TRANSFERENCE OF A LITTLEWOOD-PALEY-RUBIO INEQUALITY AND DIMENSION FREE ESTIMATES

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Dedicated to Professor Roberto Macías

Abstract. A well known result by Rubio de Francia asserts that for every finite family of disjoint intervals \( \{I_k\} \) in \( \mathbb{R} \), and \( p \) in the range \( 2 \leq p < \infty \), there exists \( C_p > 0 \) such that
\[
\left\| \sum_k r_k S_{I_k} f \right\|_{L_p^p([0,1])} \leq C_p \|f\|_{L_p^p(\mathbb{R})},
\]
where the \( r_k \)'s are the Rademacher functions.

In this note we prove that, given a compact connected abelian group \( G \) with dual group \( \Gamma \) and \( p \) in the range \( 2 \leq p < \infty \), there is a constant \( C_p \), independent of \( G \) and the particular ordering on \( \Gamma \), such that for every sequence \( \{I_k\} \) of disjoint intervals in \( \Gamma \), we have
\[
\left\| \sum_k r_k S_{I_k} f \right\|_{L_p^p([0,1])} \leq C_p \|f\|_{L_p^p(\Gamma)}.
\]

We obtain the result by a transference approach that can be used for functions taking values in Banach spaces.

1. INTRODUCTION

In [RdeF2], Rubio de Francia proved the following remarkable result establishing a Littlewood-Paley property for disjoint intervals in \( \mathbb{R} \).

Theorem 1.1. Given an interval \( I \subset \mathbb{R} \), denote by \( S_I \) the partial sum operator \( (S_I f) = \hat{f} \chi_I \), where \( \hat{f} \) stands for the Fourier transform of the function \( f \). For every \( p \) in the range \( 2 \leq p < \infty \), there exists \( C_p > 0 \) such that, for every sequence \( \{I_k\} \) of disjoint intervals, we have
\[
\left\| \sum_k |S_{I_k} f|^2 \right\|_{L_p^p(\mathbb{R})}^{1/2} \leq C_p \|f\|_{L_p^p(\mathbb{R})}, \quad f \in L_p^p(\mathbb{R}).
\]

See also [J] for the \( n \)-dimensional analogue.

By Kintchine's inequality, an equivalent formulation of this result can be given in which the expression
\[
\left\| \sum_k |S_{I_k} f|^2 \right\|_{L_p^p(\mathbb{R})}^{1/2}
\]

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is replaced by

\[ \left\| \sum_k r_k S f \right\|_{L^p_{L^p([0,1])}(\mathbb{R})}, \]

where the \( r_k \)'s are the Rademacher functions. In this form, the conclusion of Rubio de Francia’s theorem allows the possibility of extension to functions taking values in a Banach space and led to the following definition (see [B,G,T2]).

**Definition 1.3.** Let \( B \) be a Banach space and let \( p \) be in the range \( 2 \leq p < \infty \). We say that \( B \) satisfies the \( \text{LPR}_p \) property if there exists a constant \( C_{p,B} \) such that for every finite family of disjoint intervals \( \{I_k\} \) in \( \mathbb{R} \), we have

\[ \left\| \sum_k r_k S f \right\|_{L^p_{L^p([0,1])}(\mathbb{R})} \leq C_{p,B} \left\| f \right\|_{L^p_{L^p}(\mathbb{R})}. \]  

(1.4)

Although no examples are known of Banach spaces having the property \( \text{LPR}_p \) other than the classical scalar valued \( L^p \) spaces, it was shown in [B,G,T2] that such a Banach space \( B \) must satisfy \( p(B) = 2 \), where \( p(B) \) denotes the supremum of the Rademacher type of \( B \). Note also that a space having \( \text{LPR}_p \) must be a \( \text{UMD} \) space (see [Bou], [B] and [RdeF1] for the definition and properties of \( \text{UMD} \) spaces). It is possible to consider a version of \( \text{LPR}_p \) in which the group \( \mathbb{R} \) is replaced by the circle group \( \mathbb{T} \) and the partial sum operators \( S f \) are relative to disjoint intervals in \( \mathbb{Z} \). The required inequality would then take the form

\[ \left\| \sum_k r_k S f \right\|_{L^p_{L^p([0,1])}(\mathbb{R})} \leq C_{p,B} \left\| f \right\|_{L^p_{L^p}(\mathbb{T})} \]

for functions \( f \) in \( L^p_{L^p}(\mathbb{T}) \) and we might refer to this property as \( \text{LPR}_p \) relative to \( \mathbb{T} \).

However, the aim of the present note is to show by transference that \( \text{LPR}_p \) already implies \( \text{LPR}_p \) relative to \( \mathbb{T} \). In a subsequent note, it will be shown that in fact the two properties are equivalent and, indeed, are also equivalent to the corresponding property for \( L^p_{L^p}(\mathbb{Z}) \) where the partial sum operators involve disjoint arcs in \( \mathbb{T} \). This is similar to the situation regarding the definition of the \( \text{UMD} \) property in which the required boundedness of the Hilbert transform (for \( \mathbb{R} \)) is equivalent to the boundedness of both the conjugate function (for \( \mathbb{T} \)) and the discrete Hilbert transform (for \( \mathbb{Z} \)) on the corresponding \( L^p \) spaces of \( B \)-valued functions. We shall also transfer the \( \text{LPR}_p \) property from \( \mathbb{T} \) to an arbitrary compact connected abelian group \( G \), giving dimensionally independent constants when \( G = \mathbb{T}^n \).

The technique of transference used in this note has proved effective in many harmonic analysis contexts and has its origins in the work of Calderón and Zygmund on singular integrals [CZ] and Cotlar on the ergodic Hilbert transform [Co]. An early account expository account was given by Calderón [C] and this was followed by a more comprehensive survey by Coifman and Weiss [C,W].
2. The main result

We prove the following result.

**Theorem 2.5.** Let $B$ be a Banach space with property $LPR_p$ for some $p$ in the range $2 \leq p < \infty$ and corresponding constant $C_{p,B}$ in (1.4). Then, for every finite sequence $I_k$ of disjoint intervals in $\mathbb{Z},$

$$\| \sum_k r_k S_k f \|_{L^p_{r_k}(I_k)} \leq C_{p,B} \| f \|_{L^p(I)},$$

(2.6)

where $S_j$ denotes the partial sum operator on $L^p_B(I)$ defined by $(S_j f)(x) = \hat{f}_1 x$ for an interval $I$ in $\mathbb{Z}.$

Before proving this result we note that a space $B$ with property $LPR_p$ is necessarily a $UMD$ space and so the partial sum operators $S_j$ are bounded on $L^p_B(I)$ by the boundedness of the conjugate function. We shall obtain Theorem 2.5 by combining a straightforward adaptation of the techniques in Chapter 3 of [C,W] to a vector valued setting with the following transference result from [B,G,T].

**Theorem 2.7.** Let $G$ be a locally compact abelian group, let $X, Y$ be Banach spaces and let $K$ be a function in $L^1_{\mathcal{L}(X,Y)}(G).$ Assume that there exist strongly continuous representations $R$ and $\hat{R}$ of the group $G$ such that:

1. For every $u \in G,$ we have $R_u \in \mathcal{L}(X, X)$ and $\hat{R}_u \in \mathcal{L}(Y, Y);$
2. There exist constants $c_1$ and $c_2$ such that $\| R_u \|_{\mathcal{L}(X, X)} \leq c_1$ and $\| \hat{R}_u \|_{\mathcal{L}(Y, Y)} \leq c_2, u \in G$
3. $R$ and $\hat{R}$ intertwine $K$ in the sense that

$$K(u) R_v(x) = \hat{R}_v K(u)(x), \text{ } u, v \in G, \text{ } x \in X.$$

We define the operator $T_K = \int_G K(u) R_{-u} du.$ Then $T_K$ is well defined as an element of $\mathcal{L}(X, Y)$ and

$$\| T_K \| \leq \inf_{1 \leq p < \infty} (c_1 c_2 N_{p,X,Y}(K)),$$

where $N_{p,X,Y}(K)$ denotes the operator norm of the convolution operator defined by the kernel $K$ from $L^p_X$ into $L^p_Y.$

We shall apply this result in the following setting. Take $X = L^p_B(I)$, $Y = L^p_{L^p_B([0,1])}(\mathbb{T})$ and $G = \mathbb{R}.$ For $u \in \mathbb{R},$ define $R_u$ on $X$ by $(R_u f)(\theta) = f(e^{i(\theta+u)})$ for $f \in X$ and $\hat{R}_u$ on $Y$ by $(\hat{R}_u g)(\eta) = g(e^{i(\eta+u)})$ for $g \in Y.$ Note that, in this case, $\| R_u \| = \| \hat{R}_u \| = 1.$ Let $k_j \in L^1(\mathbb{R})$ for $j = 1, \ldots, J$ and, for $u \in \mathbb{R}$ a.e., define $K(u) \in \mathcal{L}(X, Y)$ by $K(u)f = (\sum_j r_j k_j(u))f.$ An easy computation gives that, in this situation, the transferred operator $T_k$ satisfies $T_k e_n x = (\sum_j r_j k_j(n)) e_n x,$ where $x \in B$ and $e_n(e^{i\eta}) = e^{i\eta}.$ Thus $T_k$ is the operator from $X$ to $Y$ corresponding to the $\left( L^p_B(I), L^p_{L^p_B([0,1])}(\mathbb{T}) \right) = (X, Y)$ multiplier $m_K = \sum_j r_j k_j \mid Z.$ An application of Theorem 2.7 gives the following.
Theorem 2.8. With the above notation, suppose that there is a constant \( C \) satisfying
\[
\| \sum r_j k_j \ast f \|_{L^p_{\text{loc}}([0,1])} \leq C \| f \|_{L^p_{\text{loc}}(\mathbb{R})}
\] (2.9)
for all \( f \in L^p_{\text{loc}}(\mathbb{R}) \). Then
\[
\| (m \hat{K} \hat{g}) \|_{L^p_{\text{loc}}([0,1])} \leq C \| g \|_{L^p_{\text{loc}}(\mathbb{T})}
\] (2.10)
for all \( g \in L^p_{\text{loc}}(\mathbb{T}) \).

Suppose now that \( B \) is a Banach space with the property \( LPR_p \), where \( 2 \leq p \leq \infty \), and let \( C_{p,B} \) be as in (1.4). For an interval \( I \) in \( \mathbb{Z} \) of the form \([m, n]\), let \( \psi_I \) denote the function on \( \mathbb{R} \) taking the value \( 1 \) on \( (m - \frac{1}{4}, n + \frac{3}{4}) \), \( \frac{1}{2} \) at \( m - \frac{1}{4} \) and at \( n + \frac{3}{4} \), and \( 0 \) elsewhere. Also, let \( I \) denote the interval \( (m - \frac{1}{4}, n + \frac{1}{4}) \) in \( \mathbb{R} \). For a half-infinite interval \( I \) in \( \mathbb{Z} \), define \( \psi_I \) and \( \hat{I} \) similarly. Note that each such function \( \psi_I \) is the normalized multiplier (in the sense of [C,W], p. 13) corresponding to \( S_I \). Now fix disjoint intervals \( I_j \) in \( \mathbb{Z} \) for \( j = 1, \ldots, J \). By the property \( LPR_p \) for \( B \), applied to the disjoint intervals \( I_j \), we have that \( m = \sum r_j \psi_{I_j} \) is an \( \left( L^p_{B}(\mathbb{R}), L^p_{B_{\text{loc}}([0,1])}([0,1]) \right) \) multiplier with multiplier norm not exceeding \( C_{p,B} \). It is straightforward to deduce that \( m|_z \) is an \( \left( L^p_{B}(\mathbb{T}), L^p_{B_{\text{loc}}([0,1])}([0,1]) \right) \) multiplier with multiplier norm not exceeding \( C_{p,B} \) by adapting the arguments given in [C,W] to prove the multiplier restriction theorem ([C,W], Theorem 3.4). Since \( \psi_I \mid_\mathbb{Z} \) equals the characteristic function of \( I \), this gives (2.6) and completes the proof of Theorem 2.5.

3. An application to dimension free estimates

Let \( G \) be a compact connected abelian group with dual group \( \Gamma \). Then \( \Gamma \) can be ordered in a non-canonical way so that it becomes an ordered group. Fix any such ordering \( \leq \) on \( \Gamma \). Let \( B \) be a UMD space and let \( 1 < p < \infty \). It is well known that, for every interval \( I \) in \( \Gamma \), the characteristic function \( \chi_I \) is a multiplier for \( L^p_B(G) \), see [Bo] and [B,G,M2]. Furthermore, the corresponding operator \( S_I \) has norm bounded by a constant dependent only on \( p \) and \( B \). The result of the previous section can be extended to \( L^p_B(G) \) as follows.

Theorem 3.11. Let \( G \) and \( \Gamma \) be as above and let \( B \) have the property \( LPR_p \) for some \( p \) in the range \( 2 \leq p < \infty \). Then there is a constant \( C_{p,B} \) with the property that, for every finite family \( \{I_k\} \) of disjoint intervals in \( \Gamma \),
\[
\| \sum_k r_k S_{I_k} f \|_{L^p_{B_{\text{loc}}([0,1])}(G)} \leq C_{p,B} \| f \|_{L^p_B(G)}
\] (3.12)
for all \( f \in L^p_B(G) \). In particular, the constant \( C_{p,B} \) can be taken to be independent of the particular order on \( \Gamma \) and can be taken to be any constant that suffices for \( L^p_B(\mathbb{T}) \).
The proof of this result uses the techniques developed in [B,G,M1] and [B,G,T1]. Firstly, the result for \( G = \mathbb{T}^n \) is obtained from the result for \( \mathbb{T} \) (that is, from Theorem 2.5) using a vector valued transference argument. The structure of the torsion-free abelian group \( \Gamma \) then yields the result for general \( G \). In particular, the transference from \( \mathbb{T} \) to \( \mathbb{T}^n \) gives the following theorem.

**Theorem 3.13.** Let \( B \) have the property \( LPR_p \) for some \( p \) in the range \( 2 \leq p < \infty \). Then there is a constant \( C_{p,B} \) with the property that, for all \( n \in \mathbb{N} \), all orderings on \( \mathbb{Z}^n \), and every finite family \( \{I_k\} \) of disjoint intervals in \( \mathbb{Z}^n \), we have

\[
\left\| \sum_k r_k S_{I_k} f \right\|_{L^p_B([0,1])} \leq C_{p,B} \left\| f \right\|_{L^p_B(\mathbb{T}^n)} \tag{3.14}
\]

for all \( f \in L^p_B(\mathbb{T}^n) \). The constant \( C_{p,B} \) can be taken to be any constant that suffices for \( L^p_B(\mathbb{T}) \).

**References**


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