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THE TERWILLIGER ALGEBRA OF THE DODECAHEDRON

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Abstract. In this paper, we consider the platonic solids as association schemes. All of them are $P-$ and $Q-$polynomial, except for the dodecahedron, which is not $Q-$polynomial. We compute explicitely the Terwilliger algebra associated to it, and show it is isomorphic to $M_6(\mathbb{C}) \oplus M_6(\mathbb{C}) \oplus M_2(\mathbb{C})$. We also show that the dodecahedron is a counterexample to a conjecture of Terwilliger.

Section 1: Introduction

For the definitions below, we follow [T1]-[T4] and [B-I].

A **symmetric association scheme of class** d is a a pair $Y = (X, \{R_i\}_{i=0}^d)$ consisting of a finite set X and symmetric relations $R_0, R_1, \ldots, R_d$ on X such that:

i) $R_0 = \Delta$, the diagonal of $X \times X$.

ii) $\{R_i\}_{i=0}^d$ is a partition of $X \times X$.

iii) Given $(x, y) \in R_h$, $|\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$ depends only on h, i and j. (This number is denoted by $p^h_{i,j}$).

Let $n = |X|$. Define the adjacency matrices A_i by the formulae:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

We then have $A_0 = I$ (by i) above) and $A_iA_j = \sum_{h=0}^d p^h_{i,j} A_h$ (by iii) above), so they span an algebra $\mathcal{A}(Y)$, called the **Bose-Mesner algebra**.

A symmetric association scheme is P-polynomial (with respect to the order R_0, R_1, ..., R_d iff $\forall\ i \exists a_i, b_i, c_i$; $a_i \neq 0 \neq c_i$ with:

$$A_1A_i = a_iA_{i-1} + b_iA_i + c_iA_{i+1} \qquad (\text{“the } P \text{ condition”})$$

Since the A_i's commute and are symmetric, there is a basis $E_0, E_1, \ldots, E_d$ of $\mathcal{A}(Y)$ consisting of idempotents. By convention, E_0 is the matrix with all entries equal to $\frac{1}{n}$. A symmetric association scheme is called Q-polynomial (with respect to the order $E_0, E_1, \ldots, E_d$) iff $\forall\ i \exists a_i, b_i, c_i$; $a_i \neq 0 \neq c_i$ with:

$$E_1 \circ E_i = a_iE_{i-1} + b_iE_i + c_iE_{i+1} \qquad (\text{“the } Q \text{ condition”})$$

where $\circ$ is Hadamard (entrywise) multiplication.

Also, given a fixed x, we can define diagonal matrices E^*_i by:

$$(E^*_i)_{yy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

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The matrices E_i^* form a basis of a space, which is called the Bose-Mesner dual with respect to x.

The **subconstituent or Terwilliger algebra** of Y with respect to x is the algebra $T(x)$ generated by the Bose-Mesner algebra and its dual with respect to x.

This algebra has been studied in many papers in recent years: see for example [Ca], [Co], [Cu1],[Cu2], [I-I-Y].

In the case of the platonic solids, because of their symmetries, the Terwilliger algebras do not depend on the base point x.

$\mathfrak{D}$ will denote the (complex) Terwilliger algebra of the dodecahedron.

In the following sections, we will compute $\mathfrak{D}$ explicitely and give its isomorphism class.

First we will define subalgebras $\mathfrak{A}, \mathfrak{A}_J$ and $\mathfrak{A}_N$ of $M_{20}(\mathbb{C})$, and show that they are isomorphic to $M_6(\mathbb{C}), M_6(\mathbb{C})$ and $M_2(\mathbb{C})$ respectively.

Then we will demonstrate that $\mathfrak{D} = \mathfrak{A} \oplus \mathfrak{A}_J \oplus \mathfrak{A}_N$.

Section 2: The Algebras $\mathfrak{A}_6$, $\mathfrak{A}_J$ and $\mathfrak{A}_N$

We have that the adjacency table for the dodecahedron can be written as:

$$\begin{array}{lccc}
x_1: & x_2 & x_3 & x_4 \\
x_2: & x_1 & x_5 & x_6 \\
x_3: & x_1 & x_7 & x_8 \\
x_4: & x_1 & x_9 & x_{10} \\
x_5: & x_2 & x_{10} & x_{11} \\
x_6: & x_2 & x_7 & x_{12} \\
x_7: & x_3 & x_6 & x_{13} \\
x_8: & x_3 & x_9 & x_{14} \\
x_9: & x_4 & x_8 & x_{15} \\
x_{10}: & x_4 & x_5 & x_{16} \\
x_{11}: & x_5 & x_{12} & x_{17} \\
x_{12}: & x_6 & x_{11} & x_{18} \\
x_{13}: & x_7 & x_{14} & x_{18} \\
x_{14}: & x_8 & x_{13} & x_{19} \\
x_{15}: & x_9 & x_{16} & x_{19} \\
x_{16}: & x_{10} & x_{15} & x_{17} \\
x_{17}: & x_{11} & x_{16} & x_{20} \\
x_{18}: & x_{12} & x_{13} & x_{20} \\
x_{19}: & x_{14} & x_{15} & x_{20} \\
x_{20}: & x_{17} & x_{18} & x_{19}
\end{array}$$

hence its adjacency matrix is:

$$A_1 = \begin{pmatrix} 0 & J & 0 & 0 & 0 & 0 \\ J & 0 & \alpha & 0 & 0 & 0 \\ 0 & \alpha^t & H & I & 0 & 0 \\ 0 & 0 & I & K & \beta & 0 \\ 0 & 0 & 0 & \beta^t & 0 & J \\ 0 & 0 & 0 & 0 & J & 0 \end{pmatrix}$$

where H, K, α and β are:

$$H = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \qquad K = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1&1&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&0&1&1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&1&0\\ 0&0&1\\ 0&0&1\\ 1&0&0 \end{pmatrix}$$

and J denotes the matrix with all entries equal to 1. (If the size is not clear, we'll write $J_{n\times m}$)

Following the structure of A_1, we will write the 20 by 20 matrices in block form:

$$M = \begin{pmatrix} B_{0,0} & B_{0,1} & B_{0,2} & B_{0,3} & B_{0,4} & B_{0,5}\\ B_{1,0} & B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5}\\ B_{2,0} & B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5}\\ B_{3,0} & B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5}\\ B_{4,0} & B_{4,1} & B_{4,2} & B_{4,3} & B_{4,4} & B_{4,5}\\ B_{5,0} & B_{5,1} & B_{5,2} & B_{5,3} & B_{5,4} & B_{5,5} \end{pmatrix}$$

where the blocks $B_{i,j}$ have size $k_i \times k_j$, where $(k_0, k_1, k_2, k_3, k_4, k_5) = (1, 3, 6, 6, 3, 1)$. We will denote by $L_{i,j}(X)$ the 20 by 20 matrix which has the k_i by k_j matrix X as the (i, j)-block, and the other blocks equal to zero.

2.1 Definition: Consider the homomorphism $\varrho : M_6(\mathbb{C}) \mapsto M_{20}(\mathbb{C})$ defined by $\varrho((m_{i,j})) = \sum_{i,j} m_{i,j} L_{i,j}(J_{k_i \times k_j})$ and let $\mathfrak{A}_J$ be the image of ϱ.

2.2 Remark:

$$\begin{aligned} a) \quad & \mathfrak{A}_J \simeq && M_6(\mathbb{C}) \\ b) \quad & \mathfrak{A}_J \subseteq && \mathfrak{D} \end{aligned}$$

2.3 Definition: Let $\mathfrak{A}_6$ be the subalgebra of $M_6(\mathbb{C})$ generated by H and K.

2.4 Lemma: $\mathfrak{A}_6$ *is isomorphic to* $\mathbb{C}S_3$ *(the group algebra of* S_3*).*

Proof: Since K can be associated with the permutation (12)(34)(56) and H with (16)(23)(45) then $H^2 = K^2 = I$ and HK is then associated with (153)(246), hence $(HK)^3 = I$, thus, $HKHKHK = I$ and since H and K are their own inverses, we get $HKH = KHK$, $(HK)^2 = KH$, and $(KH)^2 = HK$. Hence, H and K generate a group isomorphic to S_3, and thus $\mathfrak{A}_6$ is isomorphic to $\mathbb{C}S_3$ (the group algebra of S_3). In fact, $\mathfrak{A}_6$, as a vector space, has basis $\{I, H, K, HK, KH, HKH\}$.

2.5 Corollary:

$$\mathfrak{A}_6 = V \oplus span\{J\} \oplus span\{N\}$$

where $N = \frac{1}{6}(I - H - K + HK + KH - HKH)$ *and* V *is a subalgebra isomorphic to* $M_2(\mathbb{C})$*.*

Proof: This follows because S_3 has 3 irreducible representations: two one-dimensional (the trivial one and the sign) and one two-dimensional. The trivial one corresponds to J, and the sign character to N.

2.6 Observations:
By **2.4)** , $HN = \frac{1}{6}(H - I - HK + K + HKH - KH) = -N$ and $KN = -N$ too, hence, $N^2 = N$. Also, $JN = 0$.

2.7 Definition: Let $\mathfrak{A}_N$ be the algebra of matrices spanned by the matrices $L_{i,j}(N)$ with $i, j = 2, 3$

2.8 Remark: $\mathfrak{A}_N \simeq M_2(\mathbb{C})$, and $\mathfrak{A}_J\mathfrak{A}_N = \{0\}$, thus $\mathfrak{A}_N + \mathfrak{A}_J = \mathfrak{A}_N \oplus \mathfrak{A}_J$.

2.9 Definition: Let $U = \alpha V$ and $W = V\beta$.

2.10 Lemma:
a) dim $\alpha\mathfrak{A}_6 = 3$
b) $\alpha\mathfrak{A}_6 = U \oplus span\{J_{3\times 6}\}$
c) U *and* W *are two-dimensional irreducible* $\mathfrak{A}_6$*-modules.*

Proof: a) Since $\alpha K = \alpha$, we have that:

$$\begin{aligned} \alpha\mathfrak{A}_6 = &\quad span\{\alpha, \alpha H, \alpha K, \alpha HK, \alpha KH, \alpha KHK\} \\ = &\quad span\{\alpha, \alpha H, \alpha, \alpha HK, \alpha H, \alpha HK\} \\ = &\quad span\{\alpha, \alpha H, \alpha HK\} \end{aligned}$$

Hence, dim $\alpha\mathfrak{A}_6 = 3$, since $\alpha, \alpha H, \alpha HK$ are linearly independent.
b) We have by **2.5)** that $\alpha\mathfrak{A}_6 = \alpha V + \alpha span\{J_{6\times 6}\} + \alpha span\{N\}$. Since $\alpha N = 0$, and $\alpha J_{6\times 6} = J_{3\times 6}$, we have $\alpha\mathfrak{A}_6 = \alpha V + span\{J_{3\times 6}\}$.
Multiplying by J on the right, and using $VJ = 0,\ JJ = 6J$, we see that the sum is direct.
c) By a) and b), $U = \alpha V$ is irreducible of dimension 2. Similarly with W, using that $H\beta = \beta$.
QED

2.11 Definition: Let $G = \alpha\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

2.12 Lemma:

$$i) U^tU \subseteq V \qquad ii) WW^t \subseteq V$$

$$iii) UU^t \subseteq span\{I - \frac{1}{3}J\} \quad iv) W^tW \subseteq span\{I - \tfrac{1}{3}J\} \quad v) UW \subseteq span\{G - \tfrac{2}{3}J\}$$

Proof: i) Since $\alpha = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$, then $\alpha^t\alpha = I + K$.

So: $U^tU = V\alpha^t\alpha V = V(I+K)V \subseteq V$ (since V is an ideal).
ii) Similar to i), using that $\beta\beta^t = I + H$.
iii)

$$\begin{aligned} UU^t &= \alpha VV\alpha^t = \alpha V\alpha^t \\ &\subseteq \alpha\mathfrak{A}_6\alpha^t \\ &\subseteq span\{\alpha\alpha^t, \alpha H\alpha^t, \alpha HK\alpha^t\} = span\{\alpha\alpha^t, \alpha H\alpha^t\} = span\{2I, J - I\} \end{aligned}$$

However, as $\alpha V\alpha^t J = 0$, then $UU^t \subseteq span\{I, J-I\} \cap J^\perp \subseteq span\{I - \frac{1}{3}J\}$.
iv) similar to iii), using that $H\beta = \beta$, $\beta^t\beta = 2I$, and $\beta^t K\beta = J - I$
v) similar to iii) and iv), using now that $\alpha\mathfrak{A}_6\beta = span\{\alpha\beta, \alpha H\beta, \alpha HK\beta\} = span\{\alpha\beta, \alpha HK\beta\}$, $\alpha\beta = G$, and $\alpha HK\beta = 2(J - G)$.
QED

Section 3: The algebras $\mathfrak{A}, \mathfrak{S}, \mathfrak{L}$

3.1 Definition: Let $\mathfrak{A}$ be the set of matrices of the form:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_1(I - \frac{1}{3}J) & u_3 & u_4 & k_2(G - \frac{2}{3}J) & 0 \\ 0 & u_1^t & v_1 & v_2 & w_1 & 0 \\ 0 & u_2^t & v_3 & v_4 & w_2 & 0 \\ 0 & k_3(G^t - \frac{2}{3}J) & w_3^t & w_4^t & k_4(I - \frac{1}{3}J) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $v_i \in V; w_i \in W; u_i \in U; i = 1, \ldots, 4$

3.2 Proposition: $\mathfrak{A}$ *is an algebra of dimension 36.*

Proof: The fact that it is an algebra follows from **2.12)** . The dimension is clear.
QED

3.3 Definition: Let $\mathfrak{L}$ be the algebra generated by $\{L_{i,j}(X) | X \in \mathfrak{A}_6,\ i, j = 2, 3\}$.

3.4 Lemma:

$$\mathfrak{L} \subseteq \mathfrak{D}$$

Proof:

i).- $L_{2,2}(H) \in \mathfrak{D}$ because $L_{2,2}(H) = E_3^* A_1 E_3^*$.

ii).- $L_{2,3}(H) \in \mathfrak{D}$ because $L_{2,3}(H) = (E_3^* A_1 E_3^*).(E_3^* A_1 E_4^*)$ (since $E_3^* A_1 E_4^* = L_{2,3}(I)$).

iii).- $L_{3,2}(H) \in \mathfrak{D}$ because $L_{3,2}(H) = (E_3^* A_1 E_3^*).(E_4^* A_1 E_3^*)$. (since $E_4^* A_1 E_3^* = L_{3,2}(I)$).

iv).- $L_{3,3}(K) \in \mathfrak{D}$ because $L_{3,3}(K) = E_4^* A_1 E_4^*$.

v).-$L_{3,2}(K)$ and $L_{2,3}(K)$ are in $\mathfrak{D}$, by using in iv) a similar trick as the one used to go from i) to ii) and iii).

vi).- Since

$$A_1^2 = \begin{pmatrix} 3 & 0 & J & 0 & 0 & 0 \\ 0 & J+2I & \alpha H & \alpha & 0 & 0 \\ J & H\alpha^t & 3I+K & H+K & \beta & 0 \\ 0 & \alpha^t & H+K & 3I+H & K\beta & J \\ 0 & 0 & \beta^t & b^t K & 2I+J & 0 \\ 0 & 0 & 0 & J & 0 & 3 \end{pmatrix}$$

we have that $L_{3,3}(3I+H) = E_4^* A_1^2 E_4^*$ and $L_{2,2}(3I+K) = E_3^* A_1^2 E_3^*$, so they are both in $\mathfrak{D}$. Since both $L_{3,3}(I)$ and $L_{2,2}(I)$ are in $\mathfrak{D}$, then $L_{3,3}(H)$ and $L_{2,2}(K)$ are in $\mathfrak{D}$.

Hence, we have that $L_{i,j}(X)$ with $X = H$ or K, $i, j = 2, 3$ are all in $\mathfrak{D}$. Thus any sum or product of them will be in $\mathfrak{D}$, hence the lemma.

QED

3.5 Definition: Define $\mathfrak{S} = \mathfrak{A} \oplus \mathfrak{A}_J \oplus \mathfrak{A}_N$

3.6 Lemma:

$$\mathfrak{L} \subseteq \mathfrak{S}$$

Proof: $L_{i,j}(v) \in \mathfrak{A}$ if $v \in V$ and $i, j = 2, 3$. In $\mathfrak{A}_J$ we have the matrices $L_{i,j}(J)$ and in $\mathfrak{A}_N$ the matrices $L_{i,j}(N)$.

Therefore $L_{i,j}(X) \in \mathfrak{S}$ for all $X \in \mathfrak{A}_6$, for $i, j = 2, 3$.

QED

3.7 Lemma:

$$A_1 \in \mathfrak{S}$$

Proof:

$$A_1 = \begin{pmatrix} 0 & J & 0 & 0 & 0 & 0 \\ J & 0 & \alpha & 0 & 0 & 0 \\ 0 & \alpha^t & H & I & 0 & 0 \\ 0 & 0 & I & K & \beta & 0 \\ 0 & 0 & 0 & \beta^t & 0 & J \\ 0 & 0 & 0 & 0 & J & 0 \end{pmatrix} = M_1 + L_{1,2}(\alpha) + L_{2,1}(\alpha^t) + L_{3,4}(\beta) + L_{4,3}(\beta^t) + M_2$$

where

$$M_1 = \begin{pmatrix} 0 & J & 0 & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J \\ 0 & 0 & 0 & 0 & J & 0 \end{pmatrix}, \ M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H & I & 0 & 0 \\ 0 & 0 & I & K & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, $M_1 \in \mathfrak{A}_J$, hence, in $\mathfrak{S}$.

$M_2 \in \mathfrak{S}$ by **3.6)** .

Since $L_{1,2}(u) \in \mathfrak{A}\ \forall u \in U$ and $L_{1,2}(J) \in \mathfrak{A}_J$, we have that $L_{1,2}(X) \in \mathfrak{S}$ for all $X \in U \oplus span\{J\}$/ However, by 2.10) a), $\alpha\mathfrak{A}_6 = U \oplus span\{J\}$. In particular, α is generated by elements of U and J, hence, $L_{1,2}(\alpha) \in \mathfrak{S}$.

Similar computations show that the other matrices are in $\mathfrak{S}$, thus $A_1 \in \mathfrak{S}$.

Section 4: Main Theorems

4.1 Lemma: *$\mathfrak{S}$ is a subalgebra of $\mathfrak{D}$.*

Proof:

i) $\mathfrak{A}_J \subseteq \mathfrak{D}$ by **2.2)** b).

ii) $\mathfrak{A}_N \subseteq \mathfrak{D}$, by **2.5)** and **3.4)** .

iii) $\mathfrak{A} \subseteq \mathfrak{D}$ follows from the following items:

iii.a) $\mathfrak{A} \cap \mathfrak{L} \subseteq \mathfrak{D}$ since $\mathfrak{L} \subseteq \mathfrak{D}$.

iii.b) Since $L_{1,2}(\alpha) = E_2^* A_1 E_3^* \in \mathfrak{D}$ we have that $L_{1,2}(\alpha v) = L_{1,2}(\alpha) L_{2,2}(v) \in \mathfrak{D}$ for all $v \in V$ (i.e., all $\alpha v \in U$)

iii.c) A similar argument applies to , for example, matrices of the form $L_{2,1}(v\alpha^t)$ or of the form $L_{2,4}(v\beta)$.

iii.d) E_1^*, E_4^* are in $\mathfrak{D}$ by definition, and by i), we get that the matrices $L_{1,1}(I - \frac{1}{3}J)$ and $L_{4,4}(I - \frac{1}{3}J)$ are in $\mathfrak{D}$.

iii.e) For the last two generating matrices, we have, by **2.12)** v), that $UW \subseteq span\{G - \frac{2}{3}J\}$, thus, $L_{1,4}(G - \frac{2}{3}J) = L_{1,2}(u).L_{2,4}(w)$ for some $u \in U$, $w \in W$; and since the two matrices on the right are in $\mathfrak{D}$ by iii.b) and iii.c), we have that the matrix on the left is in $\mathfrak{D}$.

Similarly with $L_{4,1}(G^t - \frac{2}{3}J)$.

Hence we have in fact that all of $\mathfrak{A}$ is in $\mathfrak{D}$, and so, $\mathfrak{S} \subseteq \mathfrak{D}$.

QED

To prove $\mathfrak{D} \subseteq \mathfrak{S}$, we first need to see that we can reconstruct A_1 from all these matrices:

4.2 Theorem:

$$\mathfrak{D} = \mathfrak{S}$$

Proof: By **4.1)** , we only need to see that $\mathfrak{D} \subseteq \mathfrak{S}$:

.- First, $A_1 \in \mathfrak{S}$ by **3.7)**, hence the Bose-Mesner algebra is included in $\mathfrak{S}$ since it is generated by A_1, since the dodecahedron is $P-$polynomial.

.- E_0^* and E_5^* are in $\mathfrak{A}_J$.

.- E_1^* and E_4^* are in $\mathfrak{A} \oplus \mathfrak{A}_J$ since $E_1^* = L_{1,1}(I)$ and, within $\mathfrak{A}$ we have the matrix $L_{1,1}(I - \frac{1}{3}J)$, and in $\mathfrak{A}_J$ we have $L_{1,1}(J)$ Hence $E_1^* \in \mathfrak{A} \oplus \mathfrak{A}_J$ and the same is true with E_4^*.

.- E_2^* and E_3^* are in $\mathfrak{A} \oplus \mathfrak{A}_J \oplus \mathfrak{A}_N$, since $E_2^* = L_{2,2}(I) \in \mathfrak{S}$ (by **3.6)**) and $E_3^* = L_{3,3}(I) \in \mathfrak{S}$.

QED

4.3 Corollary: *The (complex) Terwilliger algebra of the dodecahedron is isomorphic to $M_6(\mathbb{C}) \oplus M_6(\mathbb{C}) \oplus M_2(\mathbb{C})$.*

Proof: Since $\mathfrak{D} = \mathfrak{A} \oplus \mathfrak{A}_J \oplus \mathfrak{A}_N$, and $\mathfrak{A}_J \sim M_6(\mathbb{C})$ and $\mathfrak{A}_N \sim M_2(\mathbb{C})$, the only thing left to prove is that $\mathfrak{A} \sim M_6(\mathbb{C})$. Since $\mathfrak{A} \subseteq \mathfrak{D}$ is the complement of $\mathfrak{A}_J \oplus \mathfrak{A}_N$ and $\mathfrak{D}$ is semisimple, it follows that $\mathfrak{A}$ is semisimple, so the result follows if we prove that the center of $\mathfrak{A}$ is 1-dimensional. Consider a matrix $M = (B_{i,j})$ (written in block form) which is in the center of $\mathfrak{A}$. In particular, $ML_{1,2}(u) = L_{1,2}(u)M \ \forall u \in U$, thus $B_{1,2} = B_{2,1} = 0$. Similarly with all the blocks $B_{i,j}$ with $i \neq j$. Also, since $\mathfrak{A}_6 = V \oplus span\{J\} \oplus span\{N\}$ is the decomposition into irreducible ideals of the, group algebra of $\mathfrak{S}_3$, then the center of V is 1- dimensional. If $(C_{i,j})$ is any matrix in $\mathfrak{A}$, since M is in the center we must have: $B_{i,i}C_{i,j} = C_{i,j}B_{j,j}$ for all i, j. It follows that M is a scalar multiple of a fixed diagonal matrix.

Section 5: Concluding Remarks

5.1 Remark: If $\mathcal{M}$ is a $T(x)$-irreducible module, $\mathcal{M}$ is said to be thin if dim $E_i^*\mathcal{M} \leq 1$ for all i, Y is thin with respect to x if each irreducible $T(x)$-module is thin, and Y is thin if it is thin with respect to x for all x. In [T4], Conjecture 13 of Terwilliger states that any antipodal (i.e. $p^0_{d,d} = 1$) P-polynomial association scheme must be thin.

We prove here that this conjecture is not true, and the counterexample we offer is the dodecahedron. The conjecture may still be true for association schemes with $d > 5$.

The dodecahedron is P-polynomial and antipodal.

If we denote $R_j(i) = \{k | d(i,k) = j\}$, we can see , by looking at the adjacency table of the dodecahedron, that $R_2(1) \cap R_1(7) \cap R_2(5) = \{6\}$ while $R_2(1) \cap R_2(7) \cap R_1(5) = \emptyset$, thus, by theorem 5.1 i) and iii) of [T4], the dodecahedron is not thin.

More explicitely, one can give a module $\mathcal{M}$ with dim $E_2^*\mathcal{M} = 2$: take

$$\mathcal{M} = \{ \begin{pmatrix} 0 \\ u \\ v \\ v' \\ w^t \\ 0 \end{pmatrix} \} | u \in U, w \in W, v, v' \in V\}$$

(i.e., $\mathcal{M}$ corresponds to the third column of the matrices in $\mathfrak{A}$ (see **3.1)**)
Hence, $E_2^*\mathcal{M} = W$, which has dimension 2 (see **2.10)**).

5.2 Remark: The dodecahedron is the most difficult case of the platonic solids. In the other cases, much shorter calculations show that they have Terwilliger algebras isomorphic to:
$M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$ (The Tetrahedon)
$M_4(\mathbb{C}) \oplus M_2(\mathbb{C})$ (The Cube)
$M_3(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C})$ (The Octahedron)
$M_4(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ (The Icosahedron).

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