4-STEP CARNOT SPACES AND THE 2-STEIN CONDITION

MÁRIA J. DRUETTA

Dedicated to the memory of my goddaughter María Pía

Abstract. We consider the 2-stein condition on $k$-step Carnot spaces $S$. These spaces are a subclass in the class of solvable Lie groups of Iwasawa type of algebraic rank one and contain the homogeneous Einstein spaces within this class. They are obtained as a semidirect product of a graded nilpotent Lie group $N$ and the abelian group $\mathbb{R}$. We show that the 2-stein condition is not satisfied on a proper 4-step Carnot spaces $S$.

A Riemannian manifold $M$ is said to be a 2-stein space, if there exist functions $\mu_l$, $l = 1, 2$, defined on $M$ such that
\[
\text{tr} (R_X^l) = \mu_l |X|^2 l, \quad l = 1, 2,
\]
for all $X \in TM$. Here, $R_X$ denote the Jacobi operator associated to $X$, defined by $R_XY = R(Y, X)X$ for all $Y \in TM$, where $R$ is the curvature tensor of $M$. Harmonic riemannian manifolds are necessarily 2-stein.

A $k$-step Carnot space ($k \geq 2$) is a simply connected solvable Lie group $S$, which is a semidirect product of a nilpotent Lie group $N$ and the abelian group $\mathbb{R}$. Assume that $S$ and $N$ have associated Lie algebras $s$ and $n$, respectively. $S$ has the left invariant metric induced by the one given on $s$, where $s$ is a solvable metric Lie algebra $s$ with inner product $\langle , \rangle$ such that:

(i) $s = n \oplus \mathbb{R}H$ with $n = [s, s]$ and $H \perp n$, $|H| = 1$.

(ii) $n$ has an orthogonal decomposition $n = \sum_{i=1}^{k-1} n_i$ into $(k-1)$ subspaces given by
\[
n_i = \{X \in n : \text{ad}_H(X) = i\alpha X\}, \quad i = 1, \ldots, k - 1,
\]
for some positive constant $\alpha \in \mathbb{R}$.

Note that, since the adjoint representation $\text{ad}_H$ is a derivation of $n$ the above decomposition defines a graded Lie algebra structure of $n$, that is,
\[
[n_i, n_j] \subset n_{i+j} \quad 1 \leq i, j \leq k - 1
\]
with the convention $n_i = \{0\}$ for $i > k - 1$. In particular, $n$ is a $(k - 1)$-step nilpotent Lie algebra, that is, the $(k - 1)$-th derived algebra vanishes: $n^{k-1} = [n, \ldots [n, [n, n]] \ldots] = \{0\}$.

2000 Mathematics Subject Classification. 53C30, 53C55.

Key words and phrases. 2-stein spaces, $k$-step Carnot spaces, Einstein spaces, Lie groups of Iwasawa type, Damek-Ricci spaces.
Note, that up to a constant in the metric, we may suppose that \( \alpha = 1 \). In this terminology, the 3-Carnot spaces are the well known Carnot spaces and the case of a 2-step Carnot space corresponds to the hyperbolic symmetric space \( RH^n \). Moreover, due to a result of [6], any homogeneous Einstein space of Iwasawa type and rank one is a \( k \)-step Carnot space for some \( k \geq 2 \).

We remark that the 2-stein condition was studied in [1] and [3] on Lie groups \( S \) of Iwasawa type in the case of 2-step nilpotent \( n \), and in particular, on Carnot spaces: in this class those which are 2-stein are exactly the Damek-Ricci spaces (also the harmonic ones). Furthermore, in [7] is shown that a simply connected homogenous harmonic space is a Carnot space, which in turn is equivalent to \( S \) be a Damek-Ricci space.

1. \( k \)-step Carnot spaces

Assume that \( S \) is a \( k \)-step Carnot space. Let \( \mathfrak{s} \) be the Lie algebra of \( S \): that is \( \mathfrak{s} \) is a solvable metric Lie algebra \( \mathfrak{s} = \mathfrak{n} \oplus RH \) with \( H \perp \mathfrak{n}, \|H\| = 1 \), with \( \mathfrak{n} = [\mathfrak{s}, \mathfrak{s}] \) a graded nilpotent Lie algebra as described above.

Let \( \mathfrak{z} \) denote the center of \( \mathfrak{n} \) and let \( \mathfrak{v} \) be the orthogonal complement of \( \mathfrak{z} \) with respect to the metric \( \langle \ast \rangle \) restricted to \( \mathfrak{n} \). Thus \( \mathfrak{n} \) decomposes \( \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \) and note that \( \text{ad}_{\mathfrak{H}} : \mathfrak{z} \rightarrow \mathfrak{z} \) hence \( \text{ad}_{\mathfrak{H}} : \mathfrak{v} \rightarrow \mathfrak{v} \) since \( \text{ad}_{\mathfrak{H}} \) is symmetric.

For any \( Z \in \mathfrak{z} \) the skew-symmetric linear operator \( j_Z : \mathfrak{v} \rightarrow \mathfrak{v} \) is defined by
\[
\langle j_Z X, Y \rangle = \langle [X, Y], Z \rangle \quad \text{for all } X, Y \in \mathfrak{v}, \text{ all } Z \in \mathfrak{z}.
\]
The Levi Civita connection and the curvature tensor on \( S \) can be computed by,
\[
2 \langle \nabla XY, Z \rangle = \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle
\]
\[
R(X, Y) = [\nabla_X \nabla_Y - \nabla_{[X,Y]}]
\]
for any \( X, Y, Z \) in \( \mathfrak{s} \).

Recall that since \( \mathfrak{z} \) and \( \mathfrak{v} \) are \( \text{ad}_{\mathfrak{H}} \)-invariant, they also have decompositions into eigenspaces as \( \mathfrak{z} = \sum \lambda \mathfrak{z}_\lambda, \mathfrak{v} = \sum \mu \mathfrak{v}_\mu \) with \( \mathfrak{z}_\lambda = [\mathfrak{z}, \mathfrak{z}] \cap \mathfrak{z}, \mathfrak{v}_\mu = [\mathfrak{v}, \mathfrak{v}] \cap \mathfrak{v} \) and the equality \( \lambda = \mu \) may be possible. Moreover, \( \mathfrak{n} \) is said to be 2-step nilpotent if \( [\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] \subset \mathfrak{z} \). Here, \( \{\lambda\} \) and \( \{\mu\} \) denote the eigenvalues of \( \text{ad}_{\mathfrak{H}}|_{\mathfrak{z}} \) and \( \text{ad}_{\mathfrak{H}}|_{\mathfrak{v}} \), respectively, that we assume they are natural numbers.

1.1. Properties of the operator \( j_Z, Z \in \mathfrak{z} \). We recall that for any \( Z \in \mathfrak{z}_\lambda \) so that \( j_Z \neq 0 \), \( j_Z : \mathfrak{v}_\mu \rightarrow \mathfrak{v}_{\lambda - \mu} \rightarrow \mathfrak{v}_\mu \) and \( j_Z : (\ker j_Z|_{\mathfrak{v}_\mu})^\perp \rightarrow (\ker j_Z|_{\mathfrak{v}_{\lambda - \mu}})^\perp \) isomorphically. Consequently, as follows from [5, Lemma1.1])
\[
\text{tr} \left( j_Z^2 |_{\mathfrak{v}_\mu} \right) = \text{tr} \left( j_Z^2 |_{\mathfrak{v}_{\lambda - \mu}} \right).
\]

1.2. The curvature formulas. By applying the connection formula, one obtains \( \nabla_H = 0 \) and if \( Z, Z' \in \mathfrak{z} \) and \( X, Y \in \mathfrak{v} \), then \( \nabla_Z Z' = \nabla_{Z} Z = \langle [H, Z], Z' \rangle H, \nabla_X Z = \nabla_{Z} X = -\frac{1}{2} j_Z X \) and \( \nabla_X Y = \frac{1}{2} [X, Y] + \langle [H, X], Y \rangle H \) in case of 2-step nilpotent \( \mathfrak{n} \). Consequently, by a direct computation we obtain the following formulas involving curvatures (see [2, Section 2]).
\[ (i) \ R_H = -\text{ad}_{\mathfrak{H}}^2 \]
(ii) If either \( Z \in \mathfrak{z}_\lambda \), \(|Z| = 1 \), or \( X \in \mathfrak{v}_\mu \), \(|X| = 1 \),
\[ R_Z H = -\lambda^2 H \text{ and } R_X H = -\mu^2 H. \]

(iii) If \( Z \in \mathfrak{z}_\lambda \), \(|Z| = 1 \), for any \( Z^* \in \mathfrak{z} \) and \( X \in \mathfrak{v} \), then
\[ R_Z Z^* = \lambda ((Z, \text{ad}_H Z^*)Z - \text{ad}_H Z^*) \text{ and } R_Z X = -\frac{1}{4} j_Z^2 (X) - \lambda \text{ad}_H X. \]

(iv) Let \( \mu_1 \) denote the maximum eigenvalue of \( \text{ad}_H|_{\mathfrak{v}} \). If \( X \in \mathfrak{v}_{\mu_1} \), \(|X| = 1 \), then
\[ R_X Z = \frac{1}{4} [X, j_Z X] - \mu_1 \text{ad}_H Z - \frac{1}{4} (\text{ad}_{j_Z X})^* X, \quad Z \in \mathfrak{z} \]
\[ R_X Y = -\frac{1}{2} [X, \nabla^a_X Y] - \frac{3}{4} j_{[X,Y]} X - \nabla^a_X (\nabla^a_X Y)_o - \mu_1 \text{ad}_H Y, \quad Y \perp X \in \mathfrak{v}. \]

1.3. **The Einstein condition.** Assume that \( S \) is an Einstein space, that is \( \text{Ric}(Y) = c |Y|^2 \) for all \( Y \in \mathfrak{s} \). Let \( \{Y_i\}_{i=1}^m \) and \( \{Z_i\}_{i=1}^n \) be orthonormal bases of \( \mathfrak{v} \) and \( \mathfrak{z} \), respectively, with \( k = \dim \mathfrak{z} \) and \( m = \dim \mathfrak{v} \). We use the well known formulas that hold in a solvable metric Lie algebra \( \mathfrak{s} = \mathfrak{n} \oplus \mathbb{R} H \),
\[ \text{Ric}(Z) = \frac{1}{4} \sum_{i=1}^m |j_Z X_i|^2 - \lambda \text{ tr ad}_H \text{ for } Z \in \mathfrak{z}_\lambda, \]
\[ \text{Ric}(X) = -\frac{1}{2} \sum_{i=1}^k |j_Z X_i|^2 - \mu \text{ tr ad}_H \text{ for } X \in \mathfrak{v}_{\mu_1}, \text{ if } \mathfrak{n} \text{ is 2-step-nilpotent}. \]

Note that the first Einstein condition becomes
\[ \text{tr} \left( -\frac{1}{4} j_Z^2 \right) = -\text{tr ad}_H^2 + \lambda \text{ tr ad}_H, \text{ for } Z \in \mathfrak{z}_\lambda, \ |Z| = 1. \] (1)

In general, for a unit vector \( X \in \mathfrak{v}_{\mu_1} \), \( \mu_1 \) the maximum eigenvalue of \( \text{ad}_H|_{\mathfrak{v}} \), we have
\[ \text{Ric}(X) = \frac{1}{2} \sum_{i=1}^k |j_Z X_i|^2 - \mu_1 \text{ tr ad}_H + \sum_{i=1}^m |(\nabla^a_X Y_i)_o|^2. \] (2)

The last one is a direct computation by applying the above expressions of \( R_X \) in (iv) to
\[ \text{Ric}(X) = \text{tr} R_X = \sum_{j=1}^k \langle R_X Z_j, Z_j \rangle + \sum_{i=1}^m \langle R_X Y_i, Y_i \rangle, \]

having into account that the following equality holds,
\[ \sum_{i=1}^m |[X, Y_i]|^2 = \sum_{j=1}^k |j_Z X|^2. \]

In fact, this follows by computing
\[ \sum_{i=1}^m |[X, Y_i]|^2 = \sum_{i=1}^m \sum_{j=1}^k \langle [X, Y_i], Z_j \rangle^2 = \sum_{i=1}^m \sum_{j=1}^k \langle j_Z X, Y_i \rangle^2 \]
\[ = \sum_{j=1}^k \left( \sum_{i=1}^m \langle j_Z X, Y_i \rangle^2 \right) = \sum_{j=1}^k |j_Z X|^2, \]
since \([X, Y] \in \mathfrak{z}\) for any \(X \in \mathfrak{v}_{\mu_1}\).

1.4. **The 2-stein condition.** For any \(X \in \mathfrak{s}\), the Jacobi operator \(R_X\) associated to \(X\) is the symmetric endomorphism of \(\mathfrak{s}\) defined by \(R_X Y = R(Y, X)X\). We said that \(S\) is a 2-stein space, or equivalently \(S\) (or \(\mathfrak{s}\)) satisfies the 2-stein condition, if there exist

\[
\text{tr} (R_X^l) = \mu_l |X|^{2l}, \ l = 1, 2, \text{ for all } X \in \mathfrak{s}.
\]

In particular, 2-stein spaces are Einstein: \(\text{Ric}(X) = \text{tr} R_X = \mu_1 |X|^2\), \(\mu_1\) a constant, for all \(X \in \mathfrak{s}\), and harmonic riemannian spaces are necessarily 2-stein. The 2-stein condition was studied on Carnot spaces in [1] and [3]: the 2-stein Carnot spaces are exactly the Damek-Ricci spaces (also the harmonic ones) in this class. The proposition below express the 2-stein condition in an adequate form for our purposes. Let \(\{\mu\}\) denote the set of eigenvalues of \(\text{ad}_{\mathfrak{h}}|_{\mathfrak{v}}\).

**Proposition 1.1.** Assume that \(\mathfrak{s}\) satisfies the 2-stein condition. If \(Z \in \mathfrak{z}_\lambda\) is a unit vector, then

\[
\text{tr}(\text{ad}_{\mathfrak{h}}^k)|_{\mathfrak{z}_\lambda \oplus \mathfrak{v}} = \text{tr} (-R_Z \circ \text{ad}_{\mathfrak{h}}^2)|_{\mathfrak{z}_\lambda \oplus \mathfrak{v}} + \frac{1}{2} \sum_{\mu} (\lambda - 2\mu)^2 \text{tr} \left(-\frac{1}{4} j_Z^2|_{\mathfrak{v}_\mu}\right).
\]

In particular, if \(\text{ad}_{\mathfrak{h}}|_{\mathfrak{z}} = \lambda \text{Id}\)

\[
\text{tr}(\text{ad}_{\mathfrak{h}}^k) = \text{tr} (-R_Z \circ \text{ad}_{\mathfrak{h}}^2) + \frac{1}{2} \sum_{\mu} (\lambda - 2\mu)^2 \text{tr} \left(-\frac{1}{4} j_Z^2|_{\mathfrak{v}_\mu}\right). \tag{3}
\]

**Proof.** It follows from the same argument as those used in Theorem 4.1 of [4] for \(k = 2\) (See also [5, Proposition 1.2]).

2. **The 2-stein condition on 4-step Carnot spaces**

We consider the case of a 4-step Carnot space; that is, \(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{R H}\) as above with \(\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3\). Hence, by the properties of \(\mathfrak{n}_1\), \(\mathfrak{n}_1, \mathfrak{n}_1 \subset \mathfrak{n}_2\), \(\mathfrak{n}_1, \mathfrak{n}_2 \subset \mathfrak{n}_3\), \(\mathfrak{n}_1, \mathfrak{n}_3 = 0\), \(\mathfrak{n}_2, \mathfrak{n}_2 = 0 = \mathfrak{n}_2, \mathfrak{n}_3\), \(\mathfrak{n}_3, \mathfrak{n}_3 = 0\) and \(\mathfrak{n}_3 \subset \mathfrak{z}\). We observe that: \([\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}_2 \oplus \mathfrak{n}_3\) and \([\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset \mathfrak{n}_2 \oplus \mathfrak{n}_3, \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3\) \subset \(\mathfrak{n}_3\). Thus, \(\mathfrak{n}_3 = 0\) since \([[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \subset \mathfrak{n}_2, 0 = \mathfrak{n}_2, \mathfrak{n}_3\) and \(\mathfrak{n}_3 \subset \mathfrak{z}\). Since \(\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}\) we have that \(\mathfrak{v} \subset \mathfrak{n}_1 \oplus \mathfrak{n}_2\). Moreover, from the facts that \(\mathfrak{z}\) and \(\mathfrak{v}\) are \(\text{ad}_{\mathfrak{h}}\)-invariant, it follows that

\[
\mathfrak{z} = \mathfrak{n}_1 \cap \mathfrak{z} \oplus \mathfrak{n}_2 \cap \mathfrak{z} \oplus \mathfrak{n}_3, \text{ and } \mathfrak{v} = \mathfrak{n}_1 \cap \mathfrak{v} \oplus \mathfrak{n}_2 \cap \mathfrak{v}.
\]

We set \(\dim \mathfrak{n}_i = \mathfrak{n}_i, \ i = 1, 2, 3\). Following with the notation introduced in the previous section we have,

**Lemma 2.1.** If \(S\) is a 4-step Carnot space that satisfies the 2-stein condition, then \(S\) is either a Damek-Ricci space or, up to scaling, the symmetric hyperbolic space \(S = \mathfrak{R H}^n\) (\(\mathfrak{n}_2 = 0\) or \(\mathfrak{n}_3 = 0\)) with \(n = \dim S\). If \(\mathfrak{n}_3 \neq 0\) the equalities \(\mathfrak{z} = \mathfrak{n}_3\), \(\mathfrak{v} = \mathfrak{n}_1 \oplus \mathfrak{n}_2\) hold in the decomposition of \(\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}\).
Proof. (i) \( n_1 \cap \mathfrak{j} = 0 \), if the Einstein condition is satisfied. If \( 0 \neq Z \in n_1 \cap \mathfrak{j} \), \( \langle j_Z X, Y \rangle = \langle [X, Y], Z \rangle = 0 \) for all \( X, Y \in \mathfrak{v} \) since \( [X, Y] \in \sum_{i \geq 2} n_i \) and \( \langle n_i, n_j \rangle = 0 \) if \( i \neq j \). Hence, \( j_Z = 0 \) and \( \text{tr}(j_Z^2) = 0 \), and the Einstein condition (1) implies that

\[
\text{tr} \left( \frac{1}{4} j_Z^2 \right) = - \text{tr}(\text{ad}^2_H) + \text{tr}(\text{ad}_H) = -(n_1 + 4n_2 + 9n_3) + (n_1 + 2n_2 + 3n_3)
\]

\[
= -(2n_2 + 6n_3).
\]

Thus, \( n_2 = n_3 = 0 \), and consequently \( n = n_1 \). In this case \( n \) is abelian and \( S \) corresponds to the symmetric hyperbolic space \( RH^n \) with \( n = n_1 + 1 \).

The previous fact implies that the eigenvalue \( \mu = 1 \) is not achieved in \( \mathfrak{j} \), when \( n \) is 3-step properly. Thus, \( \mathfrak{j} = n_3 \oplus n_2 \cap \mathfrak{j} \) and \( \mathfrak{v} = n_1 \oplus n_2 \cap \mathfrak{v} \).

(ii) \( n_2 \cap \mathfrak{j} = 0 \) if the 2-stein condition holds.

In fact, assume that there exists \( 0 \neq Z \in n_2 \cap \mathfrak{j} \). In this case, if \( |Z| = 1 \) condition (1) gives

\[
\text{tr} \left( \frac{1}{4} j_Z^2 \right) = - \text{tr}(\text{ad}^2_H) + 2\text{tr}(\text{ad}_H) = -(n_1 + 4n_2 + 9n_3) + 2(n_1 + 2n_2 + 3n_3)
\]

\[
= n_1 - 3n_3.
\]

Moreover, since \( Z \in n_2 \) and \( \langle j_Z X, Y \rangle = \langle [X, Y], Z \rangle = 0 \) for all \( X \in n_2 \cap \mathfrak{v}, Y \in \mathfrak{v} \), we have that \( j_Z|_{n_2 \cap \mathfrak{v}} = 0 \) and \( j_Z : n_1 \rightarrow n_1 \). Now, we apply the formula given in Proposition 1.1 for \( \lambda = 2 \), and \( \mu = 1, 2 \). We first note that,

\[
\frac{1}{2} \sum_{\mu = 1, 2} (\lambda - 2\mu)^2 \text{tr} \left( \frac{1}{4} j_Z^2 |_{\mathfrak{v}} \right) = 0.
\]

since \( j_Z|_{n_2 \cap \mathfrak{v}} = 0 \) and \( \lambda - 2\mu = 0 \) for \( \mu = 1 \). This implies that \( \text{tr}(\text{ad}^4_H)|_{\mathfrak{j} \cap \mathfrak{v}} = \text{tr}(\text{ad}^2_H)|_{\mathfrak{j} \cap \mathfrak{v}} \) and consequently, equality holds in the Cauchy-Schwartz Inequality

\[
\left( \text{tr} (-R_Z \circ \text{ad}^2_H)|_{\mathfrak{j} \cap \mathfrak{v}} \right)^2 \leq \text{tr}(\text{ad}^4_H)|_{\mathfrak{j} \cap \mathfrak{v}} \text{tr}(\text{ad}^2_H)|_{\mathfrak{j} \cap \mathfrak{v}} = \left( \text{tr} \text{ad}^4_H|_{\mathfrak{j} \cap \mathfrak{v}} \right)^2.
\]

Therefore, \( -R_Z = \text{cad}^2_H \) in \( \mathfrak{j} \cap \mathfrak{v} \) with \( c = 1 \), by the Einstein condition. The equality \( R_Z|_{n_3} = -\text{ad}^2_H|_{n_3} \) implies that \( n_3 = 0 \), since \( Z \in n_2 \cap \mathfrak{j} \) and \( R_Z|_{n_3} = -6\text{Id} \). Hence, \( n = n_3 \oplus n_2 \) with \( n_1 = \mathfrak{v}, n_2 = \mathfrak{j} \) and \( S \) is a Carnot space. In this case \( S \) is a Damek-Ricci space (see [3, Theorem 3.1]).

Assume that \( n_3 \neq 0 \). If \( s \) satisfies the 2-stein condition then \( \mathfrak{j} = n_3 \), and \( \mathfrak{v} = n_1 \oplus n_2 \).

The following proposition is basic for our purposes,
Proposition 2.2. If $S$ is a 4-step Carnot space that is 2-stein, then $n_1 = 2n_2$ and $-\frac{1}{4} j_Z^2 |_{n_2} = 3Id$ for any unit vector $Z \in \mathfrak{z}$. Equivalently, $|j_Z X|^2 = 12$ for all unit vectors $Z \in \mathfrak{z}$ and $X \in \mathfrak{n}_2$.

Proof. Next we will show that $n_1 = 2n_2$,
\[
\text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_3} \right) = 3n_2 \quad \text{and} \quad \text{tr} \left( \frac{1}{16} j_Z^2 |_{n_2} \right) = 9n_2
\]
for any unit vector $Z \in \mathfrak{n}_3$.

Let $Z \in \mathfrak{z} = \mathfrak{n}_3$ be a unit vector. The Einstein condition (1) applied to $Z$ gives,
\[
\text{tr} \left( -\frac{1}{4} j_Z^2 \right) = -(n_1 + 4n_2 + 9n_3) + 3(n_1 + 2n_2 + 3n_3)
\]

We compute separately the two terms in (3). First, using the definition of $R_Z |_{o} = -\frac{1}{4} j_Z^2 - 3 \text{ad}_H |_{o}$, it follows that
\[
\text{tr} \left( -R_Z \circ \text{ad}_H^2 |_{o} \right) = \text{tr} \left( -R_Z \circ \text{ad}_H^2 |_{n_1} \right) + \text{tr} \left( -R_Z \circ \text{ad}_H^2 |_{n_2} \right)
\]
and applying (4),
\[
\sum_{\mu=1}^{2} (\lambda - 2\mu)^2 \text{tr} \left( -\frac{1}{4} j_Z^2 |_{v_{\mu}} \right) = \frac{1}{2} \text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_1} \right) + \frac{1}{2} \text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_2} \right)
\]

Thus, (3) gives
\[
n_1 + 2^4 n_2 = \text{tr} (\text{ad}_H^4 |_{o})
\]
and consequently,
\[
\text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_1} \right) + 4\text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_2} \right) = 3n_1 + 9n_2 = 3(n_1 + 3n_2).
\]

Now, using that $(\text{tr} j_Z^2 |_{n_1}) = \text{tr} j_Z^2 |_{n_2}$ (see 1.1), (4) and (5), that is
\[
2\text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_1} \right) = \text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_1} \right) + \text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_2} \right) = 2(n_1 + n_2)
\]
\[
5\text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_1} \right) = \text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_1} \right) + 4\text{tr} \left( -\frac{1}{4} j_Z^2 |_{n_2} \right) = 3(n_1 + 3n_2),
\]
\( n_1 = 2n_2 \) is obtained, since
\[ 3(n_1 + 3n_2) = 5(n_1 + n_2) \iff 2n_1 = 4n_2 \iff n_1 = 2n_2. \]
Finally, we will show the last assertion of the proposition. Using that \( R_Z|_v = -\frac{1}{4} \mathcal{J}_Z^2 - 3 \mathrm{ad} H|_v \), again, we compute
\[
\text{tr}(R_Z|_v)^2 = \text{tr} \left( \frac{1}{4} \mathcal{J}_Z^2 + 3 \mathrm{ad} H|_v \right)^2
\]
by developing \( R_Z^2|_v = (-\frac{1}{4} \mathcal{J}_Z^2 - 3 \mathrm{ad} H|_v)^2 \). Thus,
\[
\text{tr} \left( R_Z^2|_v \right) = \text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) + 9 \text{tr} \mathrm{ad}^2 H|_v + \frac{3}{2} \text{tr} (\mathcal{J}_Z^2 \circ \mathrm{ad} H|_v)
\]
\[
= \text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) + 9 \left( \text{tr}(\mathrm{ad}^2 H|_{n_1}) + \text{tr}(\mathrm{ad}^2 H|_{n_2}) \right)
\]
\[
+ \frac{3}{2} \left( \text{tr}(\mathcal{J}_Z^2|_{n_1} \circ \mathrm{ad} H|_{n_1}) + \text{tr}(\mathcal{J}_Z^2|_{n_2} \circ \mathrm{ad} H|_{n_2}) \right)
\]
and substituting the values of \( \mathrm{ad} H|_{n_i} = \text{Id} \), we have
\[
\text{tr} \left( R_Z^2|_v \right) = \text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) + 9(n_1 + 4n_2) + \frac{3}{2} \left( \text{tr}(\mathcal{J}_Z^2|_{n_1}) + 2\text{tr}(\mathcal{J}_Z^2|_{n_2}) \right)
\]
\[
= \text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) + \frac{9}{2} \text{tr} \left( \mathcal{J}_Z^2 \right) + 9(n_1 + 4n_2),
\]
since \( \text{tr}(\mathcal{J}_Z^2|_{n_1}) = \text{tr}(\mathcal{J}_Z^2|_{n_2}) \). Note, that the same argument used to show this equality, implies
\[
\text{tr} \left( \mathcal{J}_Z^2 \right) = \text{tr} \left( \mathcal{J}_Z^2 \right).
\]
Therefore, the 2-stein condition, \( \text{tr}(R_Z^2|_v) = \text{tr}(\mathrm{ad}^4 H|_v) \), gives
\[
2\text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) + \frac{9}{2} \text{tr} \left( \mathcal{J}_Z^2 \right) + 9(n_1 + 4n_2) = n_1 + 16n_2 \quad \text{or}
\]
\[
2\text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) + \frac{9}{2} \text{tr} \left( \mathcal{J}_Z^2 \right) = -(8n_1 + 20n_2).
\]
By applying (4), the last equality is equivalent to
\[
2\text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) - 18\text{tr} \left( \frac{1}{4} \mathcal{J}_Z^2 \right) = -4(2n_1 + 5n_2), \quad \text{or}
\]
\[
2\text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) - 18(n_1 + n_2) = -4(2n_1 + 5n_2).
\]
It follows from (6) that
\[
2\text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) = 10n_1 - 2n_2 = 2(5n_1 - n_2),
\]
and from the fact \( n_1 = 2n_2 \),
\[
\text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) = \text{tr} \left( \frac{1}{16} \mathcal{J}_Z^4 \right) = 5n_1 - n_2 = 9n_2.
\]
The final assertion follows as claimed, since
\[
\text{tr} \left( -\frac{1}{4} \frac{I}{n_2} \right) = 3n_2 \quad \text{and} \quad \text{tr} \left( \frac{1}{16} \frac{I}{n_2} \right) = 9n_2
\]
implies that equality holds in the Cauchy-Schwartz Inequality
\[
\left( \text{tr} \left( -\frac{1}{4} \frac{I}{n_2} \right) \right)^2 \leq n_2 \text{tr} \left( \frac{1}{16} \frac{I}{n_2} \right),
\]
which gives \(-\frac{1}{4} \frac{I}{n_2} = 3 \cdot I_d\), or equivalently, \(|jZ|_2^2 = 12\) for all unit vector \(X \in n_2\).

Let \(s = n_1 \oplus n_2 \oplus n_3 \oplus RH\) be a 4-step Lie algebra so that \(z = n_3\), and \(v = n_1 \oplus n_2\). In what follows let \(\{Z_i\}_{i=1}^k\), \(\{Y_i\}_{i=1}^{n_1}\) and \(\{X_j\}_{j=1}^{n_2}\) be any orthonormal bases of \(z\), \(n_1\) and \(n_2\), respectively. The following basic formulas are deduced from the hypothesis \(n_3 \neq 0\) and not assuming that the 2-stein condition is satisfied. They are shown in [5, Proposition 2.3].

If \(X \in n_2\) and \(Y \in n_1\) are unit vectors, then
\[
\sum_{j=1}^{n_2} |[Y, X_j]|^2 = \sum_{k=1}^{n_3} |jz_k Y|^2 \quad (7)
\]
\[
\sum_{j=1}^{n_2} \left| \left( \nabla X \right) Y \right|^2 = \frac{1}{4} \sum_{i=1}^{n_1} |Y_i Y|^2 \quad (8)
\]
In order to show that the 2-stein condition is not satisfied by \(s\), in case that \(s\) is 4-step properly, we need to compute \(\text{Ric}(Y)\) and \(\text{Ric}(X)\) for unit vectors \(Y \in n_1\) and \(X \in n_2\).

**Lemma 2.3.** If \(Y \in n_1\) is a unit vector then,
\[
\text{tr} \text{R}_Y = -n_1 - 2n_2 - 3n_3 - \frac{1}{2} \sum_{j=1}^{n_2} |[Y, X_j]|^2 - \frac{1}{2} \sum_{i=1}^{n_1} |Y_i Y|^2.
\]

**Proof.** Let \(Y \in n_1\) be a unit vector. We first note, that it is a direct computation to see that
\[
\nabla_Y W = \frac{1}{2} [Y, W] \quad \text{for} \quad W \in n_1, \ W \perp Y
\]
\[
\nabla_Y X = \frac{1}{2} [Y, X] + \left( \nabla^p_Y \right) X, \quad \text{for} \quad X \in n_2.
\]
Therefore, for unit vectors \(Z \in z\), \(W \in n_1\) and \(X \in n_2\) using the definition of \(\text{R}_Y\) we obtain
\[
\text{R}_Y Z = -3Z + \frac{1}{2} \nabla_Y jZ Y,
\]
\[
\text{R}_Y W = -W - \frac{1}{2} \nabla_Y [W, Y] - \nabla_{[W, Y]} Y,
\]
\[
\text{R}_Y X = -2X - \nabla_Y \nabla_X Y + \frac{1}{2} \delta_{X,Y} Y, \quad \text{R}_Y H = -H.
\]
Now, it is a straightforward computation using the definition of $\nabla_{(.)}$ to see that,

$$\langle R_Y Z, Z \rangle = -3 + \frac{1}{2} \langle \nabla_Y jzY, Z \rangle = -3 + \frac{1}{4} |jzY|^2,$$

$$\langle R_Y W, W \rangle = -1 - \frac{1}{2} \langle \nabla_Y [W, Y], W \rangle - \langle \nabla_{[W,Y]} Y, W \rangle$$

$$= -1 - \frac{1}{4} |[Y, W]|^2 + \frac{1}{2} \langle [Y, W], [W, Y] \rangle$$

$$= -1 - \frac{3}{4} |[Y, W]|^2$$

and

$$\langle R_Y X, X \rangle = -2 - \langle \nabla_Y \nabla_X Y, X \rangle + \frac{1}{2} \langle j_{[X,Y]} Y, X \rangle$$

$$= -2 + \langle \nabla_X Y, \nabla_Y X \rangle + \frac{1}{2} \langle [Y, X], [X, Y] \rangle$$

$$= -2 + |\nabla_X Y|^2 + \langle \nabla_X Y, [Y, X] \rangle - \frac{1}{2} |[X, Y]|^2$$

$$= -2 + |\nabla_X Y|^2 - |[X, Y]|^2$$

$$= -2 - \frac{3}{4} |[X, Y]|^2 + |(\nabla_X Y)_o|^2.$$  

Next, by computing

$$\text{tr } R_Y|_{j \in RH} = \sum_{k=1}^{n_3} \langle R_Y Z_k, Z_k \rangle - 1 = -3n_3 + \frac{1}{4} \sum_{k=1}^{n_3} |jz_k Y|^2 - 1$$

$$\text{tr } R_Y|_{n_1} = \sum_{i=1}^{n_1} \langle R_Y Y_i, Y_i \rangle = -(n_1 - 1) - \frac{3}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2$$

$$\text{tr } R_Y|_{n_2} = \sum_{j=1}^{n_2} \langle R_Y X_j, X_j \rangle = -2n_2 + \sum_{j=1}^{n_2} |(\nabla_X Y)_o|^2 - \frac{3}{4} \sum_{j=1}^{n_2} |[X, Y_j]|^2,$$

it is immediate that

$$\text{tr } R_Y = -n_1 - 2n_2 - 3n_3 + \frac{1}{4} \sum_{k=1}^{n_3} |jz_k Y|^2 - \frac{3}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2$$

$$+ \sum_{j=1}^{n_2} |(\nabla_X Y)_o|^2 - \frac{3}{4} \sum_{j=1}^{n_2} |[X, X_j]|^2.$$  

By applying the equalities given by (7) and (8) we have,

$$\text{tr } R_Y = -n_1 - 2n_2 - 3n_3 - \frac{1}{2} \sum_{j=1}^{n_2} |[Y, X_j]|^2 - \frac{1}{2} \sum_{i=1}^{n_1} |[Y, Y_i]|^2,$$

the expression stated in the lemma.  

\[\square\]
Corollary 2.4. If $\mathfrak{s} = n_1 \oplus n_2 \oplus n_3 \oplus RH$ satisfies the Einstein condition, then for all unit vectors $Y \in n_1$ and $X \in n_2$ we have,

\begin{align*}
(i) \quad & \sum_{i=1}^{n_1} |(\nabla_X Y_i)_o|^2 - \frac{1}{2} \sum_{i=1}^{k} |j_{Z_i} X_i|^2 = n_1 - 3n_3 \\
(ii) \quad & \sum_{j=1}^{n_2} ||Y, X_j||^2 + \sum_{i=1}^{n_1} ||Y, Y_i||^2 = 4(n_2 + 3n_3).
\end{align*}

Proof. (i) and (ii) are obtained by applying the Einstein condition $\text{tr}(R_X) = -\text{tr}(\text{ad}^2_H)$ to $X \in n_2$, and $\text{tr}(R_Y) = -\text{tr}(\text{ad}^2_H)$ to $Y \in n_1$ (Lemma 2.3), respectively. In fact, they are immediate since

$$-\frac{1}{2} \sum_{i=1}^{k} |j_{Z_i} X_i|^2 - 2(n_1 + 2n_2 + 3n_3) + \sum_{i=1}^{m} |(\nabla_X Y_i)_o|^2 = -\text{tr}(\text{ad}^2_H)$$

and

$$-n_1 - 2n_2 - 3n_3 - \frac{1}{2} \sum_{j=1}^{n_2} ||Y, X_j||^2 - \frac{1}{2} \sum_{i=1}^{n_1} ||Y, Y_i||^2 = -\text{tr}(\text{ad}^2_H)$$

implying the required formulas.

Theorem 2.5. If $S$ is a 4-step Carnot space that satisfies the 2-stein condition, then $S$ is either, up to scaling, the hyperbolic space $RH^n$ ($n = \dim S$) or a Damek-Ricci space.

Proof. Let $S$ be a 4-step Carnot space with associated Lie algebra $\mathfrak{s}$. If $n_3 = 0$ then either $S$ is, up to scaling, the hyperbolic symmetric space $RH^n$ with $n = n_1 + 1$ or $n = n_2 + 1$ according to $n_2 = 0$ or $n_1 = 0$, or $S$ is a Damek-Ricci space by [3, Theorem 3.1].

Assume that $n_3 \neq 0$; we will show that $S$ cannot be 2-stein unless $n_1 = 0 = n_2$.

(i) Let $Y \in n_1$ a unit vector. If $U$ is the operator on $\mathfrak{s}$ defined by $U(\cdot) = R(\cdot, Y)H + R(\cdot, H)Y$ it follows that

$$\text{tr} \left( \text{ad}_H^4 + (R_Y \circ \text{ad}_H^2) - \frac{1}{2} U^2 \right)_{\mathfrak{s}_0} = 0. \quad (9)$$

where $\mathfrak{s}_0 = \text{span}\{Y, H\}$ is the totally geodesic subalgebra of $\mathfrak{s}$ which corresponds to the symmetric hyperbolic space $RH^2$ of constant curvature $-1$. We show this expression (9) following the same argument developed in [5, Proposition 1.2]. In order to do that we need to revise some properties fulfilled by $R_Y$ and $U$. First, note that

$$R_Y : \mathfrak{z} \to \mathfrak{z} \oplus n_1 \quad n_1 \to \mathfrak{z} \oplus n_1 \quad \text{and} \quad R_Y H = -H,$$

hence $R_Y : n_2 \to n_2$, since it is symmetric.
Next, by using the definition of $U$, for any unit vectors $Z \in \mathfrak{z}$, $W \perp Y \in \mathfrak{n}_1$ and $X \in \mathfrak{n}_2$ we compute:

$$U(Z) = \nabla_Z \nabla_Y H - \nabla_Y \nabla_Z H - \nabla_{[Z,H]} Y = -\nabla_Z Y + 3\nabla_Y Z + 3\nabla_Z Y$$

$$= 5\nabla_Z Y - \frac{5}{2} jz Y,$$  

$$U(W) = \nabla_W \nabla_Y H - \nabla_Y \nabla_W H - \nabla_{[W,Y]} H - \nabla_{[W,H]} Y$$

$$= -\nabla_W Y + \nabla_Y W + 2[W,Y] + \nabla_W Y$$

$$= \frac{1}{2}[Y,W] - 2[Y,W] = -\frac{3}{2}[Y,W] \quad \text{and}$$

$$U(X) = \nabla_X \nabla_Y H - \nabla_Y \nabla_X H - \nabla_{[X,Y]} H - \nabla_{[X,H]} Y$$

$$= -\nabla_X Y + 2\nabla_Y X + 3[X,Y] + 2\nabla_X Y$$

$$= \nabla_X Y + 2\nabla_Y X + 3[X,Y] = 3\nabla_X Y + [X,Y]$$

$$= \frac{3}{2}[X,Y] + 3(\nabla_X Y)_v + [X,Y] = \frac{5}{2}[X,Y] + 3(\nabla_X Y)_v.$$

Thus, $U : \mathfrak{n}_2 \to \mathfrak{z} \oplus \mathfrak{n}_1$ and $\mathfrak{z} \oplus \mathfrak{n}_1 \to \mathfrak{n}_2$. This property, together with the fact $R_Y : \mathfrak{z} \oplus \mathfrak{n}_1 \to \mathfrak{z} \oplus \mathfrak{n}_1$ and $\mathfrak{n}_2 \to \mathfrak{n}_2$ ($R_Y H = -H$) imply that (9) holds, since

$$\text{tr}\left(r^2 R_Y - s^2 \text{ad}_H^2\right) \circ U|_{\mathfrak{a}^\perp} = 0.$$

(ii) Next, (9) it will be applied, computing separately each term and using (7) and (8). First, from the above computations we obtain

$$\langle U^2 Z, Z \rangle = |U(Z)|^2 = \frac{25}{4} |jzY|^2 \quad \text{for all } Z \in \mathfrak{z}, \ |Z| = 1$$

$$\langle U^2 W, W \rangle = |U(W)|^2 = \frac{9}{4} |[Y,W]|^2, \ |W| = 1, \ \text{and}$$

$$\langle U^2 X, X \rangle = |U(X)|^2 = \frac{25}{4} |[X,Y]|^2 + 9 |(\nabla_X Y)_v|^2, \ \text{for } X \in \mathfrak{n}_2, \ |X| = 1.$$

Hence,

$$\text{tr}\left(U^2|_{\mathfrak{a}^\perp}\right) = \sum_{k=1}^{n_3} |U(z_k)|^2 + \sum_{i=1}^{n_1} |U(Y_i)|^2 + \sum_{j=1}^{n_2} |U(X_j)|^2$$

$$= \frac{25}{4} \sum_{k=1}^{n_3} |jz_k Y|^2 + \frac{9}{4} \sum_{i=1}^{n_1} |Y_i|^2 + \frac{25}{4} \sum_{j=1}^{n_2} |Y_j|^2 + \frac{25}{4} \sum_{j=1}^{n_2} |(\nabla_X Y)_v|^2$$

$$= \frac{2}{9} \sum_{i=1}^{n_1} |Y_i|^2 + 2\frac{25}{4} \sum_{j=1}^{n_2} |Y_j|^2,$$

which implies that

$$-\frac{1}{2} \text{tr}\left(U^2|_{\mathfrak{a}^\perp}\right) = -\frac{9}{4} \sum_{i=1}^{n_1} |Y_i|^2 - \frac{25}{4} \sum_{j=1}^{n_2} |Y_j|^2.$$
It is a straightforward computation to see that,
\[
\text{tr } (R_Y \circ \text{ad}_H^2) |_{\mathfrak{h}_0} = 9 \text{tr } R_Y |_{\mathfrak{h}} + \text{tr } (R_Y |_{\mathfrak{n}_1 \cap \mathfrak{h}^\perp}) + 4 \text{tr } (R_Y |_{\mathfrak{n}_2})
\]
\[
= 9 \left( -3n_3 + \sum_{k=1}^{n_2} \frac{1}{4} |j_{Z_k} Y|^2 \right) - (n_1 - 1) \frac{3}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2
\]
\[
+ 4 \left( -2n_2 + \sum_{j=1}^{n_2} |\nabla_{X_j} Y|^2 - \frac{3}{4} \sum_{j=1}^{n_2} |[Y, X_j]|^2 \right)
\]
\[
= -27n_3 - (n_1 - 1) - 8n_2 - \frac{3}{4} \sum_{j=1}^{n_2} |[Y, X_j]|^2 + \frac{1}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2,
\]
and (9) becomes
\[
0 = \text{tr } \left( \text{ad}_H^4 + (R_Y \circ \text{ad}_H^2) - \frac{1}{2} U \right) |_{\mathfrak{h}_0} = (n_1 - 1) + 16n_2 + 81n_3
\]
\[
-27n_3 - (n_1 - 1) - 8n_2 - \frac{3}{4} \sum_{j=1}^{n_2} |[Y, X_j]|^2 + \frac{1}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2
\]
\[
- \frac{9}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2 - \frac{25}{4} \sum_{j=1}^{n_2} |[Y, X_j]|^2
\]
\[
= 54n_3 + 8n_2 - 28 \frac{1}{4} \sum_{j=1}^{n_2} |[Y, X_j]|^2 - \frac{8}{4} \sum_{i=1}^{n_1} |[Y, Y_i]|^2.
\]
Thus,
\[
7 \sum_{j=1}^{n_2} |[Y, X_j]|^2 + 2 \sum_{i=1}^{n_1} |[Y, Y_i]|^2 = 54n_3 + 8n_2. \tag{10}
\]
(iii) Now, from the equality given by Corollary 2.4 (ii), we obtain
\[
2 \sum_{j=1}^{n_2} |[Y, X_j]|^2 + 2 \sum_{i=1}^{n_1} |[Y, Y_i]|^2 = 24n_3 + 8n_2 = 2 (12n_3 + 4n_2) \tag{11}
\]
and the expression (10)–(11) gives
\[
5 \sum_{j=1}^{n_2} |[Y, X_j]|^2 = 30n_3 \iff \sum_{j=1}^{n_2} |[Y, X_j]|^2 = 6n_3 \text{ for a unit vector } Y \in \mathfrak{n}_1.
\]
Finally, it follows from (7) that
\[
\sum_{k=1}^{n_3} |j_{Z_k} Y|^2 = 6n_3
\]
and we get a contradiction, since $|j_{Z} Y|^2 = 12$ for any unit vectors $Z \in \mathfrak{z}$ and $Y \in \mathfrak{n}_1$. In order to show this last assertion, note that $j_Z : \mathfrak{n}_2 \to \ker(j_Z |_{\mathfrak{n}_1})^\perp \subset \mathfrak{n}_1$ isomorphically (1.1) and by Proposition 2.2, a unit vector $Y \in \mathfrak{n}_1$ is expressed as $Y = \frac{1}{\sqrt{2}} j_{Z} X$ with $X \in \mathfrak{n}_2$, $|X| = 1$ (\( Y = \sum_{i=1}^{n} a_i j_{Z} X_i \) where $j_{Z} X_i =$ \( \frac{1}{\sqrt{2}} j_{Z} X_i \)).
$-|j_Z X_i|^2 X_i, |X_i| = 1$, in terms of the orthonormal basis $\{X_i\}$ and $\{\frac{1}{\sqrt{12}}j_Z X_i\} : i = 1, \ldots, m)$. Hence $|j_Z Y|^2 = 12$ and we have that $\sum_{k=1}^{n_3} |j_{Z_k} Y|^2 = 12n_3$.  

3. References


