathfrak{r}) = \mathfrak{r}.$$

Let $\mathfrak{m}_0 \subseteq \mathfrak{g}$ be the real form given by the fixed point set of γ. Observe that if $\mathfrak{r}$ is a bistable subalgebra of $\mathfrak{g}$ then $\mathfrak{m}_0 \cap \mathfrak{r}$ is a real form of $\mathfrak{r}$. Since there exists a compact

Lie group with Lie algebra $\mathfrak{m}_0$, it follows that every subalgebra of $\mathfrak{m}_0$ is reductive. Thus each bistable subalgebra of $\mathfrak{g}$ is a complex reductive Lie algebra.

We define a point $y \in Y$ to be *special* if the corresponding parabolic subalgebra $\mathfrak{p}_y$ contains a bistable Cartan subalgebra of $\mathfrak{g}$. One knows that each point in Y is G_0-conjugate (and K-conjugate) to a special point [8]. Therefore, in order to describe the stabilizers $G_0[y]$ and K_y it suffices to consider special points. We fix a special point $y \in Y$. Let $\mathfrak{l}$ be the γ-stable Levi factor of $\mathfrak{p}_y$ and put

$$\mathfrak{j} = \mathfrak{l} \cap \tau(\mathfrak{l}).$$

Thus $\mathfrak{j}$ is the largest bistable subalgebra of $\mathfrak{p}_y$. We refer to $\mathfrak{j}$ as the *full bistable subalgebra* of $\mathfrak{p}_y$. Let $\mathfrak{u}_y$ be the nilradical of $\mathfrak{p}_y$ and define the following two complex subalgebras of $\mathfrak{p}_y$:

$$\mathfrak{u}(\tau) = \mathfrak{l} \cap \tau(\mathfrak{u}_y) \oplus \mathfrak{u}_y \cap \tau(\mathfrak{p}_y) \quad \text{and} \quad \mathfrak{u}(\theta) = \mathfrak{l} \cap \theta(\mathfrak{u}_y) \oplus \mathfrak{u}_y \cap \theta(\mathfrak{p}_y)\ .$$

Lemma 4.2 *Suppose $y \in Y$ is special. Then, using the notations introduced above, we have the following.*

(a) $\mathfrak{j} \oplus \mathfrak{l} \cap \mathfrak{u}(\tau)$ *and* $\mathfrak{j} \oplus \mathfrak{l} \cap \mathfrak{u}(\theta)$ *are parabolic subalgebras of* $\mathfrak{l}$ *with Levi factor* $\mathfrak{j}$ *and corresponding nilradicals* $\mathfrak{l} \cap \mathfrak{u}(\tau)$ *and* $\mathfrak{l} \cap \mathfrak{u}(\theta)$*, respectively.*

(b) $\mathfrak{l} = \mathfrak{j} \oplus \mathfrak{l} \cap \mathfrak{u}(\tau) \oplus \mathfrak{l} \cap \mathfrak{u}(\theta)$.

(c) $\mathfrak{u}(\tau)$ *and* $\mathfrak{u}(\theta)$ *are the nilradicals of* $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y)$ *and* $\mathfrak{p}_y \cap \theta(\mathfrak{p}_y)$*, respectively.*

(d) $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y) = \mathfrak{j} \oplus \mathfrak{u}(\tau)$ *and* $\mathfrak{p}_y \cap \theta(\mathfrak{p}_y) = \mathfrak{j} \oplus \mathfrak{u}(\theta)$.

Proof: Observe that $\mathfrak{j}$ normalizes both $\mathfrak{l} \cap \tau(\mathfrak{u}_y)$ and $\mathfrak{l} \cap \theta(\mathfrak{u}_y)$. Let $\mathfrak{c}$ be a bistable Cartan subalgebra contained in $\mathfrak{p}_y$. Thus $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{j}$. Let $\Sigma^+(\mathfrak{c}, \mathfrak{j})$ be a positive system of roots for $\mathfrak{c}$ in $\mathfrak{j}$ and let $\Sigma(\mathfrak{c}, \mathfrak{l} \cap \tau(\mathfrak{u}_y))$ and $\Sigma(\mathfrak{c}, \mathfrak{l} \cap \theta(\mathfrak{u}_y))$ denote the roots of $\mathfrak{c}$ in $\mathfrak{l} \cap \tau(\mathfrak{u}_y)$ and $\mathfrak{l} \cap \theta(\mathfrak{u}_y)$, respectively. Observe that the conjugation $\gamma = \tau\theta$ acts as -1 on the roots of $\mathfrak{c}$ in $\mathfrak{g}$. It follows that

$$\Sigma^+(\mathfrak{c}, \mathfrak{j}) \cup \Sigma(\mathfrak{c}, \mathfrak{l} \cap \tau(\mathfrak{u}_y)) \quad \text{and} \quad \Sigma^+(\mathfrak{c}, \mathfrak{j}) \cup \Sigma(\mathfrak{c}, \mathfrak{l} \cap \theta(\mathfrak{u}_y))$$

are each positive system of roots for $\mathfrak{c}$ in $\mathfrak{l}$ and that

$$\Sigma(\mathfrak{c}, \mathfrak{l} \cap \tau(\mathfrak{u}_y)) = -\Sigma(\mathfrak{c}, \mathfrak{l} \cap \theta(\mathfrak{u}_y)).$$

Thus (a) and (b) follow.

We establish (c) and (d) for the Lie algebra $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y)$. The proof for $\mathfrak{p}_y \cap \theta(\mathfrak{p}_y)$ is identical. Observe that $\mathfrak{u}(\tau)$ is a subalgebra of $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y)$ normalized by $\mathfrak{j}$. From part (a) it follows that $\mathfrak{l} \cap \tau(\mathfrak{u}_y) \oplus \mathfrak{u}_y$ is contained in the nilradical of a Borel subalgebra of $\mathfrak{g}$. Therefore $\mathfrak{u}(\tau)$ is a nilpotent ideal of $\mathfrak{j} \oplus \mathfrak{u}(\tau)$. We can thus deduce the direct sum decomposition

$$\mathfrak{p}_y \cap \tau(\mathfrak{p}_y) = \mathfrak{j} \oplus \mathfrak{u}(\tau)$$

as well as the fact that $\mathfrak{u}(\tau)$ is contained in the derived algebra of $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y)$ directly from the root space decomposition for the adjoint action of $\mathfrak{c}$ in $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y)$. Thus (c) and (d) follow. ■

We now show that the above lemma descends to level of the groups. In particular, suppose $y \in Y$ is special and let $L \subseteq P_y$ be the γ-stable Levi factor. Let

$$J = L \cap \tau(L).$$

Observe that J is the largest subgroup of P_y invariant under both τ and θ. Also observe that the corresponding Lie algebra

$$\mathfrak{j} = \mathfrak{z} \oplus \mathrm{Lie}(J).$$

is the full bistable subalgebra of $\mathfrak{p}_y$.

Lemma 4.3 *Suppose $y \in Y$ is special. Then, using the notations from above we have the following.*
(a) The unipotent radicals of $P_y \cap \tau(P_y)$ and $P_y \cap \theta(P_y)$ are $exp(\mathfrak{u}(\tau))$ and $exp(\mathfrak{u}(\theta))$, respectively.
(b) J is a Levi factor of both $P_y \cap \tau(P_y)$ and $P_y \cap \theta(P_y)$.

Proof: We establish (a) and (b) for the group $P_y \cap \tau(P_y)$. The proof for $P_y \cap \theta(P_y)$ is identical. Since $\mathfrak{u}(\tau)$ is a complex subalgebra of the nilradical of a Borel subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ it follows that

$$\exp(\mathfrak{u}(\tau)) \subseteq G$$

is a connected, simply connected, Zariski closed, nilpotent subgroup of G. In addition, $\mathfrak{u}(\tau)$ is the nilradical of $\mathfrak{p}_y \cap \tau(\mathfrak{p}_y)$, so that $\exp(\mathfrak{u}(\tau))$ is normal in $P_y \cap \tau(P_y)$. On the other hand, J is a complex reductive group since J is Zariski closed in G and invariant under γ. Therefore it follows from the previous lemma that

$$J \cap \exp(\mathfrak{u}(\tau)) = \{e\}$$

and that $J{\cdot}\exp(\mathfrak{u}(\tau))$ is an open subgroup of $P_y \cap \tau(P_y)$. This establishes (a).
To establish (b), let Q be a Levi factor of $P_y \cap \tau(P_y)$ containing J^{id}. Suppose $l \in L$, $u \in U_y$ and $g = l \cdot u$ normalizes the identity component J^{id}. Then u centralizes the group $l \cdot J^{\mathrm{id}} \cdot l^{-1}$. Since this last group contains a Cartan subgroup of G^{id} it follows that u is trivial. Therefore Q is contained in L. Similarly one shows Q is contained in $\tau(L)$. This proves (c). ■

Observe that J is the normalizer of $\mathfrak{j}$ in each of the two groups $P_y \cap \tau(P_y)$ and $P_y \cap \theta(P_y)$.
We are now ready to describe the stabilizers $G_0[y]$ and K_y. Let $\mathfrak{j}_0 = \mathfrak{g}_0 \cap \mathfrak{j}$ and let $\mathfrak{u}(\tau)_0 = \mathfrak{g}_0 \cap \mathfrak{u}(\tau)$. Then $\mathfrak{j}_0$ and $\mathfrak{u}(\tau)_0$ are real forms of $\mathfrak{j}$ and $\mathfrak{u}(\tau)$, respectively. Observe that $\mathfrak{K}_0 \cap \mathfrak{p}_y = \mathfrak{K}_0 \cap \mathfrak{j}$ is the set of γ-fixed vectors in $\mathfrak{j}_0$ as well as a real form of $\mathfrak{K} \cap \mathfrak{j}$. It follows from Lemma 4.3 that

$$\mathfrak{g}_0 \cap \mathfrak{p}_y = \mathfrak{j}_0 \oplus \mathfrak{u}(\tau)_0 \quad \text{and} \quad \mathfrak{K} \cap \mathfrak{p}_y = \mathfrak{K} \cap \mathfrak{j} \oplus \mathfrak{K} \cap \mathfrak{u}(\theta).$$

We now show that this result descends to the groups. Let

$$J_0 = \text{the normalizer of } \mathfrak{j} \text{ in } G_0[y]$$

and recall that $K[y]$ is the Zariski closure of $K_0[y]$ in K_y.

Theorem 4.4 *Suppose $y \in Y$ is special. Then, using the notations from above, we have the following..*
(a) *$exp(\mathfrak{u}(\tau)_0)$ is a closed, connected, simply connected, nilpotent normal subgroup of $G_0[y]$ and $G_0[y]$ is a semidirect product of J_0 with $exp(\mathfrak{u}(\tau)_0)$.*
(b) *J_0 is a reductive Lie group with Lie algebra $\mathfrak{j}_0$, maximal compact subgroup $K_0[y]$ and corresponding Cartan involution given by the restriction of θ to J_0.*
(c) *$exp(\mathfrak{K} \cap \mathfrak{u}(\theta))$ is a closed, connected, simply connected, nilpotent normal subgroup of K_y and K_y is a semidirect product of $K[y]$ with $exp(\mathfrak{K} \cap \mathfrak{u}(\theta))$.*
(d) *$K[y]$ is the normalizer of $\mathfrak{j}$ in K_y*

Proof: Let J_τ be the fixed point set of τ in J. Then J_τ is a reductive Lie group and a real form of J. Since J is the normalizer of $\mathfrak{j}$ in $P_y \cap \tau(P_y)$ and since J is τ-stable, it follows that

$$\mathrm{Ad}(J_0) = \mathrm{Ad}(G_0[y]) \cap J_\tau$$

is an open subgroup of J_τ. Using Lemma 3.1, it follows that $\mathrm{Ad}(G_0[y])$ can be written as the semidirect product

$$\mathrm{Ad}(G_0[y]) = \mathrm{Ad}(J_0) \cdot \exp(\mathfrak{u}(\tau)_0).$$

Therefore (a) follows, since $\exp(\mathfrak{u}(\tau)_0)$ is a simply connected nilpotent Lie group and since the kernel of the adjoint morphism is contained in J_0. Similarly, one checks that K_y is a semidirect product of the normalizer of $\mathfrak{j}$ in K_y with $\exp(\mathfrak{K} \cap \mathfrak{u}(\theta))$.
To establish (b), we first note that J_0 has Lie algebra

$$\mathfrak{j}_0 = \mathfrak{z}_0 \oplus \mathrm{Lie}(J_\tau)$$

where $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$ is the Lie algebra of the kernel of adjoint morphism. Next, we observe that $[J_\tau^{\mathrm{id}}, J_\tau^{\mathrm{id}}]$ is a connected semisimple Lie group with finite center and that the adjoint morphism defines a finite covering

$$\mathrm{Ad} : [J_0^{\mathrm{id}}, J_0^{\mathrm{id}}] \to [J_\tau^{\mathrm{id}}, J_\tau^{\mathrm{id}}].$$

Therefore $[J_0^{\mathrm{id}}, J_0^{\mathrm{id}}]$ has finite center. Since $\mathrm{Ad}(K_0[y]) \subseteq L \cap \tau(L)$ and since J (and therefore J_0) are θ–stable, it remains to show that J_0 is closed under the Cartan decomposition determined by θ. Let $\mathfrak{s}_0$ be the -1 eigenspace of θ in $\mathfrak{g}_0$ and suppose $k \in K$, $\xi \in \exp(\mathfrak{s}_0)$ and that $g = k \cdot \exp(\xi) \in J_0$. Then $\exp(2\xi) \in J_0$. Hence $\mathrm{Ad}(\exp(2\xi))$ normalizes $\mathfrak{p}_y$ and $\mathfrak{j}$. Since

$$\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$$

is a semisimple operator it follows that $\mathrm{Ad}(\exp(\xi))$ normalizes $\mathfrak{p}_y$ and $\mathfrak{j}$. Therefore $\exp(\xi) \in J_0$ and $k \in J_0$. This establishes (b).

To establish (d), let N be the normalizer of $\mathfrak{j}$ in K_y. Observe that N is θ-stable, since J is. Arguing as above, in the case of J_0, one can show that N is a reductive Lie group with Cartan decomposition

$$N = K_0[y] \cdot \exp(i(\mathfrak{K}_0 \cap \mathfrak{p}_y)).$$

This proves the theorem. ■

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