REVISTA DE LA
UNION MATEMATICA ARGENTINA
Volumen 42, Número 2, 2001, Páginas 17-33

The dimensions of Hausdorff and Mendès France. A comparative study

R. Hansen*and M. Piacquadio†

Depto. de Matemática, Fac. de Ingeniería, Universidad de Buenos Aires
Av. Paseo Colón 850 (1063), Buenos Aires, Argentina

Abstract

This paper contains a comparative study of two families of simple planar curves. On the one hand, we have the fractal curves on the unit interval, with self-similar structure, which have associated a Hausdorff dimension. On the other hand, we have the opposite: a class of locally rectifiable unbounded curves, which have another "fractional dimension" defined by M. Mendès France. We propose a geometrical constructive process that will allow us to obtain —as the limit of a sequence of polygonal curves— one curve of the first family, by contracting transformations; and another of the second family, by expansing transformations. Thanks to this process of linking curves from both families, we are able to compare their dimensions —our aim in this work—, and to obtain interesting results such as the equality of the latter in the case of strict self-similarity.

...The reader may feel surprised that there is no mention of Benoît Mandelbrot in these notes. His objects are fractals, i.e., locally irregular. Mine, on the contrary are locally smooth. The curves I discuss are locally rectifiable. My topic could be thought of "anti-Mandelbrotian" within "Mandelbrotmania". I was, I am, and I hope to remain influenced by B. Mandelbrot. [1]

Michel Mendès France.

1 Introduction

In this paper we will study two families of non-intersecting planar curves. The first of these families, which we call $\mathcal{F}_H$, is composed of fractal curves F with a self-similar structure defined by N contractions of ratios $a_1, a_2, ..., a_N$ $(0<a_i<1,\ 1\leq i\leq N)$ and satisfying the *Closed-set criterion* ([2]), which guarantees their non-overlapping

*e-mail: rhansen@fi.uba.ar
†e-mail: gdutt@fi.uba.ar

character. These fractals are also called *Moran fractals*. The Hausdorff dimension of one of these curves is the unique value $\boldsymbol{d}=\dim_H(F)$ that satisfies the equation:

$$\sum_{i=1}^{N} a_i^{\boldsymbol{d}} = 1 \,. \tag{1}$$

The second family, that we will call $\mathcal{F}_{MF}$, contains curves Γ that are: simple, unbounded and locally rectifiable, i.e. any arc of the curve Γ has finite length. We are interested in the "fractional dimension" defined by Mendès France for this type of curve [1].

The two families have no curve in common; moreover, their curves have absolutely different geometrical properties; however, we will see that there exists a geometrical constructive process that allow us to link curves of both families, and therefore, to compare their respective dimensions —such is the aim in this work.

2 The Mendès France dimension of the curve

For a curve $\Gamma\epsilon\mathcal{F}_{MF}$, we fix an origin and we consider the first portion Γ_L of Γ of length L. Let $\varepsilon>0$ be given, and let us consider the set:

$$\Gamma_L(\varepsilon) = \{P \in I\!R^2 / \mathrm{dist}(P, \Gamma_L) < \varepsilon/2\} \,.$$

This set is also known as the *ε–Minkowski sausage of* Γ_L. Let C_L be the length of the boundary of the convex hull of Γ_L. Then, the Mendès France dimension of a curve Γ is, by definition:

$$\dim_{MF}(\Gamma) = \lim_{\varepsilon\searrow 0} \liminf_{L\nearrow\infty} \frac{\log A(\Gamma_L(\varepsilon))}{\log C_L} \,,$$

where $A(\Gamma_L(\varepsilon))$ denotes the area of $\Gamma_L(\varepsilon)$. There is a remark in [1] showing that the value of $\liminf_{L\nearrow\infty} \frac{\log A(\Gamma_L(\varepsilon))}{\log C_L}$ **does not** depend critically on ε, so we will either take ε away, or replace it by a suitable value in order to make calculations; hence we can write:

$$\dim_{MF}(\Gamma) = \liminf_{L\nearrow\infty} \frac{\log A(\Gamma_L(\varepsilon))}{\log C_L} \,. \tag{2}$$

This remark is very important, because, intuitively, it says that it doesn't matter how "wide" the ε–Minkowski sausage is, but how the sausage "fills up" the plane according to the development of Γ_L when L grows. Therefore, we are dealing with a type of dimension which does not look at the curve with a "zoom lens" —as the Hausdorff dimension does; on the contrary, this dimension "zooms out" looking from afar at the behavior of the curve when its length tends to infinity.

To illustrate this idea, let us consider two well known curves. First, the *Archimedean spiral* of step equal to r (Fig. 1). When the length L tends to infinity, we have to step away from the plane again and again to observe its behavior, because its convex hull also grows. And if we continue moving away, soon we won't be able to distinguish the step r, and we will see the spiral as filling up $I\!R^2$ completely. Let us now take $\varepsilon=r$, and let us consider the corresponding r–Minkowski sausage of Γ, $\Gamma(r)$. We can easily see that it covers all the plane; then, the dimension of Mendès France of Γ is 2. Instead, if we consider a *logarithmic spiral* (Fig. 2), it doesn't matter which is the value of ε chosen; no matter how far away we are from $I\!R^2$, we always see an arc —the same arc— of the curve Γ, which has dimension of Hausdorff equal to 1, and its $\Gamma(\varepsilon)$ will always appear equally "thin". It comes therefore, as no surprise that the dimension of Mendès France of this curve is unity.

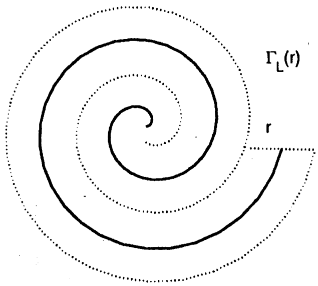

Figure 1

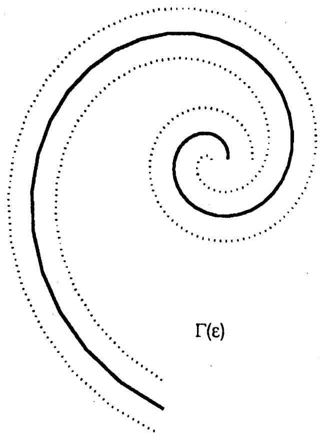

Figure 2

3 The strict self–similar curves

The Hausdorff dimension of a fractal is not, in general, easy to compute, unless the fractal has, for example, some self–similar structure. Among these cases, we have the strict self–similar case with N contracting transformations of ratios $a_1=a_2=...=a_N=1/n$ —for example the well–known *von Koch curve*, with $n=3$, $N=4$ (Fig. 3).
The process by which we obtain such a curve consists of replacing the unit interval $I=[0,1]$ by a polygonal p_1 made out of N segments, all of them with length equal to $1/n$ $(N>n)$. Successively, the polygonals p_2, p_3,..., etc., are obtained by making the same $(n,\ N)$ substitution on each segment of the preceding polygonal. Repeating this replacement process *ad infinitum*, we obtain a bounded continuous curve F of infinite length and infinitely "wrinkled" that belongs to $\mathcal{F}_H$ and whose Hausdorff dimension is, by Eq. (1):

$$\dim_H(F) = \frac{\log N}{\log n} \quad .$$

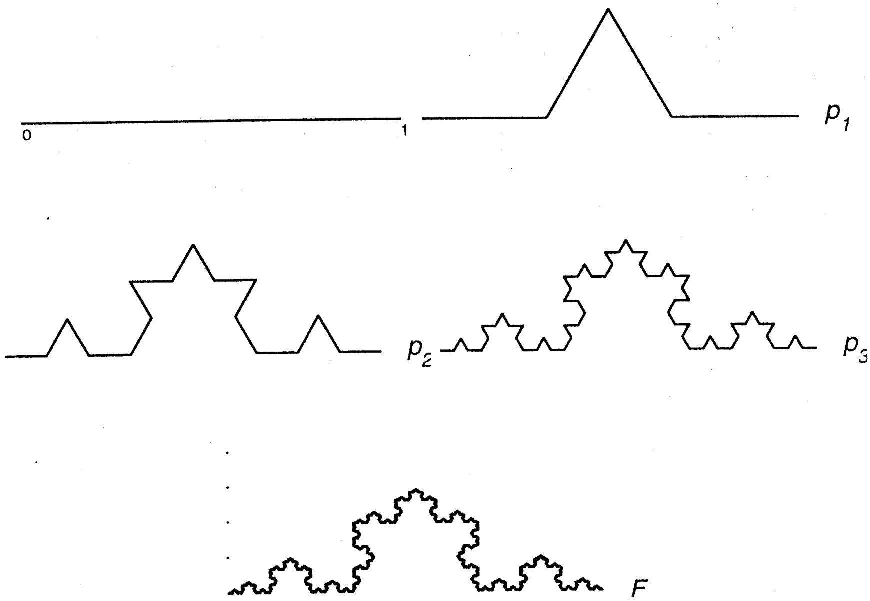

Figure 3

Now (see Fig. 4), if we start again with the interval [0,1], but in the first step we construct a polygonal p_1' with N unit segments, and whose shape is the same as p_1, then the diameter of p_1' will be n times larger than the diameter of p_1: p_1' will be p_1 expanded by a ratio of n to 1, n being the inverse of the unique contraction factor involved in the fractal construction. In the second step, we construct a polygonal p_2' identical to p_2, but with diameter n^2 times larger than that of p_2, and so on. In this way we obtain a continuous unbounded curve Γ, locally rectifiable, $\Gamma \epsilon \mathcal{F}_{MF}$. We will call Γ strictly self-similar. We associate F with Γ, and we will compare the dimensions $\dim_H(F)$ and $\dim_{MF}(\Gamma)$.

For any k–step, the segments of the polygonal p_k' are unity. Let ℓ_k be the length of p_k', then $A(\Gamma_k(\varepsilon)) \approx \varepsilon \times \ell_k = \varepsilon \times N^k$. If C_k is the length of the boundary of the convex hull of p_k', then $C_k \approx \text{const.} \times \text{diam}(p_k') \approx \text{const.} \times n^k$. Thus, the Mendès France dimension of the curve Γ is:

$$\dim_{MF}(\Gamma) = \lim_{k\to\infty} \frac{\log A(\Gamma_k(\varepsilon))}{\log C_k} = \lim_{k\to\infty} \frac{\log(\varepsilon \times N^k)}{\log(\text{const.} \times n^k)} = \frac{\log N}{\log n} .$$

As we can note, in the strict self-similar case, the Mendès France dimension of Γ is equal to the Hausdorff dimension of the corresponding fractal F.

One question that arises in a natural way: does equality hold for processes other than the (n,N) ones? If this equality does not hold, which, then, would be the relationship between these two dimensions if, for instance, we take away strict self–similarity and allow N contraction factors $a_i<1$, $1\leq i\leq N$ to be **different**?

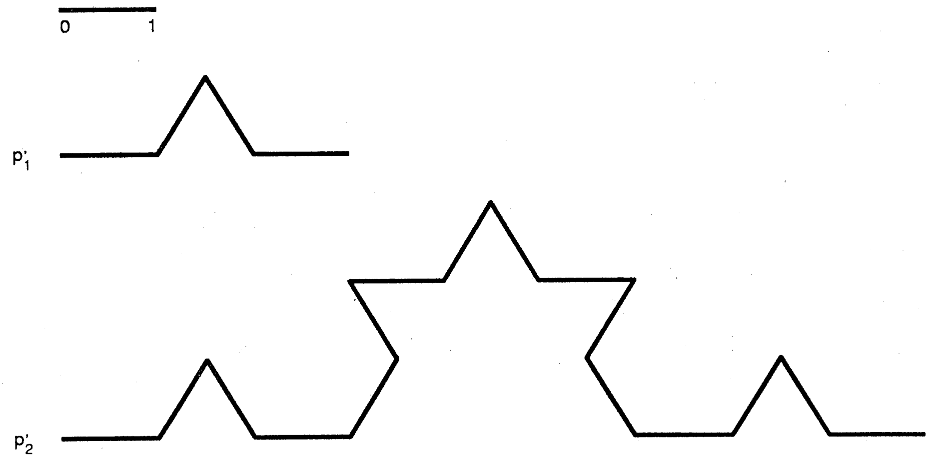

Figure 4

4 Self–similar curves

In the case of strict self–similarity, the expansion ratio used in the construction of polygonals p'_k is the inverse of the **unique** contraction factor of the N transformations generating the fractal. Instead, if we allow the N contraction ratios a_i to be different, as shwon in Fig. 5, we have the possibility of constructing a curve choosing an expansion ratio among the reciprocals $1/a_1, 1/a_2, ..., 1/a_N$, and then we obtain N curves $\Gamma^{a_1}, \Gamma^{a_2}, ..., \Gamma^{a_N}$, all of them the limit curve of a sequence $\{p_k^{a_i}\}_{k\to\infty}(1\leq i\leq N)$of polygonals, and all of them in $\mathcal{F}_{MF}$.

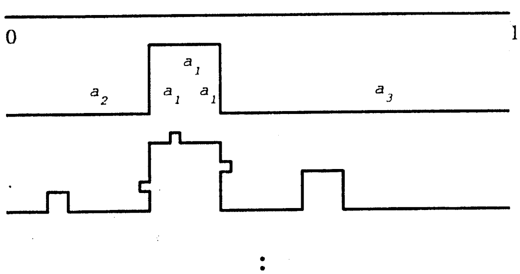

Figure 5: N=3, $a_1<a_2<a_3<1$.

In this section we will analyze the geometric differences among $\Gamma^{a_1}, \Gamma^{a_2}, ..., \Gamma^{a_N}$; we will compare the different $\dim_{MF}(\Gamma^{a_i})$ $(1 \leq i \leq N)$, and we will compare the latter with the **unique** Hausdorff dimension of the corresponding fractal curve F.

Let us consider an iterative replacement process that generates a fractal curve F with ratios $a_1 \leq a_2 \leq ... \leq a_N < 1$, then each of its reciprocals $1/a_1 \geq 1/a_2 \geq ... \geq 1/a_N$ produces a different curve. If we take $1/a_1$ —the largest factor— we can see, as shown in Fig. 6, that in every polygonal $p_k^{a_1}$ the shortest segment is always unity, and $p_k^{a_1}$ adds new segments larger than the segments in $p_{k-1}^{a_1}$. Let us look at Γ^{a_1}: in $p_1^{a_1}$ the shortest segments at the top of the figure have length 1, the other segments are larger than or equal to it in length. $p_2^{a_1}$ shows, clearly, that the shortest segments are again at the top of the figure, all other segments being larger than or equal to them in length. This means that in the limit curve Γ^{a_1}, the length of all segments will be larger than or equal to 1. If we take $1/a_N$ —the smallest factor— we can see (in Fig. 6: $a_N = a_3$) that in every polygonal $p_k^{a_N}$ the largest segments are always unity, and $p_k^{a_N}$ adds new segments smaller than the segments in $p_{k-1}^{a_N}$. So, the limit curve Γ^{a_N} will have arbitrarily small segments, and the length of each segment will be always smaller than or equal to 1. Finally, if we take an intermediate factor $1/a_i$, $i \neq 1, N$ (in Fig. 6: $a_i = a_2$), each k iteration produces polygonal curves $p_k^{a_i}$ that have both larger and shorter segments than those in the preceding polygonal, and therefore the limit curve Γ^{a_i} will have arbitrarily both small and long segments.

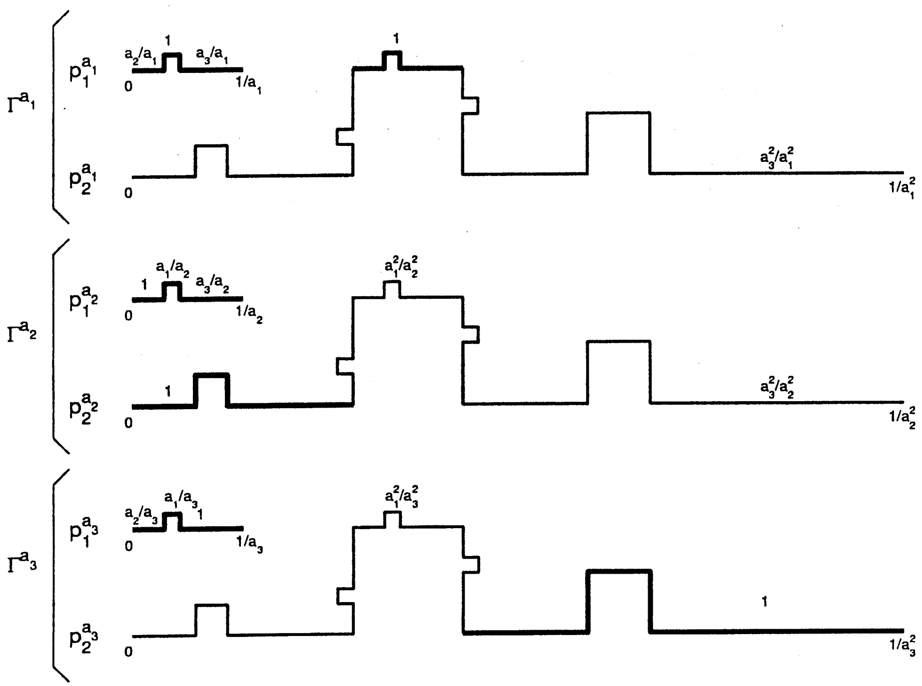

Figure 6: N=3, $1/a_1 > 1/a_2 > 1/a_3$.

This fact causes the limit curves to be very different from one another. Among them the **only one** that is a *resolvable curve* is Γ^{a_1}, the first one; the others are *non-resolvable curves.* By a resolvable curve we mean the following: we know that, for any curve $\Gamma \epsilon \mathcal{F}_{MF}$, if we take a closed ball in $I\!R^2$ with centre on Γ, and we run this ball along the curve, the ball always contains a finite arc of Γ. But if this arc increases its length as the ball runs, i.e., if the arc inside the ball becomes more and more "wrinkled" and larger, then we say that Γ is non-resolvable. Otherwise Γ is resolvable —the formal definition is in [1].

For a resolvable case —Γ^{a_1}— it is easy to calculate the Mendès France dimension, because we can have a very good approximation of $A(\Gamma_k(\varepsilon))$ as $\varepsilon \times \ell_k$. Then, for all resolvable curves Γ, we have, by Eq. (2):

$$\dim_{MF}(\Gamma) = \lim_{k\to\infty} \frac{\log \ell_k}{\log C_k} \quad .$$

If we bear in mind that in each step k, the length of polygonal p'_k is:

$$\ell_k = \left(\frac{1}{a_1}\right)^k \left(\sum_{i=1}^{N} a_i\right)^k = \left(\frac{\sum_{i=1}^{N} a_i}{a_1}\right)^k$$

and

$$C_k \approx \text{const.} \times \left(\frac{1}{a_1}\right)^k$$

we then have

$$\dim_{MF}(\Gamma^{a_1}) = \frac{\log\left(\frac{\sum_{i=1}^{N} a_i}{a_1}\right)}{\log\left(\frac{1}{a_1}\right)} \quad . \tag{3}$$

Notice that it is the diametr —and not the shape of C_k— the relevant magnitude in equation (3). If we want to calculate the dimension for the other non resolvable curves, we find that it is very difficult to estimate $A(\Gamma_k(\varepsilon))$ in the general case. Nevertheless, if we call $\Gamma^{a_1}, \Gamma^{a_2}, ..., \Gamma^{a_N}$ the different curves, we are able to affirm that Γ^{a_1} is the one with the minimal Mendès France dimension —its segments are the longest—, Γ^{a_N} is the one with the maximal dimension —its segments are the shortest ones, hence the curve is the most wrinkled of them all— , and the others have intermediate dimensions. The larger the dimension, the smaller the expansion factor.

This is the Theorem 4.1, which will be proven later.

Theorem 4.1 *Let $F \epsilon \mathcal{F}_H$ be a fractal curve constructed with ratios $a_1 \leq a_2 \leq ... \leq a_N < 1$, and the expansion factors are the recripocals: $1/a_1 \geq 1/a_2 \geq ... \geq 1/a_N$; then:*

$$\dim_{MF}(\Gamma^{a_1}) \leq \dim_{MF}(\Gamma^{a_2}) \leq ... \leq \dim_{MF}(\Gamma^{a_N}) \quad .$$

In order to relate these dimensions to the Hausdorff dimension —Theorem 4.2— we will study the geometric differences of the expanded curves Γ^{a_i}, $1\leq i\leq N$. Consider one of these curves Γ^{a_i}, $i\neq N$ —so $a_i<a_N$. Then, in accordance with what we said above, Γ^{a_i} has segments as large as we want. Now, to fix ideas, let us consider $\varepsilon=1$ and the 1–Minkowski sausage. Let us suppose that on each segment of this curve we make an *ad infinitum* iteration of the corresponding process that generates the fractal F. This new fractal curve, Υ^{a_i}, just does not belong to $\mathcal{F}_{MF}$, so there is no Mendès France dimension associated with it. The curve does not belong to $\mathcal{F}_H$ either; however, it has the same Hausdorff dimension $\boldsymbol{d}$ as the fractal F, i.e., $\boldsymbol{d}=\dim_H(F)$ —because now there is a fractal like F where before there was a segment of Γ^{a_i}. The important thing to note here is that this curve Υ^{a_i} is not covered by the 1–Minkowski sausage, since, if we take some segment in Γ^{a_i} with length ℓ very very large, then the fractal that replaces it is not covered any more by the rectangle with area equal to $\varepsilon\times\ell=1\times\ell$. And this fact is true for **all values** of $\varepsilon>0$.

Let us take, now, the curve Γ^{a_N} expanded by $1/a_N$. All its segments have length smaller than or equal to unity. Let us consider again the 1–Minkowski sausage of Γ^{a_N} and let us make the same *ad infinitum* iteration of the corresponding fractal F on each segment of Γ^{a_N}. Thinking in the same manner as before, this new curve Υ^{a_N} has a Hausdorff dimension equal to $\dim_H(F)$, but now Υ^{a_N} is completely covered by the 1–Minkowski sausage of Γ^{a_N}. If we took some $\varepsilon<1$ in place of $\varepsilon=1$, and we take into account the simple example of Fig. 6, we conclude that Υ^{a_N} will also be covered by the ε–sausage of Γ^{a_N}, except for those arcs corresponding to the few segments in Γ^{a_N} of length larger than ε —i.e. except for the polygonals of the first iterations.

This means that Γ^{a_N} is **the only curve** of all expanded curves that "shares" the $\Gamma(\varepsilon)$ with its corresponding Υ^{a_N} for every ε, as if "ε were not able to distinguish" between a segment and a fractal built on it.

Since Γ^{a_N} is the only curve, among all of Γ^{a_i}, that exclusively "gets wrinkled" while its convex hull increases, this curve is the only one whose shape becomes the fractal form of F, as we move away from the plane in which the curves Γ^{a_i} are drawn.

This fact suggests that both dimensions, $\dim_H(F)$ and $\dim_{MF}(\Gamma^{a_N})$, are the same; and this is the Theorem 4.2.

Theorem 4.2 *Let $F\epsilon\mathcal{F}_H$ be a fractal constructed by the contraction factors $a_1\leq a_2\leq\ldots\leq a_N<1$, and let Γ^{a_N} be the limit curve constructed by the expansion factor $1/a_N$. Then, we have:*

$$\dim_H(F) \ = \ \dim_{MF}(\Gamma^{a_N}) \ .$$

Now, we are able to give the proofs of the Theorems 4.1 and 4.2 respectively.

Proof of the Theorem 4.1. Let i be a fixed value, $1\leq i\leq N$; let us suppose that $a_i=a_{i+1}^m$, where $m>1$. Let us consider the corresponding polygonal p_k of Γ^{a_i} for a step k. Its length is:

$$\ell_k \ = \left(\frac{\sum_{j=1}^N a_j}{a_i}\right)^k = \ \left(\frac{\sigma}{a_i}\right)^k \quad ,$$

where $\sigma=\sum_{j=1}^{n} a_j$. The diameter in the k–step is

$$\operatorname{diam}(p_k) \approx \left(\frac{1}{a_i}\right)^k .$$

Let us now consider the step mk. The corresponding polygonal of $\Gamma^{a_{i+1}}$, p'_{mk}, has length

$$\ell'_{mk} = \left(\frac{\sigma}{a_{i+1}}\right)^{mk} ,$$

and diameter

$$\operatorname{diam}(p'_{mk}) \approx \left(\frac{1}{a_{i+1}}\right)^{mk} .$$

Then:

$$\ell'_{mk} = \frac{\sigma^{mk}}{a_{i+1}^{mk}} = \frac{\sigma^k}{a_i^k}\sigma^{(m-1)k} = \ell_k\sigma^{(m-1)k} ,$$

and then we have

$$\ell'_{mk} > \ell_k .$$

Therefore, for a fixed value of ε, we have

$$A(\Gamma_k^{a_i}(\varepsilon)) < A(\Gamma_{mk}^{a_{i+1}}(\varepsilon)) ,$$

and also

$$\operatorname{diam}(p'_{mk}) = \operatorname{diam}(p_k) ;$$

so we have

$$\frac{1}{a_{i+1}^{mk}} = \left(\frac{1}{a_{i+1}^{m}}\right)^k = \left(\frac{1}{a_i}\right)^k .$$

Thus

$$\frac{\log A(\Gamma_k^{a_i}(\varepsilon))}{\log\left(\frac{1}{a_i}\right)^k} < \frac{\log A(\Gamma_{mk}^{a_{i+1}}(\varepsilon))}{\log\left(\frac{1}{a_{i+1}}\right)^{mk}}$$

and taking limits when k tends to infinity, we have:

$$\dim_{MF}(\Gamma^{a_i}) \leq \dim_{MF}(\Gamma^{a_{i+1}}) .$$

In the case $a_i=a_{i+1}^m$ with non integer m, the calculation is the same, considering the step $[mk]$. ∎

Proof of the Theorem 4.2. It is known that the Hausdorff dimension of a fractal F with these characteristics can be expressed by:

$$\dim_H(F) = \lim_{\varepsilon\to 0}\left(2 - \frac{\log A(F(\varepsilon))}{\log(\varepsilon)}\right) , \tag{4}$$

$A(F(\varepsilon))$ being the area of the ε–Minkowski sausage of F ([2]). Let p_k be the polygonals of successive k–steps in the fractal iteration of F, and let p'_k be the corresponding polygonals of successive steps in the iteration of the limit curve Γ. For a certain value of k, we have that p_k and p'_k are "alike", that is, they have identical shape but different size. Then, if ℓ_k=length of p_k and ℓ'_k=length of p'_k, we have

$$\ell'_k = \left(\frac{1}{a_N}\right)^k \ell_k \ .$$

Besides, if we consider the area of the ε–sausage of p'_k, we have that the corresponding "alike" sausage of p_k satisfies:

$$A(p'_k(\varepsilon)) = \left(\frac{1}{a_N}\right)^{2k} A(p_k(\varepsilon \times a_N^k)) \ . \tag{5}$$

On the other hand, we have that for every polygonal p_k, the largest segment has length equal to a_N^k; so, for every k we can write:

$$A(p_k(a_N^k)) \approx A(F(a_N^k)) \ . \tag{6}$$

Notice that $A(p_k(\lambda^k)) \approx A(F(\lambda^k))$ is not valid for all k when $\lambda < a_N$; for, as k grows, so does the difference between λ^k and a_N^k, breaking down the comparability stated in (6).
Therefore, taking ε=1, from (5) and (6) we have:

$$A(p'_k(1)) \approx \left(\frac{1}{a_N}\right)^{2k} A(F(a_N^k)) \ .$$

Then:

$$\begin{aligned} \dim_{MF}(\Gamma) &= \lim_{k\to\infty} \frac{\log A(p'_k(1))}{\log\left(\frac{1}{a_N}\right)^k} = \lim_{k\to\infty} \frac{\log\left(\left(\frac{1}{a_N}\right)^{2k} A(F(a_N^k))\right)}{\log\left(\frac{1}{a_N}\right)^k} = \\ &= \lim_{k\to\infty}\left(2 - \frac{\log A(F(a_N^k))}{\log(a_N^k)}\right) \end{aligned}$$

and then, by (2):

$$\dim_{MF}(\Gamma) = \dim_H(F) \ .$$

■

The following theorem states that the inequalities from the Theorem 4.1 are strict inequalities.

Theorem 4.3 *Let τ, $i \in \mathbb{N}$ be such that $\tau < i$ and $a_\tau < a_i$; let Γ^{a_τ} and Γ^{a_i} be the expanded limit curves corresponding to the factors $1/a_\tau$ and $1/a_i$ respectively. Then the following inequality holds:*

$$\dim_{MF}(\Gamma^{a_\tau}) < \dim_{MF}(\Gamma^{a_i}) \ .$$

First we need two results:

Lemma 4.1 *For $k \geq 2$, $k \in I\!N$, and $x \in [0,1]$:*

$$\sum_{i=0}^{[\frac{k}{2}]} \binom{k}{i} x^i \geq \frac{1}{2}(1+x)^k \quad ,$$

Proof. In fact,

$$\binom{k}{0} x^0 \geq \binom{k}{k} x^k$$

$$\binom{k}{1} x^1 \geq \binom{k}{k-1} x^{k-1}$$

$$\vdots$$

$$\binom{k}{\frac{k}{2}-1} x^{\frac{k}{2}-1} \geq \binom{k}{\frac{k}{2}+1} x^{\frac{k}{2}+1}.$$

Then,

$$\sum_{i=0}^{\frac{k}{2}-1} \binom{k}{i} x^i \geq \sum_{i=\frac{k}{2}+1}^{k} \binom{k}{i} x^i \quad ,$$

and therefore

$$\sum_{i=0}^{\frac{k}{2}} \binom{k}{i} x^i > \sum_{i=\frac{k}{2}+1}^{k} \binom{k}{i} x^i.$$

To both sides of the last inequality we add the first side, and we obtain:

$$2\sum_{i=0}^{\frac{k}{2}} \binom{k}{i} x^i > \sum_{i=0}^{k} \binom{k}{i} x^i \quad ,$$

that is to say

$$\sum_{i=0}^{\frac{k}{2}} \binom{k}{i} x^i > \frac{1}{2}(1+x)^k.$$

■

If we take the k^{th}–step in the construction of the limit curve Γ^{a_i} —i fixed, expansion factor $1/a_i$— we see that the segments of the polygonal p'_k have lengths $\frac{A}{B}$; the denominator B is always a_i^k, the numerator A is always a product of powers of $a_1, a_2, ..., a_N$ in such a way that the sum of their exponents is equal to k:

$$A = a_1^{j_1} a_2^{j_2} ... a_i^{j_i} ... a_N^{j_N} \quad , \qquad \sum_{\alpha=1}^{N} j_\alpha = k \quad .$$

For one such configuration $(j_1, j_2, ..., j_N)$ we pose the question: how many segments of **this** length are there in step k? Answer: the number of such segments is the numerical coefficient of the term whose "literal" part is $a_1^{j_1} a_2^{j_2} ... a_i^{j_i} ... a_N^{j_N}$, in the development of $(a_1 + a_2 + ... + a_N)^k$. That is to say, there are:

$$\binom{k}{j_1}\binom{k-j_1}{j_2}\binom{k-j_1-j_2}{j_3}...\binom{k-j_1-...-j_{N-2}}{j_{N-1}}\binom{k-j_1-...-j_{N-1}}{j_N}.$$

Let us now consider those segments for which the numerator A has the following configuration: half —or less— of the factors that appear in A are equal to a_1, and the rest of them —to complete a total of k factors— are combinations of $a_2, a_3, ..., a_N$. That is to say, $j_1 \in [0, \frac{k}{2}]$.

Lemma 4.2 *If $j_1 \in [0, \frac{k}{2}]$, then, the number of segments wich have the configuration descripted above, is larger than or equal to $\frac{1}{2}N^k$, where N^k is the total number of segments appearing in step k.*

Proof. Indeed, let us fix exponent j_α, $\alpha \neq N-1$, j_{N-1} will run from 0 to $k-j_1-...-j_{N-2}$; and let us count the corresponding number of such segments:

$$\sum_{j_{N-1}=0}^{k-j_1-...-j_{N-2}} \binom{k}{j_1}\binom{k-j_1}{j_2}...\binom{k-j_1-...-j_{N-2}}{j_{N-1}} =$$

$$= \binom{k}{j_1}\binom{k-j_1}{j_2}...\binom{k-j_1-...-j_{N-3}}{j_{N-2}} \sum_{j_{N-1}=0}^{k-j_1-...-j_{N-2}} \binom{k-j_1-...-j_{N-2}}{j_{N-1}} =$$

$$= \binom{k}{j_1}\binom{k-j_1}{j_2}...\binom{k-j_1-...-j_{N-3}}{j_{N-2}} 2^{k-j_1-...-j_{N-2}}.$$

Now, let us fix all j_α, except j_{N-2} which will be allowed to run from 0 to $k-j_1-...-j_{N-3}$, and let us obtain the total of the corresponding segments:

$$\sum_{j_{N-2}=0}^{k-j_1-...-j_{N-3}} \binom{k}{j_1}\binom{k-j_1}{j_2}...\binom{k-j_1-...-j_{N-3}}{j_{N-2}} 2^{k-j_1-...-j_{N-3}} 2^{-j_{N-2}} =$$

$$= 2^{k-j_1-...-j_{N-3}} \binom{k}{j_1}...\binom{k-j_1-...-j_{N-4}}{j_{N-3}} \sum_{j_{N-2}=0}^{k-j_1-...-j_{N-3}} \binom{k-j_1-...-j_{N-3}}{j_{N-2}} 2^{-j_{N-2}} =$$

$$= 2^{k-j_1-...-j_{N-3}} \binom{k}{j_1}...\binom{k-j_1-...-j_{N-4}}{j_{N-3}} \left(1+\frac{1}{2}\right)^{k-j_1-...-j_{N-3}}.$$

Next, let us fix all j_α, except j_{N-3}, that we will run from 0 to $k-j_1-...-j_{N-4}$, doing the same as before, and we again obtain the total number of corresponding segments:

$$\binom{k}{j_1}\binom{k-j_1}{j_2}...\binom{k-j_1-...-j_{N-5}}{j_{N-4}} 4^{k-j_1-...-j_{N-4}}.$$

Proceeding in this way with the rest of the j_α, let us fix j_1 and let us run j_2 from 0 to $k-j_1$. Counting as above, the corresponding number of such segments is:

$$\binom{k}{j_1}(N-1)^{k-j_1}, \qquad (j_1 = j_{N-(N-1)}) \ .$$

Finally, running j_1 from 0 to $\frac{k}{2}$ we obtain the total number of segments, and by the Lemma 4.1, we have:

$$\sum_{j_1=0}^{\frac{k}{2}} \binom{k}{j_1}(N-1)^{k-j_1} = (N-1)^k \sum_{j_1=0}^{\frac{k}{2}} \binom{k}{j_1}(N-1)^{-j_1} \geq$$

$$\geq (N-1)^k \frac{1}{2}\left(1+\frac{1}{N-1}\right)^k = \frac{1}{2}N^k \ ,$$

■

We can prove the theorem now.

Proof of Theorem 4.3. We will make here a simplification, considering the case τ=1 and $1<i\leq N$, since the proof for the general case —such as it is at present— would exceed the limits of this work.

We want to prove that if i>1, then $\dim_{MF}(\Gamma^{a_i}) > \dim_{MF}(\Gamma^{a_1})$. We will suppose, without loss of generality, that $a_1<a_2\leq a_i$, i fixed between 2 and N.

Let p_k^i be the polygonal of the corresponding k^{th} step of the limit curve Γ^{a_i}. The diameter of this polygonal is equal to $\left(\frac{1}{a_i}\right)^k$.

As we said before, in this polygonal there are N^k segments whose lengths can be written thus:

$$\frac{a_1^{j_1}a_2^{j_2}\ldots a_i^{j_i}\ldots a_N^{j_N}}{a_i^k} \ , \qquad \sum_{\alpha=1}^{N} j_\alpha = k \ .$$

Let $m \in I\!N$, $m<k$, and let p_m^i be the corresponding polygonal. This polygonal has N^m segments, and a diameter equal to $\left(\frac{1}{a_i}\right)^m$.

Let $\overline{p}_m^i$ be a polygonal with the same shape as p_m^i but expanded by a ratio equal to $\left(\frac{1}{a_i}\right)^{k-m}$. The diameter of $\overline{p}_m^i$ is, now, $\left(\frac{1}{a_i}\right)^k$, and the lengths of its segments are of the form:

$$\left(\frac{1}{a_i}\right)^{k-m} \frac{a_1^{j_1}a_2^{j_2}\ldots a_N^{m-(j_1+\ldots+j_{N-1})}}{a_i^m} \ ,$$

where j_α are not necessarily the same as before.

Now let us choose a number m —later we will exhibit the explicit value of this m— such that any segment with length equal to

$$\left(\frac{1}{a_i}\right)^{k-m} \frac{a_1^{\frac{m}{2}}a_2^{\frac{m}{2}}a_3^0\ldots a_N^0}{a_i^m} = \frac{a_1^{\frac{m}{2}}a_2^{\frac{m}{2}}}{a_i^k}$$

—that is to say $j_1=j_2=\frac{m}{2}$ and $j_3=...=j_N=0$— is comparable to unity.
This entails that segments, whose lengths have a "configuration" such that $j_1\leq\frac{m}{2}$, the remaining j_α arbitrary —the sum always being m— become larger than or equal to unity.
Now then, the number of these segments is, by virtue of Lemma 4.2, larger than or equal to the half of the total number of segments, which is N^m.
In other words, if ℓ_m is the length of p^i_m, and $\overline{\ell}_m$ is the length of $\overline{p}^i_m$, we have:

$$\overline{\ell}_m=\left(\frac{1}{a_i}\right)^{k-m} \quad , \quad \ell_m=\left(\frac{1}{a_i}\right)^{k-m}\frac{\left(\sum_{j=1}^N a_j\right)^m}{a_i^m}=\frac{\left(\sum_{j=1}^N a_j\right)^m}{a_i^k} \quad ,$$

and also:

$$\overline{\ell}_m<\ell_k \quad .$$

Therefore, taking $\varepsilon=1$, it is true that:

$$A(\overline{p}^i_m(1))\leq A(p^i_k(1)) \quad .$$

But, since more than half of the total number of segments are segments larger than or equal to unity, it follows that:

$$1\times\frac{1}{2}\overline{\ell}_m \ \leq \ A(\overline{p}^i_m(1))$$

and then

$$\frac{1}{2}\frac{\left(\sum_{j=1}^N a_j\right)^m}{a_i^k} \ \leq \ A(p^i_k(1)) \quad .$$

Therefore,

$$\frac{\log\left(\frac{1}{2}\frac{\left(\sum_{j=1}^N a_j\right)^m}{a_i^k}\right)}{\log\left(\frac{1}{a_i^k}\right)} \leq \frac{\log A(p^i_k(1))}{\log\left(\frac{1}{a_i^k}\right)} \quad . \tag{7}$$

Next, we will calculate the explicit value of m —which we have chosen in order to satisfy the last inequality. The value m was chosen requiring that:

$$\frac{a_1^{\frac{m}{2}}a_2^{\frac{m}{2}}}{a_i^k} \approx 1 \ ;$$

then,

$$\frac{a_1^{\frac{m}{2}}a_2^{\frac{m}{2}}}{a_i^k}=\frac{a_1^{\frac{m}{2}}\left(a_1^{\log_{a_1}(a_2)}\right)^{\frac{m}{2}}}{a_i^k}=$$

$$=\frac{a_1^{\frac{m}{2(\log_{a_1}(a_2)+1)}}}{a_i^k}=\frac{\left(a_1^{\frac{m(1+\log_{a_1}(a_2))}{2k}}\right)^k}{a_i^k}\approx 1 \ .$$

That is to say:

$$\frac{a_1^{m(1+\log_{a_1}(a_2))}}{2k} \approx a_i \ ;$$

therefore

$$\frac{m(1+\log_{a_1}(a_2))}{2k} \approx \frac{1}{\log_{a_i}(a_1)} \ ,$$

that is to say

$$\frac{m}{k} \approx \frac{2}{\log_{a_i}(a_1)(1+\log_{a_1}(a_2))} = \frac{2}{\log_{a_i}(a_1)+\log_{a_i}(a_1)\log_{a_1}(a_2)} \ ,$$

so that

$$\frac{m}{k} \approx \frac{2}{\log_{a_i}(a_1 a_2)} \ . \tag{8}$$

Going back to inequality (7), we have that:

$$\frac{\log\left(\frac{1}{2}\right) + m\ \log\left(\sum_{j=1}^{N} a_j\right) + k\ \log\left(\frac{1}{a_i}\right)}{k\ \log\left(\frac{1}{a_i}\right)} \leq \frac{\log A(p_k^i(1))}{k \log\left(\frac{1}{a_i}\right)} \ .$$

Replacing $\frac{m}{k}$ by the expression (8), we obtain:

$$\frac{\log\left(\frac{1}{2}\right)}{k\ \log\left(\frac{1}{a_i}\right)} + \frac{2}{\log_{a_i}(a_1 a_2)} \frac{\log\left(\sum_{j=1}^{N} a_j\right)}{\log\left(\frac{1}{a_i}\right)} + 1 \leq \frac{\log A(p_k^i(1))}{k \log\left(\frac{1}{a_i}\right)} \ ,$$

and taking limits when k tends to infinity, we have that:

$$\frac{\log\left(\sum_{j=1}^{N} a_j\right)}{\log\left(\frac{1}{\sqrt{a_1 a_2}}\right)} + 1 \leq \dim_{MF}(\Gamma^{a_i}) \ .$$

Finally, since $a_1 < a_2$, we have $\frac{1}{a_1} > \frac{1}{\sqrt{a_1 a_2}}$, so that:

$$1 + \frac{\log\left(\sum_{j=1}^{N} a_j\right)}{\log\left(\frac{1}{a_1}\right)} < 1 + \frac{\log\left(\sum_{j=1}^{N} a_j\right)}{\log\left(\frac{1}{\sqrt{a_1 a_2}}\right)} \ .$$

Therefore, taking equality (3) into account, we have:

$$\dim_{MF}(\Gamma^{a_1}) < \dim_{MF}(\Gamma^{a_i}) \ .$$

■

5 Remark

The preceding theorem's proof is based on arguments and ideas that are strongly geometrical; however, if we consider the extremal cases τ=1 and i=N —minimal and maximal dimensions respectively— a completely different —and much shorter!— proof of the corresponding inequality can be given.

Remark 5.1

$$\dim_{MF}(\Gamma^{a_1}) \; < \; \dim_{MF}(\Gamma^{a_N}) \quad , \tag{9}$$

Proof. Indeed, for the first curve Γ^{a_1} we have a formula that allows us to calculate its dimension, and for Γ^{a_N} —which is the only curve whose dimension is equal to the dimension of the associated fractal, by virtue of Theorem 4.2— we have an implicit equation that is satisfied by the Hausdorff dimension of the fractal. The Hausdorff dimension is the value $\boldsymbol{d_H}$=$\dim_H(F)$ that satisfies:

$$\sum_{i=1}^{N} a_i^{\boldsymbol{d_H}} = 1 \; ;$$

on the other hand, if $\boldsymbol{d}$=$\dim_{MF}(\Gamma^{a_1})$, then $\boldsymbol{d}$ satisfies:

$$\sum_{i=1}^{N} \frac{a_i}{a_1} a_1^{\boldsymbol{d}} = 1 \quad ,$$

since we have

$$\sum_{i=1}^{N} a_i a_1^{\boldsymbol{d}-1} = 1 \quad ,$$

if and only if

$$a_1^{\boldsymbol{d}} \sum_{i=1}^{N} \frac{a_i}{a_1} = 1 \quad .$$

Taking logarithms in this last equation, we obtain:

$$\boldsymbol{d} \; \log(a_1) \; + \; \log\left(\frac{\sum_{i=1}^{N} a_i}{a_1}\right) = 0 \quad ,$$

and then:

$$\boldsymbol{d} \; = \; \frac{\log\left(\frac{\sum_{i=1}^{N} a_i}{a_1}\right)}{\log\left(\frac{1}{a_1}\right)} \; = \; \dim_{MF}(\Gamma^{a_1}) \quad .$$

Let us consider now the following functions:

$$f(x) = \left(\sum_{i=1}^{N} a_i a_1^{x-1}\right) - 1 \quad \text{and} \quad g(x) = \left(\sum_{i=1}^{N} a_i^x\right) - 1 \quad .$$

According to what we just wrote, we have:

$$f(\boldsymbol{d}) = 0 \quad \text{and} \quad g(\boldsymbol{d_H}) = 0 \ .$$

Besides, f and g are decreasing functions ($a_i<1$, $i=1,\ldots,N$). Let us compare any term in the expression of f with the corresponding term in g —except the first term which is equal in both functions:
for $x>1$, we have

$$a_1^{x-1} < a_i^{x-1} \qquad (i \neq 1) \ ,$$

so

$$\frac{a_i}{a_1} a_1^x \leq a_i^x \ ;$$

therefore

$$f(x) < g(x) \quad \text{for } x > 1 \ ;$$

in particular

$$f(\boldsymbol{d_H}) < g(\boldsymbol{d_H}) = f(\boldsymbol{d}) \ ,$$

and therefore

$$\boldsymbol{d} < \boldsymbol{d_H} \ .$$

■

Note. From this remark we conclude that the study of Hausdorff and Mendès France dimensions of curves Γ^{a_i}, $1\leq i\leq N$, associated with the same fractal curve F, has a very different nature for the case $i=1$, $i=N$, and for the case $i\neq 1,N$. Because of this we gave the proof of Theorem 4.3 restricting the general situation $1\leq\tau<i\leq N$ to $\tau=1$, since, as stated above, the proof of the general case, such as it is at present, would exceed the limits of this paper.

Acknowledgment. The authors wish to thank E. Cesaratto for the valuable help in the preparation of all the figures in this article.

References

[1] Mendès France, M., *The Planck Constant of a Curve*, Fractal Geometry and Analysis, Kluwer Academic Publishers, (1991) pp. 325-366.

[2] Tricot, C., *Curves and Fractal Dimension*, Springer-Verlag, New York, (1995) p. 119 and 187.

Recibido : 30 de diciembre de 1999.
Aceptado : 19 de junio de 2001.