GROUP ACTIONS ON ALGEBRAS AND MODULE CATEGORIES

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INTRODUCTION AND NOTATIONS

Let \( k \) be a field and \( A \) a finite dimensional (associative with 1) \( k \)-algebra. By \( \text{mod}_A \) we denote the category of finite dimensional left \( A \)-modules. In many important situations we may suppose that \( A \) is presented as a quiver with relations \((Q, I)\) (e.g. if \( k \) is algebraically closed, then \( A \) is Morita equivalent to \( kQ/I \)). We recall that if \( A \) is presented by \((Q, I)\), then \( Q \) is a finite quiver and \( I \) is an admissible ideal of the path algebra \( kQ \), that is, \( J^m \subset I \subset J^2 \) for some \( m \geq 2 \), where \( J \) is the ideal of \( kQ \) generated by the arrows of \( Q \), see [6].

It is convenient to consider \( A = kQ/I \) as a \( k \)-linear category with objects \( Q_0 \) (= vertices of \( Q \)) and morphisms given by linear maps \( Q(x, y) = e_yAe_x \), where \( e_x \) is the trivial path at \( x \) (for \( x, y \in Q_0 \)). In this categorical approach we do not need to assume that \( Q \) is finite (therefore the \( k \)-algebra \( kQ/I \) may not have unity). Occasionally we write \( A_0 = Q_0 \) if we do not need to explicit the quiver \( Q \).

The purpose of these notes is to present an introduction to the study of actions of groups \( G \subset \text{Aut}(Q, I) \) on \( A = kQ/I \) and their module categories \( \text{MOD}_A \) and to consider associated constructions that have proved useful in the Representation Theory of Algebras.

A symmetry of the quiver \( Q \) is a permutation of the set of vertices \( Q_0 \) inducing an automorphism of \( Q \). We denote by \( \text{Aut}(Q) \) the group of all symmetries of \( Q \). Those symmetries \( g \in \text{Aut}(Q) \) inducing a morphism \( g : kQ \to kQ \) such that \( g(I) \subset I \) form the group \( \text{Aut}(Q, I) \). In natural way, any \( g \in \text{Aut}(Q, I) \) induces an automorphism of the module category \( \text{mod}_A \) and on the Auslander-Reiten quiver \( \Gamma_A \) of \( A \) (since the action commutes with the Auslander-Reiten translation \( \tau_A \) of \( \Gamma_A \)).

In section 1, we present some basic facts about the actions of groups \( G \subset \text{Aut}(Q, I) \) on \( A = kQ/I \) (orbits, stabilizers, Burnside’s lemma) and show that for a representation-finite standard algebra \( A \), we have \( \text{Aut}(Q, I) = \text{Aut} \Gamma_A \), where \( \text{Aut} \Gamma_A \) is formed by the symmetries of \( \Gamma_A \) commuting with the translation \( \tau_A \). We recall that \( A \) is standard if \( A \) is representation-finite and for a choice of representatives of the isoclasses of indecomposables, the induced full subcategory of \( \text{mod}_A \) (denoted by \( \text{ind}_A/ \cong \)) is equivalent to \( k(\Gamma_A) \) which is the quotient of the path algebra \( k\Gamma_A \) by the ideal generated by the meshes \( \sum_{i=1}^{s}(\sigma\alpha_i) \cdot \alpha_i \) of the almost split
sequences $0 \rightarrow \tau_A X \xrightarrow{(\sigma_i)} \bigoplus_{i=1}^{s} Y_i \xrightarrow{(\alpha_i)} X \rightarrow 0$. We recall that for $\text{char } k \neq 2$, any representation-finite algebra $A = kQ/I$ is standard.

In section 2 we consider relations between the structure of $\text{Aut} (Q, I)$ and the Coxeter polynomial of $A$ (recall that, $p_A(t) = \det (t \text{id} - \varphi_A)$ is the Coxeter polynomial associated to the Coxeter matrix $\varphi_A$ which is $\mathbb{Z}$-invertible in case $g\ell \dim A < \infty$).

In section 3 we present the main constructions associated to groups acting on algebras:

- if $G$ acts freely on $A$ (that is, $G \subset \text{Aut} (Q, I)$ and $g(x) = x$ for a vertex $x$, implies $g = 1$), then the Galois covering $F: A \rightarrow A/G$ is a $G$-invariant functor of $k$-categories.

- if $B$ is a $G$-graded $k$-category, the smash product $B \# G$ is a $k$-category accepting the free action of $G$. Moreover, $(B \# G)/G \cong B$.

Galois coverings were introduced by Bongartz and Gabriel [6, 7, 2] for the study of representation type of algebras. Smash products is a well-known construction in $k$-categories. Indeed, if $G$ acts freely on $A$, then $A/G$ is a $G$-graded category such that $(A/G) \# G \cong A$.

In section 4, following [2], we introduce functors relating the module categories of $A$ and $A/G$ when $G$ acts freely on $A$. The main results in these notes (section 5) relate the representation types of $A$ and $A/G$ (which was the original purpose of the introduction of Galois coverings). Indeed, given a Galois covering $F: A \rightarrow A/G = B$ with $B$ a finite dimensional $k$-algebra, then $B$ is representation-finite if and only if $A$ is locally representation-finite (that is, for each $i \in A_0$, there are only finitely many indecomposable $A$-modules $X$, up to isomorphism, with $X(i) \neq 0$). The proof of this result was partially given in [7] and completed in [10], and provides an efficient tool to deal with representation-finite algebras.

The representation-infinite situation is more involved. We recall that $A$ is said to be tame if for every $d \in \mathbb{N}$ there are finitely many $A - k[t]$-bimodules $M_1, \ldots, M_{s(d)}$ which are finitely generated as right $k[t]$-modules and such that any indecomposable $A$-module $X$ with dimension $d$ is of the form $M_i \otimes_{k[t]} (k[t]/(t-\lambda))$ for some $1 \leq i \leq s(d)$ and some $\lambda \in k$. We say that a tame algebra $A$ is domestic (resp. of polynomial growth) if $s(d)$ can be chosen $\leq e$ (resp. $\leq d^m$ for some $m$) for all $d$. It is not hard to show that $A$ is tame if $A/G$ is tame for a group acting freely on $A$. The converse was shown to be false in [8]; nevertheless there are many interesting, general situations where it holds true.

In section 5 we give examples of results [5, 12] showing that for a Galois covering $F: A \rightarrow A/G$, the category $A$ is tame if and only if $A/G$ is tame, provided certain restrictions on the group $G$ or on the category $A/G$ are satisfied.

We denote by $K_0(A)$ the Grothendieck group of $A$, freely generated by representatives $S_1, \ldots, S_n$ of the simple $A$-modules, where $n = n(Q)$ is the number of vertices of $Q$. We denote $P_i$ (resp. $I_i$) the projective cover (resp. injective envelope) of $S_i$. With the categorical approach an $A$-module is a functor $X: A \rightarrow \text{Mod}_k$, and
a morphism $f: X \to Y$ is a natural transformation. In case $\operatorname{gf \dim} A < \infty$, the Coxeter matrix $\varphi_A: K_0(A) \to K_0(A)$ is defined by $\varphi_A([P_i]) = -[I_i]$.

These notes follow closely the lectures given at the Workshop on Representation Theory in Mar del Plata, Argentina in March 2006. The intention of the lectures was to present an elementary introduction to the topic which would serve as a source of motivation and information on the techniques used. While we cannot provide complete proofs of every result, we tried to sketch some representative arguments. We thank the organizers of the Workshop for his hospitality.

1. **The group of automorphisms of an algebra**

1.1. Let $A = kQ/I$ be a finite dimensional $k$-algebra:

$\operatorname{Aut} A$ denotes the group of automorphisms of $A$. By $\operatorname{Aut}(Q)$ we denote the group of symmetries of $Q$ and by $\operatorname{Aut}(Q, I)$ the group of symmetries of $Q$ fixing $I$ (that is, $g \in \operatorname{Aut}(Q)$ induces $g: kQ \to kQ$ such that $g(I) = I$).

**Lemma.** $\operatorname{Aut}(Q, I)$ is a subgroup of $\operatorname{Aut} A$. \hfill \Box

Each symmetry $g \in \operatorname{Aut}(Q)$ gives rise to a matrix $g \in \mathbb{G}^{\ell \mathbb{Z}}(n(Q))$ sending $S_i$ to $S_{g(i)}$. This representation $\gamma: \operatorname{Aut}(Q) \to \mathbb{G}^{\ell \mathbb{Z}}(n(Q))$ is called the canonical representation.

1.2. Let $G \subset \operatorname{Aut}(Q, I)$ be a subgroup. For a given $i \in Q_0$, $G_i$ denotes the orbit of $i$ and subgroup of $G$, $G_i = \{g \in G: gi = i\}$ denotes the stabilizer of $i$. Clearly $G_i$ is a subgroup of $G$.

**Lemma.** The mapping $G_i \to G/G_i$, $gi \mapsto gG_i$ is a bijection from the orbit to the set of left cosets $G/G_i$.

$$n(Q) = |G| \sum \frac{1}{|G_i|}$$

where the sum runs over representatives of the orbits. \hfill \Box

1.3. Let $K_0(A) = \mathbb{Z}^{n(Q)}$ be the Grothendieck group of $A$. Consider

$$\operatorname{Inv}_G(A) = \{v \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}: v^g = v \text{ for } g \in G\}$$

the $\mathbb{Q}$-space of $G$-invariant vectors. Then $t_0(G)$ the number of orbits of $G$ in $Q_0$ equals $\dim \operatorname{Inv}_G(Q)$.

Let $S_1, \ldots, S_m$ be a set of representatives of the irreducible $\mathbb{C}$-representations of $G$ (here $m$ is the number of conjugacy classes of $G$). Let $S_1$ be the trivial representation.

Consider $\chi_{\beta}$ the character corresponding to $S_{\beta}$ (that is, $\chi_{\beta}: G \to \mathbb{C}^\ast$, $g \mapsto \operatorname{tr} S_{\beta}(g)$). The characters $1 = \chi_1, \ldots, \chi_m$ form an orthonormal basis of the class group $X(G)$, with the scalar product $(\chi, \chi') = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi'(g)}$.

**Lemma.** [Burnside’s lemma]. $t_0(G) = \frac{1}{|G|} \sum_{g \in G} \chi_{\gamma}(g)$ where $\chi_{\gamma}: G \to \mathbb{Q}^\ast$, is the character of the canonical representation $\gamma$.  

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Proof. Observe that $\chi_\alpha(g)$ is the number of fixed points of $g$.

$$
\sum_{g \in G} \chi_\alpha(g) = \sum_{g \in G} \sum_{i \in Q_0} 1 = \sum_{g \in G} \sum_{i \in Q_0} 1 = |G| \sum_{i \in Q_0} |G_i| = |G| |t_0(G)|
$$

where $Q_g = \{ j \in Q_0 : gj = j \}$.

\[ \square \]

1.4. Let $G$ be a subgroup of $\text{Aut} (Q, I)$ then $G$ acts on $\text{Mod}_A$ as follows:

$$
X \in \text{Mod}_A \text{ and } g \in G, \text{ then } X^g \in \text{Mod}_A
$$
such that for $i \xrightarrow{\alpha} j$, we have $X^g(i) = X(gi) \xrightarrow{X(g\alpha)} X(gj)$. Similarly, for $f \in \text{Hom}_A(X, Y)$, we define $f^g \in \text{Hom}_A(X^g, Y^g)$.

Clearly, this action preserves indecomposable modules and induces an action of $G$ of the Auslander-Reiten quiver $\Gamma_A$, satisfying:

(a) the action preserves projective, injective and simple modules;

(b) the action preserves Auslander-Reiten sequences (in particular, $(\tau_A X)^g = \tau_A X^g$);

(c) $G$ is a subgroup of $\text{Aut} \Gamma_A$, the group of automorphisms of the quiver $\Gamma_A$ (commuting with the $(\tau_A)$-structure).

For (c), let $g \in G$ be an element inducing a trivial action on $\text{Aut} \Gamma_A$; we shall prove $g = 1$. Indeed, $P_i = P_i^g = P_{gi}$ implies $gi = i$ for every vertex $i \in Q_0$.

Let $i \xrightarrow{\alpha_s} j$, $s = 1, \ldots, m$ be all arrows between $i$ and $j$, then $g$ establishes a permutation of the $\alpha_s$. Let $X_s$ be the 2 dimensional $A$-module with $X_s(\alpha_i) = \delta_{is} : k \rightarrow k$. Since $X_s$ is indecomposable, $X_s^g = X_s$ or $g\alpha_s = \alpha_s$. Therefore $g = 1$.

Proposition. Let $A = kQ/I$ be an algebra satisfying:

(a) $A$ is representation finite, (b) $A$ is standard
then $\text{Aut} (Q, I) = \text{Aut} \Gamma_A$.

Proof. We already know that $\text{Aut} (Q, I)$ is a subgroup of $\text{Aut} \Gamma_A$. Let $g \in \text{Aut} \Gamma_A$ and consider the induced automorphism $\bar{g}$ of the mesh category $k(\Gamma_A)$. Clearly, $\bar{g}$ restricts to an automorphism of $A$ (considering the full embedding $i \rightarrow P_i$, $i \in Q_0$). By definition there is an automorphism $h \in \text{Aut} (Q, I)$ inducing $\bar{g}$, and therefore inducing $g$. \[ \square \]

2. THE CANONICAL REPRESENTATION AND THE COXETER MATRIX

2.1. Let $A = kQ/I$ and assume that $g\ell \dim A < \infty$. For example, this happens if $A$ is triangular, that is, $Q$ has no oriented cycles.

Recall that $\varphi_A : K_0(A) \xrightarrow{\sim} K_0(A), \dim P_i \xrightarrow{\dim} \dim I_i$ is the Coxeter matrix if $A$, which is $\mathbb{Z}$-invertible. In case $A = kQ$ (i.e. $I = 0$), then $(\dim X) \varphi_A = \dim \tau_A X$ for any non-projective indecomposable $X$.

The characteristic polynomial $p_A(t) = \det (t \text{Id} - \varphi_A)$ is called the Coxeter polynomial.

Examples:

(a) $Q$ Dynkin type, then $\text{Spec} \varphi_A \subset \mathbb{S}^1 \setminus \{1, -1\}$. 

(b) If $Q$ extended Dynkin type, then $\text{Spec } \varphi_A \subset S^1$ and 1 is a root of multiplicity 2.

(c) $p_A(t) = 1 + t - t^3 - t^4 - t^5 - t^6 - t^7 + t^9 + t^{10}$ is irreducible (over $\mathbb{Z}[t]$).


(a) $\varphi_A$ is an automorphism of the representation $\gamma$.
(b) If $\text{Aut}(Q, I)$ is not trivial, then $p_A(t)$ is not irreducible.

Proof. (a): It suffices to observe that $g\varphi_A = \varphi_A g$ for any $g \in G$.
(b): Let $R_1, \ldots, R_m$ be a set of representatives of the irreducible $\mathbb{Q}$-representations of $G = \text{Aut}(Q, I)$. Let $R_1$ be the trivial representation. Up to conjugation (with $L$)

$$
\gamma^L = \bigoplus_{\alpha=1}^m R^r(\alpha)
$$

$R_\alpha : G \to GL_\mathbb{Q}(\dim R_\alpha)$, then $n = \sum_{\alpha=1}^m r(\alpha) \dim R_\alpha$.

By Schur’s lemma $\varphi_A^L = \begin{bmatrix} \varphi_1 & 0 \\ \ddots & \ddots \\ 0 & \varphi_m \end{bmatrix}$, where $\varphi_\alpha : R^r(\alpha) \to R^r(\alpha)$ is an automorphism. Hence $p_A(t) = p_{\varphi_1}(t) \ldots p_{\varphi_m}(t)$. Therefore if $p_A(t)$ is irreducible then $r(\alpha) = 0$ for $\alpha \geq 2$.

If $G \neq (1)$, then $t_0(G) < n$. Moreover, the characters

$$
\chi_\alpha : G \to \mathbb{Q}^*, \quad g \mapsto \text{tr } R_\alpha(g), \quad \alpha = 1, \ldots, m
$$

form an orthonormal basis of $X(G)$ with scalar product

$$
(\chi, \chi') = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g)}.
$$

Then

$$
r(\alpha) = (\chi_\gamma, \chi_\alpha) = \frac{1}{|G|} \sum_{g \in G} \chi_\gamma(g) \overline{\chi_\alpha(g)}
$$

for $\alpha = 1$, $r(1) = \frac{n}{|\alpha|} \sum_{g \in G} \chi_\gamma(g) = t_0(G)$.

Finally, $n = \sum_{\alpha=1}^m r(\alpha) \dim R_\alpha = t_0(G) + \sum_{\alpha=2}^m r(\alpha) \dim R_\alpha$, with implies the existence of $\alpha \geq 2$ with $r(\alpha) > 0$. Therefore $p_A(t)$ is not irreducible. \qed
2.2. Example: Consider $G = A_5 \subset \text{Aut } Q$ and $\gamma: A_5 \to \text{GL}(6)$ the canonical representation. It is not hard to calculate the character table of $A_5$:

<table>
<thead>
<tr>
<th>Conjugacy classes</th>
<th>${1}$</th>
<th>$(123)$</th>
<th>$(12)(34)$</th>
<th>$(12345)$</th>
<th>$(13524)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>Gx_i</td>
<td>$</td>
<td>1</td>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>5</td>
<td>$-1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>3</td>
<td>0</td>
<td>$-1$</td>
<td>$\alpha_2$</td>
<td>$\alpha_1$</td>
</tr>
</tbody>
</table>

where $\chi_i$ corresponds to $S_i$ irreducible $\mathbb{R}$-representation of $G$ (with $S_1$ the trivial representation).

Then

$$\gamma^T = \bigoplus_{\beta=1}^{s} S^n_{\beta}, \quad n(1) = t_0(G) = 2$$

$$n(2) = (\chi_1, \chi_2) = \frac{1}{60} \sum_{g \in G} \chi_{\gamma}(g)\overline{\chi_2}(g) = \frac{1}{60} [24 + 60 - 24] = 1$$

Since $\dim \gamma = 6$, $\gamma^T = S_1 \oplus S_1 \oplus S_2$ which is also a $Q$-decomposition. Moreover, $p_A(t) = (1 - 3t + t^2)(1 + t)^4$ is an irreducible factorization.

3. Constructions of algebras associated to groups of automorphisms (coverings and smash products)

3.1. Let $A$ be a $k$-category given as $A = kQ/I$. Let $G$ be a subgroup of $\text{Aut } (Q, I) \subset \text{Aut } A$. We say that $G$ acts freely on $A$ if $gi = i$ for some $i \in Q_0$ implies $g = 1$.

**Lemma.** Let $G$ be a group acting freely on $A = kQ/I$. Then there exists a $k$-category $B = k\bar{Q}/\bar{I}$ and a functor $F: A \to B$ satisfying:

(a) $(G$-invariant$)$: $Fg = F$ for every $g \in G$
(b) $(universal \ G$-invariant$)$: for any functor $F': A \to B'$ which is $G$-invariant, there exists a unique functor $\bar{F}: B \to B'$ such that $F' = \bar{F}F$
(c) $\bar{Q}_0$ is formed by the $G$-orbits of vertices in $Q_0$ and for any $a = Gi$ and $b = Gj$

$$B(a, b) = \bigoplus_{g \in G} A(i, gj)$$
Proof. Let $B$ be defined as in (c) with composition maps

$$B(a, b) \otimes B(b, c) = \left( \bigoplus_{g \in G} A(i, gj) \right) \otimes \left( \bigoplus_{h \in G} A(j, h\ell) \right) \to \bigoplus_{g'} A(i, g'j) = B(a, c)$$

$$(f_g) \otimes (f_h^*) \mapsto \left( \sum_{g' = g} (f_h^*)g f_g \right)$$

Define $F: A \to B$, $f \in A(i, j) \subset \bigoplus_g A(i, gj) = B(a, b)$ in the unique possible way (since $gj = j$ implies $g = 1$).

3.2. If $G$ acts freely on $A = kQ/I$, the functor $F: A \to B$ as in (2.1) is called a Galois covering defined by $G$ and $B = A/G$ is a Galois quotient of $A$.

We say that a $k$-category $B$ is $G$-graded if for each pair of objects $a, b$ there is a vector space decomposition $B(a, b) = \bigoplus_{g \in G} B^g(a, b)$ such that the composition induces linear maps

$$B^g(a, b) \otimes B^h(b, c) \to B^{gh}(a, c)$$

Lemma. $B = A/G$ is a $G$-graded $k$-category.

3.3. For a $G$-graded $k$-category $B$ we define the smash product as the $k$-category $B \# G$ with objects $B_0 \times G$ and for any pair $(a, g), (b, h) \in B_0 \times G$, the morphisms are

$$(B \# G)((a, g), (b, h)) = B^{g^{-1}h}(a, b)$$

with composition given by:

$$(B \# G)((a, g), (b, h)) \otimes (B \# G)((b, h), (c, t)) \to (B \# G)((a, g), (c, t))$$

$$B^{g^{-1}h}(a, b) \otimes B^{h^{-1}t}(b, c) \mapsto B^{g^{-1}t}(a, c)$$

Theorem [4]. The category $B \# G$ accepts a free action of $G$ such that $(B \# G)/G \cong B$.

Moreover, if $G$ acts freely on $A$, then $(A/G) \# G \cong A$.

Proof. Clearly there is a $G$-invariant functor $\tilde{F}: B \# G \to B$ inducing a functor $\tilde{F}: (B \# G)/G \to B$ which is the identity on objects. Check that $\tilde{F}$ is an isomorphism.

Assume $G$ acts freely on $A$ and let $f: A \to A/G$ be the induced Galois covering. Since $G$ acts freely, there is a bijection $(A/G)_0 \times G \overset{\sim}{\to} A_0$ which commutes with the $G$-action. Moreover, if $a, b \in (A/G)_0$, $f(a, 1) = i$, $f(b, 1) = j \in A_0$ then

$$(A/G \# G)((a, g), (b, h)) = A/G g^{-1}h(a, b) = A(i, g^{-1}hj) = A(gi, hj) = A(f(a, g), f(b, h))$$

is compatible with the composition.

In other words, smash products and Galois quotients are inverse operations in the class of $k$-categories.
3.4. Galois correspondence. Let $B$ be a $k$-category. There is a $1-1$ correspondence

$$(A \xrightarrow{F} B \text{ Galois covering defined by } G) \mapsto G \text{ group grading } B,$$

such that for any two Galois coverings $A \xrightarrow{F} B$ defined by $G$ and $A' \xrightarrow{F'} B$ defined by $H$, there exist a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{k} & & \downarrow{F'} \\
A' & \xleftarrow{H} & B
\end{array}
$$

if and only if there exists a normal subgroup $H \trianglelefteq G'$, such that $G'/H = G$.

A Galois triple $(A, F, G)$ of $B$ is a Galois covering $F: A \to B$ defined by the (free) action of a group $G$. There is a universal object in this set of Galois triples. Indeed, let $B = kQ/I$ and consider $W$ the set of all walks in $Q$ starting and ending at a fixed vertex $b$. Let $\sim$ be the equivalence relation induced by the following elementary relations:

(i) $\alpha \alpha^{-1} \sim e_y$ and $\alpha^{-1} \alpha \sim e_x$ for any arrow $x \xrightarrow{\alpha} y$;

(ii) if $\sum_{i=1}^{s} \lambda_i w_i \in I(x, y)$ with $\lambda_i \in k^*$, such that for any $L \subseteq \{1, \ldots, s\}$ we have $\sum_{i \in L} \lambda_i \notin I(x, y)$, then $w_i \sim w_j$ for $i, j$;

(iii) if $w \sim w'$, then $ww'' \sim w'w''$, whenever the products are defined.

Then $\tilde{G} = W/\sim$ has a group structure such that $B$ is $\tilde{G}$-graded. The group $\tilde{G}$ is called the fundamental group of $B$.

**Proposition** [9]. Let $\tilde{B} = B \# \tilde{G}$ and $\tilde{F}: \tilde{B} \to B$ be defined by $\tilde{G}$. The triple $(\tilde{B}, \tilde{F}, \tilde{G})$ is a universal Galois covering, that is, for any Galois covering $F: A \to B$ defined by the action of a group $G$, there exists a covering $\tilde{F}: \tilde{B} \to A$ defined by $H \trianglelefteq \tilde{G}$ such that $\tilde{G}/H = G$. \hfill \Box

4. Actions induced on module categories

4.1. Let $F: A \to B$ be a Galois covering of $k$-categories defined by the action of $G$. We shall denote by

- $\text{MOD}_A$ the category of left $A$-modules;
- $\text{Mod}_A$ those $X \in \text{MOD}_A$ with $\dim_k X(i) < \infty$ for every $i \in A_0$;
- $\text{mod}_A$ those $X \in \text{MOD}_A$ with $\sum_{i \in A_0} \dim_k X(i) < \infty$.

There are naturally defined functors:

- $F_*: \text{MOD}_B \to \text{MOD}_A$, $(Y: B \to \text{Mod}_k) \mapsto (Y \circ F: A \to \text{Mod}_k)$,
- $F_\lambda: \text{MOD}_A \to \text{MOD}_B$, $(X: A \to \text{Mod}_k) \mapsto F_\lambda X(a) = \bigoplus_{g \in G} X(gi)$,

called the pull-up functor;

$$
F_\lambda(\bigoplus_{i \in A_0} X(i)) = \bigoplus_{i \in \tilde{G} \setminus \{1\}} F_\lambda X(i).
$$

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and such that for \( f = (f_g) \in B(a, b) = \bigoplus_{g \in G} A(i, g) \), then \( F_\lambda X(f) : F_\lambda X(a) \to F_\lambda X(b) \), sends \((a_g)\) to \( \left( \sum_{h} X(f_h^{g^{-1}})(a_h) \right) \), called the push-down functor. We observe that \( F_\lambda \) is a left adjoint to \( F \). Similarly, there is a right adjoint \( F_\rho : \text{MOD}_A \to \text{MOD}_B \) to \( F \). For modules \( X \in \text{mod}_A \), the modules \( F_\lambda X \) and \( F_\rho X \) coincide.

Recall that \( G \) acts on \( \text{MOD}_A \) and \( X \in \text{MOD}_A \) is \( G \)-stable if \( X^g = X \) for every \( g \in G \). The category of \( G \)-stable \( A \)-modules is denoted by \( \text{MOD}_A^G \).

4.2. Proposition [7, 2]. Let \( F : A \to B \) be a Galois covering defined by the action of \( G \). Then the following happens:

(a) The categories \( \text{MOD}_A^G \) and \( \text{MOD}_B \) are equivalent.

(b) For any \( X \in \text{MOD}_A \) and \( g \in G \), we have \( F_\lambda X^g \cong F_\lambda X \). Moreover, \( F.F_\lambda X \sim_{g \in G} X^g \) as \( A \)-modules.

(c) Let \( H \) be a subgroup of \( G \) and \( X \in \text{MOD}_A^H \). Then \( F_\lambda X \) has a natural structure as \( kH \)-module. If \( H \) is a finite group and \( \text{char} \ k \not\mid |H| \), then \( F_\lambda X \) decomposes as a direct sum of at least \( |H| \) factors.

Proof. (a): clear.

(b): Observe that \( F_\lambda X(a) = \bigoplus_{h \in G} X(hi) \sim_{g \in G} F_\lambda X^g(a) = \bigoplus_{h \in G} X(hgi) \) canonically.

Hence \( F.F_\lambda X(i) = F_\lambda X(Fi) = \bigoplus_{h \in G} X(hi) = \bigoplus_{h \in G} X^h(i) \) and correspondingly in morphisms.

(c): Choose a set \( W \) of representatives in \( G \) of the right cosets \( G/H \). Then \( F_\lambda X(a) = \bigoplus_{g \in G} X(i) = kH \otimes_k \left( \bigoplus_{w \in W} X(wi) \right) \) and correspondingly in morphisms.

Hence we get \( \varphi : kH \to \text{End}_B(F_\lambda X) \) a group homomorphism such that for each idempotent \( e \) of \( kH \), \( \varphi(e) \) is idempotent and there is a factorization of \( F_\lambda X \).

In case \( H \) is a finite group with \( \text{char} \ k \not\mid |H| \), by Mashké theorem, the group algebra \( kH \) is semisimple (with \( |H| \) idempotents). The result follows. \( \square \)

4.3. Assume \( G \) acts freely on \( A \) and \( F : A \to B = A/G \) is the corresponding Galois covering.

Let \( X \in \text{MOD}_A \), the stabilizer \( G_X \) is the subgroup of \( G \) formed by those \( g \in G \) such that \( X^g \cong X \). That is, \( X \in \text{MOD}_A^H \) if \( H \subset G_X \).

Proposition [7].

(a) If \( X \in \text{ind}_A \) and \( G \) is torsion free, then \( G_X = (1) \).

(b) If \( X \in \text{ind}_A \) and \( G_X = (1) \), then \( F_\lambda X \) is indecomposable and for any module \( Y \in \text{mod}_A \) with \( F_\lambda X \simeq F_\lambda Y \), then \( Y \simeq X^g \) for some \( g \in G \).

Proof. (a): Let \( g \in G_X \) for some \( X \in \text{ind}_A \), then \( g \) establishes a permutation of \( \text{supp} \ X \) (a finite set). Then for some \( s \in \mathbb{N} \), \( 1 = g^s \) on \( \text{supp} \ X \). Since \( G \) acts freely on \( A \), then \( g^s = 1 \). Since \( G \) is torsion free, then \( g = 1 \) and \( G_X = (1) \).
(b): Assume $F_{\lambda}X \cong Z \oplus Z'$, then $\bigoplus_{g \in G} X^g = F.F_{\lambda}X \cong F.Z \oplus F.Z'$. Assume $X$ is a direct summand of $F.Z \in \text{MOD}_A^G$, then $\bigoplus_{g \in G} X^g \subset F.Z$ and $F.Z' = 0$. Therefore $F_{\lambda}X$ is indecomposable.

If $F_{\lambda}X \cong F_{\lambda}Y$, then $Y$ is indecomposable and $Y \cong X^g$ for some $g \in G$. □

5. COVERINGS AND THE REPRESENTATION TYPE OF AN ALGEBRA

5.1. Examples:

(a) Let $B = k\langle x, y \rangle/(x^2, y^3, xy, yx)$

Let $A = kQ/I$ be the infinite $k$-category with quiver

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\text{y} & \text{x} & \text{y} & \text{x} & \text{y} & \\
\vdots & \text{y} & \text{x} & \text{y} & \text{x} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

$Q$:

and $I$ generated by all relations of the form $x^2, y^3, xy, yx$. Then we get a Galois covering $F: A \to B$ defined by the action of $\mathbb{Z} \times \mathbb{Z}$ on $A$.

The module

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
X : k & k & k & \\
\downarrow & \downarrow & \downarrow & \\
k & k & 0 & \\
\downarrow & \downarrow & \downarrow & \\
k & 0 & \\
\vdots & \vdots & \vdots & \\
\end{array}
\]

is indecomposable.

It is stable under the action of $\mathbb{Z} \leftarrow \mathbb{Z} \times \mathbb{Z}$, $n \mapsto (n, n)$.

The universal Galois covering of $B$ is given by $\tilde{B} = k\tilde{Q}/\tilde{I}$ where the free group in two generators $F_2$ acts on $\tilde{B}$. The normal subgroup $H = \langle xy - yx \rangle$ of $F_2$ acts on $\tilde{B}$ inducing the covering $F: A \to B$.

Observe that for $X \in \text{ind}_A$, we have $G_X = (1)$ and therefore $F_{\lambda}X \in \text{ind}_B$.

(b) [13] Let $B$ be a standard $k$-algebra of finite representation type. Then the Auslander-Reiten quiver $\Gamma_B$ is finite and equipped with the mesh relations: $\sum_{i=1}^n \beta_i \circ \sigma \beta_i = 0$, for each almost split sequence $0 \to \tau_B X \xrightarrow{(\sigma \beta_i)} Y_i \xrightarrow{(\beta_i)} X \to 0$. 

There is an universal Galois cover $\tilde{\Gamma}_B \xrightarrow{\pi} \Gamma_B$ of translation quivers defined by the action of a free group $G$. Moreover, $\tilde{\Gamma}_B$ is the Auslander-Reiten quiver of a $k$-category $\tilde{B}$ such that $k(\tilde{\Gamma}_B) = \text{ind}_{\tilde{B}}$.

We illustrate the situation in the following example:

A full subcategory $B'$ of $B = kQ/I$ is convex in $B$ if $B' = kQ'/I'$ for a quiver $Q'$ which is path closed in $Q$ (i.e. $i_0 \to i_1 \cdots \to i_s \to i_{s+1}$ with $i_0, i_{s+1} \in Q'$ implies $i_1, \ldots, i_s \in Q'$).

**Theorem.** Let $B$ be a representation-finite $k$-algebra. Then

(a) [13] There exists a universal Galois covering $\tilde{\Gamma}_B \to \Gamma_B$ defined by the action of a free group $G$. Moreover $k(\tilde{\Gamma}_B) = \text{ind}_{\tilde{B}}$ for a $k$-category $\tilde{B}$. If $B$ is standard, then $\tilde{B} \to B$ is a universal Galois covering.

(b) [3] For every finite convex subcategory $C$ of $B$, we have $\tilde{\Gamma}_C = \Gamma_C$ and $C$ is representation-directed (i.e. $C$ is representation-finite and $\Gamma_C$ is a preprojective component).
5.2. We say that the group $G$ acts freely on indecomposable classes of $A$-modules if $X^g \simeq X$ for $X \in \text{ind}_A$ implies $g = 1$. Observe that for the algebra $B$ with quiver

the indecomposable representation $X$ has non-trivial stabilizer.

**Theorem.** Let $F : A \to A/G = B$ be a Galois covering defined by a group $G$

a) [2] If $G$ acts freely on indecomposable classes of $A$-modules, then $F_\lambda$ induces an injection $(\text{ind}_A/ \cong )/G \hookrightarrow \text{ind}_B/ \cong$ and preserves Auslander-Reiten sequences.

b) [2, 10] If $B$ is finite (hence a $k$-algebra), then $B$ is representation-finite if and only if (i) $G$ acts freely on $\text{ind}_A/ \cong$ and (ii) $A$ is locally representation-finite (i.e. for each $i \in A_0$, \{ $X \in \text{ind}_A : X(i) \neq 0$ $\}$ is finite). In that case, $\Gamma_A/G \simeq \Gamma_B$. 

Proof. (a): Let $X \in \text{ind}_A$, then we know $G_X = (1)$ and $F_\lambda X \in \text{ind}_B$. Moreover $F_\lambda X \cong F_\lambda Y$ implies $Y \cong X^g$ for some $g \in G$. For an almost split sequence $\eta: 0 \to \tau_A X \xrightarrow{\alpha} E \xrightarrow{\beta} X \to 0$ with $X \in \text{ind}_A$ we get an exact sequence

$$F_\lambda \eta: 0 \to F_\lambda \tau_A X \to F_\lambda E \to F_\lambda X \to 0$$

with $F_\lambda X$, $F_\lambda \tau_A X$ indecomposable. For $f: Y \to F_\lambda X$ a non-invertible epi in $\text{mod}_B$, we get a map

$$
\begin{array}{ccc}
F.Y \\
\oplus \left( \bigoplus_{g \in G} E^g = F.F_\lambda E \xrightarrow{F.F_\lambda \beta} F.F_\lambda X = \bigoplus_{g \in G} X^g \right) \\
\downarrow \pi_1 \\
E \\
\downarrow \beta \\
X
\end{array}
$$

There is some $g \in \text{Hom}_A(F.Y, E)$ such that $\beta g = \pi_1 F.(f)$. From the adjunction $(F, F_\rho)$ we get

$$\text{Hom}_A(F.Y, E) \xrightarrow{\sim} \text{Hom}_B(Y, F_\rho E) = \text{Hom}_B(Y, F_\lambda E), \ g \mapsto g'$$

Then $\beta g' = f$ and $F_\lambda \eta$ is almost split sequence in $\text{mod}_B$.

(b): Assume $B$ is representation-finite and $Y_1, \ldots, Y_s$ are representatives of $\text{ind}_B/ \cong$. Let $X \in \text{ind}_B$ and suppose $g \in G$ is such that $X^g \cong X$. We shall prove that $g = 1$. Let $\tilde{B} \xrightarrow{F} B$ be a covering defined by a free group $H$ as in (5.1), that is, we get commutative diagrams:

$$
\begin{array}{ccc}
\tilde{B} & \xleftarrow{k(\pi)} & \text{ind}_{\tilde{B}} = k(\tilde{\Gamma}_B) \\
F & & \downarrow k(\pi) \\
B & \xleftarrow{k(\pi)} & \text{ind}_B = k(\Gamma_B)
\end{array}
\quad
\begin{array}{ccc}
\tilde{\Gamma}_B & \xrightarrow{\tilde{g}} & \tilde{\Gamma}_B \\
\downarrow \pi & & \downarrow \pi \\
\Gamma_B & \xrightarrow{g} & \Gamma_B
\end{array}
$$

Then $g$ induces an automorphism $\tilde{g}$ on $\tilde{\Gamma}_B$ such that $g\pi = \pi\tilde{g}$ and $\tilde{g}X' = X'$ for some $X' \in \tilde{\Gamma}_B$ with $\pi X' = X$. We get an automorphism $g'$ of $\tilde{B}$ acting freely (since $g = Fg'$ acts freely on $B$).

Let $C$ be the full subcategory of $\tilde{B}$ formed by $\text{supp} X$. It is easy to see that $C$ is convex in $\tilde{B}$ and therefore $\Gamma_C$ is a preprojective component. Since $g' X' = X'$, then we get $h \in \text{Aut} C$ with $(X')^h \cong X'$. Since $h$ commutes with $\tau_C$, there is a projective $C$-module $P_x$ with $P_x^h = P_x$, which is a contradiction unless $h = 1$.

We check that $A$ is locally representation-finite: let $i \in A_0$ and let $(X_\alpha)_{\alpha \in \Delta}$ be all indecomposable $A$-modules which are direct summands of $F.Y_i$ for some $1 \leq j \leq s$ and $X_\alpha(i) \neq 0$. This set is finite [indeed, $\bigoplus X_\alpha = F.Y_j$, then $\bigoplus F_\lambda X_\alpha = F_\lambda F.Y_j = \bigoplus Y^{(i)}$ and $X_\alpha \in \text{ind}_A$, for all $\alpha$. Since $\dim_k F.Y_j(i) = \dim_k Y^{(i)}(Fi) < \infty$, the set is finite].
Finally, let \( Y \in \text{ind}_B \). Then \( F_\lambda F.Y = \oplus F_\lambda X_\alpha \) and
\[
\text{Hom}_A(F,Y,F.Y) \cong \text{Hom}_B(Y,F_\rho F.Y) = \text{Hom}_B(Y,F_\lambda F.Y)
\]
hence \( Y \cong F_\lambda X_\alpha \) for some \( \alpha \). Therefore \( \Gamma_B = \Gamma_{A/G} \).

5.3. Let \( F : A \to A/G \) be a Galois covering with group \( G \). In case \( A/G \) is a representation-finite \( k \)-algebra, we have seen that \( F_\lambda \) covers all indecomposable modules. The situation is different for representation-infinite algebras. We shall briefly discuss the new occurring phenomena. By \( \text{ind}_1 A/G \) we denote the full subcategory of \( \text{ind}_{A/G} \) formed by all objects isomorphic to \( F_\lambda M \) for some \( M \in \text{ind}_A \) (these modules are called \( A/G \)-modules of the first kind). The remaining indecomposable modules (called of the second kind) form the category \( \text{ind}_2 A/G \).

If \( M \in \text{mod}^H \) is such that \( \text{supp} M \) is not finite but \( \text{supp} M/H \) is finite, then \( M \) is called a weakly \( G \)-periodic module.

**Lemma** [5]. Let \( F : A \to A/G \) be a Galois covering such that \( G \) acts freely on \( \text{ind}_A / \cong \). For \( X \in \text{ind}_{A/G} \) the following conditions hold:

1. \( X \in \text{ind}_1 A/G \) if and only if \( F.X \cong \bigoplus_{i \in I} Z_i \), where all \( Z_i \in \text{mod}_A \).

2. \( X \in \text{ind}_2 A/G \) if and only if \( F.X \cong \bigoplus_{i \in I} Y_i \), where each \( Y_i \) is weakly \( G \)-periodic.

**Proof.** (1): If \( X \cong F_\lambda M \) for some \( M \in \text{mod}_A \), then \( F.X \cong \bigoplus_{g \in G} M^g \). Assume now that \( Z \in \text{ind}_A \) is a direct summand of \( F.X \). Let \( j \in \text{Hom}_A(Z,F.X) \) and \( p \in \text{Hom}_A(F.X,Z) \) be such that \( pj = 1_Z \). We get morphisms \( j^g : Z^g \to (F.X)^g \cong F.X \) and \( p^g : (G.X)^g \to Z^g \) which yield a pair of maps \( j' : \bigoplus_{g \in G} Z^g \to F.X \) and \( p' : F.X \to \bigoplus_{g \in G} Z^g \). Since \( Z^g \not\cong Z \) for \( g \neq h \), then the endomorphism \( p' \circ j' \) of \( \bigoplus_{g \in G} Z^g \) is invertible.

Since \( F \) induces an equivalence of categories \( \text{MOD}_{A/G} \cong \text{MOD}_{A}^G \), then \( F_\lambda Z \) is a direct summand of \( X \). Thus \( X \cong F_\lambda Z \).

(2): Let \( X \in \text{mod}_2 A/G \). By the proof of (1), \( F.X = \bigoplus_{i \in I} Y_i \) where \( Y_i \in \text{Ind}_A \setminus \text{ind}_A \). Then we have to show that an indecomposable direct summand \( Y \) of \( F.X \) is weakly \( G \)-periodic. Indeed, let \( \mathcal{U} \) be a set of representatives of the cosets of \( G \) with respect to \( G_Y \). Then \( \bigoplus_{g \in \mathcal{U}} Y^g \) is a direct summand of \( F.X \). Since \( \text{supp} Y \subset \text{supp} X \) is contained in a finite number of \( G \)-orbits, then \( (\text{supp} Y)/G_Y \) is finite. The converse follows from (1).

5.4. A category \( A = kQ/I \) is called locally support finite if for each \( x \in Q_0 \), the full subcategory \( A_x \) of \( A \), consisting of the vertices of all \( \text{supp} M \) with \( M \in \text{ind}_A \) and \( M(x) \neq 0 \), is finite.
Proposition. Let \( F: A \to A/G \) be a Galois covering. Then

a) If \( G \) acts freely on \( \text{Ind}_A / \cong \), then \( F_\lambda: \text{mod}_A \to \text{mod}_{A/G} \) induces a bijection \( F_\lambda: (\text{ind}_A / \cong)/G \to (\text{ind}_{A/G} / \cong) \).

b) If \( A \) is locally support finite, then \( \text{Ind}_A = \text{ind}_A \). In particular if \( G \) acts freely on \( \text{ind}_A / \cong \), then \( F_\lambda \) induces a bijection between \( (\text{ind}_A / \cong)/G \) and \( (\text{ind}_{A/G} / \cong) \). Moreover, in this case \( A/G \) is tame (resp. domestic, polynomial growth) if and only if so is \( A \). For the Auslander-Reiten quivers we have \( \Gamma_{A/G} = \Gamma_A/G \).

Proof. (a): Follows from (4.2) since there are no weakly \( G \)-periodic indecomposable \( A \)-modules.

(b): Let \( Y \in \text{Ind}_A \) and consider \( x \in Q_0 \) with \( Y(x) \neq 0 \). Let \( \hat{A}_x \) be the (finite) full subcategory of \( A \) consisting in the objects \( y \in Q_0 \) such that \( A(y, z) \neq 0 \) or \( A(z, y) \neq 0 \) for some \( z \in A_x \). Let \( Z \) be an indecomposable direct summand of the restriction \( Y|\hat{A}_x \) such that \( Z(x) \neq 0 \). Hence \( \text{supp} Z \subset A_x \) and it is easy to check that \( Z \) is a direct summand of \( Y \) in \( \text{mod}_A \).

Thus \( Y = Z \in \text{ind}_A \). The last claim is easy to prove. \( \square \)

Example: The category \( A_{\alpha, \beta} \) given by the quiver with relations

\[
\begin{align*}
\bullet \xrightarrow{p_1} & \bullet \xrightarrow{a_1} \bullet = & \sigma_{i+1} \sigma_i &= \alpha \nu_{i+1} \gamma_i \\
\bullet \xrightarrow{\gamma_1} & \bullet \xrightarrow{p_0} \bullet = & \nu_{i+1} \rho_i &= \beta \sigma_{i+1} \nu_i \\
\bullet \xrightarrow{\gamma_1} & \bullet \xrightarrow{a_0} \bullet = & \gamma_{i+1} \nu_i &= \rho_{i+1} \rho_i \\
\bullet \xrightarrow{\rho_{i+1}} & \bullet \xrightarrow{\gamma_i} \bullet = & \rho_{i+1} \gamma_i &= \gamma_{i+1} \sigma_i \\
\end{align*}
\]

with \( (\alpha, \beta) \neq (1, 1) \), is locally support finite (why?). Moreover the group \( Z \) generated by the action \( (a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1}) \) acts freely on \( A_{\alpha, \beta} \) and on \( \text{ind}_{A_{\alpha, \beta}} / \cong \). Hence the Galois covering \( F: A_{\alpha, \beta} \to \tilde{A}_{\alpha, \beta} /Z \) yields a bijection \( F_\lambda: (\text{ind}_{A_{\alpha, \beta}} / \cong)/Z \to (\text{ind}_{\tilde{A}_{\alpha, \beta}} / \cong) \). The algebra \( \tilde{A}_{\alpha, \beta} \) is given by the quiver with relations

\[
\begin{align*}
\rho & \bigcirc \ a \bigcirc \ b \bigcirc \sigma \\
\sigma^2 &= \alpha \gamma \nu \\
\rho \nu &= \beta \sigma \nu \\
\nu \gamma &= \rho^2 \\
\gamma \rho &= \sigma \gamma
\end{align*}
\]

Since \( A_{\alpha, \beta} \) is tame (resp. polynomial growth for \( \alpha \beta \neq 1 \)), so is \( \tilde{A}_{\alpha, \beta} \).

5.5. Given a natural number \( p \geq 0 \), a group \( G \) is said to be \( p \)-residually finite if for each finite subset \( S \subset G \setminus \{1\} \) there is a normal subgroup of finite index \( H \triangleleft G \) such that \( H \cap S = \emptyset \) and \( p \nmid (G/H) \). For example, free groups are \( p \)-residually finite.

Theorem [12]. Let \( F: (\tilde{Q}, \tilde{I}) \to (Q, I) \) be a Galois covering given by the action of a \( p \)-residually finite group \( G \), where \( p = \text{char} k \). Assume that \( \tilde{A} = k\tilde{Q}/\tilde{I} \) is locally support finite. Then \( \tilde{A} \) is tame if and only if \( A = kQ/I \) is tame.
Proof. Without loss of generality, we assume that \( Q \) is finite. We denote also by \( F: \hat{A} \to A \) the induced covering functor. We divide the proof in several steps.

1. Let \( F_\lambda: \text{mod}_\hat{A} \to \text{mod}_A \) be the push-down functor. Consider a sequence \((\Gamma_n)_{n \in \mathbb{N}}\) of finite full subcategories of \( \hat{A} \) such that \( \bigcup_n \Gamma_n = \hat{A} \) and if \( \hat{A}(x, \Gamma_n) \neq 0 \) or \( \hat{A}(\Gamma_n, x) \neq 0 \), then \( x \in \Gamma_{n+1} \). The restriction functor \( \varepsilon^n: \text{mod}_\hat{A} \to \text{mod}_{\Gamma_n} \) has a left adjoint \( \varepsilon^n_\lambda: \text{mod}_{\Gamma_n} \to \text{mod}_\hat{A} \) such that \( \varepsilon^n \circ \varepsilon^n_\lambda = \text{id}_{\text{mod}_{\Gamma_n}} \). Therefore, the functor

\[
F_n = F_\lambda \varepsilon^n_\lambda: \text{mod}_{\Gamma_n} \to \text{mod}_A
\]

is right exact, \( n \in \mathbb{N} \).

2. Let \( Y \in \text{ind}_A \), we shall prove that \( Y \) is a direct summand of \( F_\lambda F.Y \). Indeed, since \( \hat{A} \) is locally support-finite, by (5.3), the pull-up \( F.Y \) decomposes as \( F.Y \simeq \bigoplus_{i \in L} X_i \), where \( X_i \in \text{ind}_\hat{A} \). Thus \( F_\lambda F.Y \simeq \bigoplus_{i \in L} F_\lambda X_i \). We proceed in two steps.

2.1 There exists a normal subgroup \( H \) of \( G \) with finite index not divisible by \( p \), such that the Galois covering \( \bar{F}: \hat{A} \to \hat{A}/H \) induces a bijection \( \bar{F}_\lambda: (\text{ind}_\hat{A}/H) \to (\text{ind}_{\hat{A}/H})/\simeq \).

Since \( A \) is finite, the set of vertices of \( \bar{Q} \) is a disjoint union of a finite number of \( G \)-orbits \( Gx_i, i = 1, \ldots, s \). With the notation introduced above, we consider the full subcategory \( R_i = A_x, \) of \( \hat{A} \). Let \( S \) be the set of all elements \( g \in G/\{1\} \) such that \( g(\hat{R}_i) \cap \hat{R}_i \neq \emptyset \), for some \( i \in \{1, \ldots, s\} \). Since \( G \) acts freely on \( \bar{Q} \), then \( S \) is finite and there is a normal subgroup \( H \) of \( G \) such that \( G/H \) is finite of order not divisible by \( p \) and \( H \cap S = \emptyset \). Hence \( h(\hat{R}_i) \cap \hat{R}_i = \emptyset \), \( i = 1, \ldots, s \). By (5.1), the induced Galois covering \( \bar{F}: \hat{A} \to \hat{A}/H \) yields an injection \( \bar{F}_\lambda(\text{ind}_{\hat{A}/H})/\simeq \to (\text{ind}_{\hat{A}/H})/\simeq \). We show that this map is also surjective. Let \( M \in \text{ind}_{\hat{A}/H} \) and \( b = \bar{F}(x_j) \) a vertex of \( \bar{Q}/H \) such that \( M(b) \neq 0 \). The restriction \( F' = \bar{F}_j: \hat{A}/H \to F(\hat{R}_j) \) is a bijection and there is an indecomposable \( \hat{R}_j \)-module \( Y = F'N \) with \( Y(x_j) \neq 0 \) and such that \( N \) is an indecomposable direct summand of the restriction \( M' \) of \( M \) to \( F(\hat{R}_j) \). We conclude that \( Y \in \text{ind}_{\hat{A}/H} \) and \( N \in \text{ind}_{\hat{A}/H} \), showing that \( M = N = F_\lambda Y \).

2.2 For the proof of (2), consider a normal subgroup \( H \) of \( G \) as in (2.1) and a factorization of Galois coverings

\[
\begin{align*}
\hat{A} & \xrightarrow{F} A = \hat{A}/G \\
& \downarrow F \\
\hat{A} & \xrightarrow{F'} \hat{A}/H
\end{align*}
\]

Then \( F_\lambda = F'_\lambda \bar{F}_\lambda \) and \( F. = \bar{F}.F' \). For an indecomposable \( Y \in \text{ind}_A \), we get

\[
F_\lambda F.Y = F'_\lambda(\bar{F}_\lambda F.)F'Y
\]

Consider the indecomposable decomposition \( F'.Y = \bigoplus_{i=1}^t Y_i \) with \( \bar{F}_\lambda F.Y_i \simeq Y_i \) for \( i = 1, \ldots, t \). Then \( F_\lambda F.Y \simeq F'_\lambda F'.Y \) and the result follows from (4.2). In particular, \( Y \) is a direct summand of \( F_\lambda X_j \) for some \( j \in L \).
(3) There is some $n \in \mathbb{N}$ and a function $f: \mathbb{N} \to \mathbb{N}$ such that 

$$F_n: \text{mod}_{\Gamma_n} \to \text{mod}_A$$

has the property that for every $Y \in \text{ind}_A$ with $\dim_k Y = d$, there exists some $X \in \text{ind}_{\Gamma_n} \text{ with } \dim_k X \leq f(d)$ and such that $Y$ is a direct summand of $F.X$.

Indeed, the set of vertices $\tilde{Q}_0$ is the disjoint union $\bigcup_{i=1}^{s} Gx_i$, then there is some $n \in \mathbb{N}$ such that $\Gamma_n$ contains all the categories $\tilde{R}_i$ for $R_i = \tilde{A}_x_i$, $i = 1, \ldots, s$. By (2), $Y$ is a direct summand of $F.X$ for some $Z \in \text{ind}_{\Gamma_n}$ and $X = \varepsilon^n Z^g \in \text{ind}_{\Gamma_n}$ satisfies that $Y$ is a direct summand of $F.X$. Clearly, $f(d) = |\Gamma_n|d$ (where $|\Gamma_n|$ denotes the number of objects of $\Gamma_n$) satisfies the desired property.

(4) Assume that $\tilde{A}$ is tame, we show that $A$ is tame. Indeed, let $n \in \mathbb{N}$ and $f: \mathbb{N} \to \mathbb{N}$ be as in (3). Since $F_n$ is right exact (1), there is a $A - F_n$-bimodule $N$ such that $F_n = N \otimes_{\Gamma_n} (-)$. Let $z \in \mathbb{N}Q_0$.

Since $\Gamma_n$ is tame, there is a family $M_1, \ldots, M_t$ of $\Gamma_n - k[T]$-bimodules which are finitely generated free as $k[T]$-right modules and such that any indecomposable $\Gamma_n$-module $X$ with $\dim_k X \leq f(d)$ is isomorphic to $M_i \otimes_{k[T]} S$ for some $1 \leq i \leq t$ and $S$ a simple $k[T]$-module. By (3), for every $Y \in \text{ind}_A$ with $\dim_k Y = d$, there exists some $1 \leq i \leq t$ and $S$ a simple $k[T]$-module such that $Y$ is a direct summand of $(N \otimes_{\Gamma_n} M_i) \otimes_{k[T]} S$. It is easy to see that this (apparently) weaker property is equivalent to the tameness of $A$.

(5) Assume now that $A$ is tame. It is enough to show that each $\Gamma_n$ is tame. Consider the right exact functors 

$$H_n = \varepsilon^n F.: \text{mod}_A \to \text{mod}_{\Gamma_n}.$$ 

Let $Y \in \text{ind}_{\Gamma_n}$. Thus $X = F_\lambda \varepsilon^n Y \in \text{mod}_A$ and

$$H_n X = \varepsilon^n \left( \bigoplus_{g \in G} (\varepsilon^n Y)^g \right) \cong \bigoplus_{g \in S} \varepsilon^n (\varepsilon^n Y)^g = Y \bigoplus_{g \in S \setminus \{1\}} \varepsilon^n (\varepsilon^n Y)^g,$$

where $S$ is the finite set of $g \in G$ such that $\text{supp} (\varepsilon^n Y)^g \cap \Gamma_n \neq \emptyset$. As in (3), (4) we get that $\Gamma_n$ is tame.

\section*{5.6.} We consider several examples:

(a) Let $F: \tilde{A} \to A$ be a Galois covering induced from a Galois covering of quivers with relations and defined by the action of a group $G$. Assume that $A$ is locally support-finite.

- If $G$ is a finite group such that $p = \text{char} k$ does not divide the order of $G$, then (5.5) applies and $\tilde{A}$ is tame if and only if so is $A$.
- If $G$ is a free group, then both (5.2) and (5.7) apply.

(b) Consider example (5.4). For $(\alpha, \beta) \neq (1, 1)$, the category $A_{\alpha, \beta}$ is tame and locally support finite. Hence the algebra $\tilde{A}_{\alpha, \beta}$ is tame.
(c) [8] Consider the Galois covering

\[ F: A = A^{(2)}_{1,1} \to \tilde{A} = A^{(2)}_{1,1}/\mathbb{Z}_2 \]

given in (2.2.c) and assume that \( \text{char } k = 2 \). We know that \( A \) is tame. We show that \( \tilde{A} \) is a wild algebra.

Set \( x_0 = \alpha_0 + \beta_0, \ y_0 = \beta_0, \ x_1 = \alpha_1 + \beta_1, \ y_1 = \beta_1. \) Then \( \tilde{A} \) is isomorphic to the algebra \( A' \) given by the quiver with relations.

\[
A': \quad \begin{array}{ccc}
& x_0 & \rightarrow & y_0 \\
\downarrow \quad \downarrow & z_1 & \rightarrow & y_1 \\
& x_1 & \rightarrow & y_1 \end{array}
\]

\[
\begin{cases}
x_1x_0 = 0 \\
y_1x_0 = x_1y_0
\end{cases}
\]

The universal covering \( \tilde{A}' \) constructed as in (2.3) admits a full convex subcategory \( \Delta \) as follows

\[
\Delta:
\]

Since \( \Delta \) is wild (observe that the Tits form \( q_\Delta \) takes value \(-1\) in the indicated vector), then \( \tilde{A}' \) is wild. Then \( A \) is also wild.

(d) The algebra \( \tilde{A} \) of example (c) provides another example of a wild algebra whose Tits form \( q_{\tilde{A}} \) is weakly non-negative. Indeed,

\[ q_{\tilde{A}}(a,b,c) = (a - b + c)^2. \]

5.7. We consider briefly the situation of coverings \( F: A \to A/G \) where \( A \) is not necessarily locally support-finite.

Let \( F: A \to A/G \) be a Galois covering. A line in \( A \) is a full convex subcategory isomorphic to the path category of a linear quiver (of type \( A_n, A_\infty \) or \( A_\infty^\infty \)). A line \( L \) is \( G \)-periodic if its stabilizer \( GL = \{ g \in G: gL = L \} \) is non-trivial (then \( L \) is of type \( A_\infty^\infty \)).

Let \( L \) be a \( G \)-periodic line in \( A \), we construct an indecomposable weakly \( G \)-periodic \( A \)-module \( B_L \) by setting \( B_L(x) = k \) for \( x \in L \) and \( B_L(x) = 0 \) for \( x \not\in L \) and \( B_L(\alpha) = \text{id} \) for any arrow \( \alpha \) in \( L \). Then \( G_{B_L} \simeq G_L \) is isomorph to \( \mathbb{Z} \). Let \( W_x \) be a set of representatives of the \( G_{B_L} \)-orbits in \( Gx \), for any object \( x \) in \( A \). Then \( F_{\lambda}B_L \) is a \( A/G - k[T, T^{-1}] \)-bimodule such that for each \( x \in A \), \( F_{\lambda}B_L(x) \) is a free \( k[T, T^{-1}] \)-module of rank \( \sum_{y \in W_x} \dim_k B_L(y) \). We consider the functor

\[ \phi^L = F_{\lambda}B_L \otimes_{k[T, T^{-1}]} (-): \text{mod}_{k[T, T^{-1}]} \to \text{mod}_{A/G}. \]
Let $\mathcal{L}$ be the set of all lines in $A$ and $\mathcal{L}_0$ be the set of representatives of the $G$-orbits in $\mathcal{L}$.

**Theorem** [5]. Let $F: A \to A/G$ be a Galois covering such that $G$ acts freely on $(\text{ind}_A)/\simeq$. Assume that for any weakly $G$-periodic $A$-module $X$, $\text{supp} X$ is a line. Then

a) Every module in $\text{ind}_2 A/G$ is of the form $\phi^L(V)$ for some $L \in \mathcal{L}_0$ and some indecomposable $k[T,T^{-1}]$-module $V$.

b) $\Gamma_{A/G} = \Gamma_A/G \vee \bigvee_{L \in \mathcal{L}_0} \Gamma_{k[T,T^{-1}]}$, where $\Gamma_{k[T,T^{-1}]}$ is the translation quiver of finite dimensional indecomposable $k[T,T^{-1}]$-modules, consisting of a $k^*$-family of stable tubes of rank one.

c) $A$ is tame if and only if so is $A/G$.

\[\square\]

**REFERENCES**


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