THE HILBERT TRANSFORM AND SCATTERING

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To the memory of Mischa Cotlar, my teacher and my friend

ABSTRACT. Through the prism of abstract scattering, and the invariant forms acting in them, we discuss the Hilbert transform in weighted $L^p$ spaces in one and several dimensions.

1. Introduction

It all started with the study of the Hilbert transform in terms of scattering...

In the late seventies, Cotlar and I began a systematic study of algebraic scattering systems, and the invariant forms acting on them.

In the late eighties we started working in multidimensional scattering—although many did not consider such approach as relevant.

In the late nineties our outlook was finally vindicated. Multidimensional abstract scattering systems appeared as counterparts of the input-output conservative linear systems.

2. The Hilbert Transform

The Hilbert transform operator

$H : f \mapsto Hf = f \ast k$

is given by convolution with the singular kernel

$k(x) = p.v. \frac{1}{x}$

for $x \in \mathbb{R}$. (Analogues in $\mathbb{T}$ and $\mathbb{R}^n$, $n > 1$.)

The basic result of Marcel Riesz (1927) is

$H$ is bounded on $L^p$, $1 < p < \infty$.

Similar boundedness properties are valid in the “weighted” cases, both for $H$ and for the iterated $H = H_1H_2...H_n$,

$H : L^p(\mu) \to L^p(\nu)$, $1 < p < \infty$

where $\mu, \nu$ are general measures.

3. **The scattering property of the analytic projector**

Given $f \in L^2$, we can decompose $f$ as $f_1 + f_2$, where $f_1$ is analytic and $f_2$ is antianalytic.

Under this decomposition the Hilbert transform can be written as

$$Hf = -if_1 + if_2.$$

The **analytic projector** $P$, associated with the Hilbert transform operator $H$, is defined as

$$Pf = P(f_1 + f_2) = f_1, \quad P = \frac{1}{2}(I + iH).$$

The crucial observation is that $P$ supports the **shift operator** $S : f(x) \mapsto e^{ix}f(x)$.

Then, the **range** of $P$ is the set $W_1$ of analytic functions, and its **kernel** is the set $W_2$ of antianalytic functions:

$$SW_1 \subset W_1 \quad \text{and} \quad S^{-1}W_2 \subset W_2.$$  

The “scattering property” of the 1-dimensional Hilbert transform provides the framework for a theory of invariant forms in scattering systems, leading to two-weight $L^2$-boundedness results for $H$.

The scattering properties are also essential to providing the two-weight $L^2$-boundedness of the **product** Hilbert transform in product spaces, where the analytic projectors supporting the $n$-dimensional shifts are at the basis of the lifting theorems in abstract scattering structures.

Notice that this fact, valid for the product Hilbert transforms, is not valid for the $n$-dimensional Calderón–Zygmund singular integrals, which do not share the scattering property.

4. **$H$ is bounded on $L^2$**

In fact, $H$ is an **isometry** on $L^2$, since

$$\|Hf\|_2 = \|f\|_2$$

and this follows easily from the Plancherel Theorem for the Fourier transform.

The result can also be obtained through the Cotlar Lemma on Almost Orthogonality, which extends the Hilbert transforms into ergodic systems.

These are two different ways to deal with the boundedness of $H$ in $L^2$. The same happens for $p \neq 2$. 

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5. Two ways for $H : L^1$ to $L^p$, $1 < p < \infty$

Start checking that $H$ is weakly bounded in $L^1$, and apply the Marcinkiewicz Interpolation Theorem between $p = 1$ and $p = 2$, and then, by duality, pass from $p = 2$ to $p < \infty$.

The “Magic Identity” for $H$, given by

$$(Hf)^2 = f^2 + 2H(f.Hf) \quad (*)$$

is valid for all functions $f$, smooth and with compact support.

Using extrapolation, since for $f \in L^2$, $(*)$ implies $Hf \in L^2$, then $f \in L^4$ implies $Hf \in L^4$, and $f \in L^{2^k}$ implies $Hf \in L^{2^k}$, $\forall k \geq 1$.

The boundedness of $H$ in $L^{p'}$, $1/p + 1/p' = 1$, $p = 2^k$, follows by duality, and interpolation gives the boundedness of $H$ in $L^p$, $1 < p < \infty$.

By polarization, the Magic Identity for an operator $T$ becomes

$$T(f.Tg + T.f.g) = T f.T g - f.g.$$  

The Magic Identity, and similar ones have been used extensively in harmonic analysis, in particular by Coifman and Meyer. Cotlar and Sadosky, and Rubio de Francia, used the Identity in dealing with the weighted Hilbert transform in Banach lattices.

Gian-Carlo Rota used three different “magic identities” in his work in combinatorics, and his school encompassed all particular cases in a general inequality.

Gohberg and Krein showed that the polarized Magic Inequality holds in the space $S^2$, and deduced the theorem of Krein and Macaev in a way similar to the passage from $L^2$ to $L^p$ described before.

The “magic identities” hold in a variety of non-commutative situations, starting with the non-commutative Hilbert transforms in von Neumann algebras, and that theory has been developed in the last years.

6. $H$ is bounded in $L^2(\omega)$, $0 \leq \omega \in L^1$


$$\omega = e^{u + H v},$$

where $u, v$ are real-valued bounded functions, such that $\|v\|_\infty < \pi/2$

is equivalent to

\[ \log \omega = u + Hv \in BMO \]

(with a special BMO norm).

6.2. **Hunt-Muckenhoupt-Wheeden theorem (1973).**

\[ \omega \in A_2 \]

is equivalent to

\[
(\frac{1}{|I|} \int_I \omega)(\frac{1}{|I|} \int_I \frac{1}{\omega}) \leq C, \quad \forall I \text{ interval}
\]

6.3. **Remark.** Take note that, although both conditions are necessary and sufficient, the first one is good to produce such weights, while the second one is good at checking them.

7. **Definition of \( u \)-boundedness**

An operator \( T \) acting in a Banach lattice \( X, T : X \rightarrow L^0(\Omega) \) is \( u \)-bounded if

\[
\forall f \in X, \|f\| \leq 1, \exists g \in X, g \geq |f|, \|g\| + \|Tg\| \leq C.
\]

The \( u \)-boundedness of operators is considerably weaker than boundedness. For example,

\[ T \text{ is } u \text{-bounded on } L^\infty \text{ if and only if } T1 \in L^\infty \]

Now we can translate another equivalence for the Helson-Szegő theorem for \( p=2 \):

There exist \( w \sim \omega \), real-valued functions, such that

\[ |H w(x)| \leq C w(x) \text{ a.e.} \]

(Here \( w \sim \omega \) means \( c \omega(t) \leq w(t) \leq C \omega(t), \forall t \)

is equivalent to

\[ T_\omega : f \mapsto \omega^{-1} H(\omega f) \]

is \( u \)-bounded in \( L^\infty \).
8. $H$ is bounded in $L^p(\mathbb{T}; \omega)$, $1 < p < \infty$


\[ \omega \in A_p \]

is equivalent to

\[ \left( \frac{1}{|I|} \int_I \omega \right) \left( \frac{1}{|I|} \int_I \omega^{-1/(p-1)} \right)^{p-1} \leq C, \; \forall I \text{ interval} \]

8.2. Cotlar-Sadosky Theorem (1982).

\[ \omega \in A_p, \]

defined by

\[ T_\omega = \omega^{-2/p} H(\omega^{2/p} f), \quad p \geq 2, \]
\[ T_\omega = \omega^{2/p} H(\omega^{-2/p} f), \quad p < 2, \]

is $u$-bounded in $L^{p^*}$ where $1/p^* = |1 - 2/p|$.

9. $H$ is bounded in weighted $L^2(\mathbb{T}^2; \omega)$

The following are equivalent:

1. The double Hilbert transform $H = H_1 H_2$ is bounded in $L^2(\mathbb{T}^2; \omega)$

2. $\omega = e^{u_1 + H_1 v_1} = e^{u_2 + H_2 v_2}$, $u_1, u_2, v_1, v_2$, are real-valued bounded functions, $\|v_i\|_\infty < \pi/2$, $i = 1, 2$

3. $\log \omega \in bmo$ (with a special $bmo$ norm)

4. $\exists w_1, w_2, w_1 \sim \omega \sim w_2$, such that

\[ |H_1 w_1(x)| \leq C w_1(x), \; |H_2 w_2(x)| \leq C w_2(x) \text{ a.e.} \]

5.

\[ T_1 : f \mapsto \omega^{-1} H_1(\omega f) \]

and

\[ T_2 : f \mapsto \omega^{-1} H_2(\omega f) \]

are simultaneously $u$-bounded in $L^\infty(\mathbb{T}^2)$
6. \( \omega \in A^*_\infty \)

10. \( H \) BOUNDED IN WEIGHTED \( L^p(\mathbb{T}^2; \omega) \), \( 1 < p < \infty \)

The following are equivalent:

1. The double Hilbert transform \( H = H_1 H_2 \) is bounded in \( L^p(\mathbb{T}^2; \omega) \)

2. \( T_1 \) and \( T_2 \) are simultaneously \( u \)-bounded in \( L^{p^*} \), where for \( i = 1, 2 \),

\[
T_i : f \mapsto \omega^{-2/p} H_i(\omega^{2/p} f), \text{ if } 2 \leq p < \infty
\]

are simultaneously \( u \)-bounded in \( L^{p^*(\mathbb{T}^2)} \).

3. \( \omega \in A^*_\infty \)

11. THE LIFTING THEOREM FOR INVARIANT FORMS IN ALGEBRAIC SCATTERING STRUCTURES

Let \( V \) be a vector space, and \( \sigma \) be a linear isomorphism in \( V \).

The subspaces of \( V \), \( W_+ \), and \( W_- \), are linear subspaces satisfying

\[
\sigma W_+ \subset W_+, \quad \sigma^{-1} W_- \subset W_-
\]

Let \( B_1, B_2 : V \times V \to \mathbb{C} \), be positive \( \sigma \)-invariant forms, such that

\[
\forall B_0 : W_+ \times W_- \to \mathbb{C} \ni \forall (x, y) \in W_+ \times W_-
\]

\[
B_0(\sigma x, y) = B_0(x, \sigma^{-1} y),
\]

\[
|B_0(x, y)| \leq B_1(x, x)^{1/2} B_2(y, y)^{1/2}
\]

Then,

\[
\exists B' : V \times V \to \mathbb{C} \ni \forall (x, y) \in V \times V
\]

such that

\[
B'(\sigma x, \sigma y) = B'(x, y),
\]

or, equivalently,

\[
B'(\sigma x, y) = B'(x, \sigma^{-1} y).
\]

Furthermore,

\[
|B'(x, y)| \leq B_1(x, x)^{1/2} B_2(y, y)^{1/2}
\]

such that

\[
B'(x, y) = B_0(x, y), \forall (x, y) \in W_+ \times W_-.
\]
12. The Lifting Theorem for trigonometric polynomials

Assume now that $V = \mathcal{P}$ is the set of trigonometric polynomials on $T$, and $\mathcal{P}_1, \mathcal{P}_2$ are the sets of analytic and antianalytic polynomials in $\mathcal{P}$, where $\sigma = S$ is the shift operator.

The Herglotz-Bochner theorem then translates to

$B$ is positive and $S$-invariant in $\mathcal{P} \times \mathcal{P}$ if and only if it exists a measure $\mu \geq 0$, such that

\[ B(f, g) = \int f \overline{g} \, d\mu \quad \forall f, g \in \mathcal{P}. \]

Since $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, the domain of $B$ splits in four pieces $\mathcal{P}_i \times \mathcal{P}_j$ for $i, j = 1, 2$.

A weaker concept of $S$-invariance is

\[ B(Sf, Sg) = B(f, g), \]

for all $(f, g)$ in each quarter $\mathcal{P}_i \times \mathcal{P}_j$, $i, j = 1, 2$.

Then the Lifting theorem asserts that $B$ is positive in $\mathcal{P} \times \mathcal{P}$ and $S$-invariant in each quarter if and only if there exists $\mu = (\mu_{ij}) \geq 0$ such that, for all $f_1, f_2 \in \mathcal{P}_1 \times \mathcal{P}_\infty$,

\[ B(f_1 + f_2, g_1 + g_2) = \sum_{i,j=1,2} \int f_i \overline{g}_j \, d\mu_{ij}. \]

Here $(\mu_{ij}) \geq 0$ means that the (complex) measures satisfy $\mu_{11} \geq 0, \mu_{22} \geq 0, \mu_{21} = \overline{\mu_{12}}$ and

\[ |\mu_{12}(D)|^2 \leq \mu_{11}(D) \mu_{22}(D), \quad \forall D \subset T. \]

13. The Lifting Theorem in $\mathcal{P} \times \mathcal{P}$

Let $B$ be an $S$-invariant in $\mathcal{P} \times \mathcal{P}$.

Then

\[ B \geq 0 \iff \mu \geq 0, \]

but, when $B$ is $S$-invariant in each $\mathcal{P}_i \times \mathcal{P}_j$,

\[ B \geq 0 \iff \sum_{i,j} \int f_i \overline{f}_j \, d\mu_{ij} \geq 0 \]

only for $f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2$, which is far less than

\[ \mu \geq 0 \iff \sum_{i,j} \int f_i \overline{f}_j \, d\mu_{ij} \geq 0 \]

for $f_1, f_2 \in \mathcal{P}$.

Let $B_1 = B|\mathcal{P}_1 \times \mathcal{P}_1, B_2 = B|\mathcal{P}_2 \times \mathcal{P}_2, B_0 = B|\mathcal{P}_1 \times \mathcal{P}_2$. 
Then $B_1, B_2$ are positive, while $B_0$ is not positive but bounded,

$$|B_0(f_1, f_2)| \leq B_1(f_1, f_1)^{1/2} B_2(f_2, f_2)^{1/2}.$$ 

Since $\mu_{11} \geq 0, \mu_{22} \geq 0$,

$$|B_0(f_1, f_2)| \leq \|f_1\|_{L^2(\mu_{11})} \|f_2\|_{L^2(\mu_{22})}.$$ 

By the definition of $B_0$ it follows

$$B_0(f_1, f_2) = \int f_1 \bar{f}_2 d\mu_{12}$$

only for $f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2$.

And since

$$|\mu_{12}(D)|^2 \leq \mu_{11}(D) \mu_{22}(D), \quad \forall D \subset \mathbb{T},$$

defining

$$B'(f_1, f_2) := \int f_1 \bar{f}_2 d\mu_{12}, \quad \forall f_1, f_2 \in \mathcal{P}$$

it is $\|B'\| = \|B_0\|$.

13.1. $H$ is bounded in $L^2(\mathbb{T}; \nu, \mu)$. Let the operator $H$ be defined in $\mathcal{P}$ as

$$H(f_1 + f_2) = -if_1 + if_2.$$ 

The two-weight inequality for $H$ is

$$\int_{\mathbb{T}} |Hf|^2 d\mu \leq M^2 \int_{\mathbb{T}} |f|^2 d\nu$$

or, equivalently,

$$(\ast) = \sum_{i,j=1,2} \int_{\mathbb{T}} f_i \bar{f}_j d\rho_{ij} \geq 0, \quad f_1 \in \mathcal{P}_1, f_2 \in \mathcal{P}_2$$

where

$$\rho_{11} = \rho_{22} = M^2 \nu - \mu, \quad \rho_{12} = \rho_{21} = M^2 \nu + \mu.$$ 

Defining by $B(f, g) = B(f_1 + f_2, g_1 + g_2) = (\ast)$, $B$ is $S$-invariant in each quarter and is also non-negative.

By the Lifting theorem, there exists $(\mu_{ij}), i, j = 1, 2$, such that

$$\hat{\rho}_{ij}(n) = \hat{\mu}_{ij}(n), \forall n \in \mathbb{Z},$$

while $\hat{\rho}_{12}(n) = \hat{\mu}_{12}(n)$, is only for $n < 0$.

By the F. and M. Riesz theorem,

$$\mu_{11} = \mu_{22} = M^2 \nu - \mu, \quad \mu_{12} = \mu_{21} = M^2 \nu + \mu - h,$$

and $h \in H^1(\mathbb{T})$. 

Then, the necessary and sufficient condition for boundedness in $L^2(\mathbb{T}; \nu, \mu)$, with norm $M$, is that for all $0 \subset T$ Borel sets,

$$|(M^2 \nu + \mu)(D) - \int_D h \, dt| \leq (M^2 \nu - \mu)(D)$$

where $h \in H^1(\mathbb{T})$.

In particular, $\mu$ is an absolutely continuous measure, $d\mu = w \, dt$ for some $0 \leq w \in L^1$.

When $\mu = \nu = \omega \, dt$ we return to the previous case:

The Hilbert transform $H$ is a bounded operator in $L^2(\omega)$ with norm $M$ if and only if

$$|(M^2 + 1) \omega(t) - h(t)| \leq (M^2 - 1) \omega(t), \text{ a.e. } T,$$

for $h \in H^1(\mathbb{T})$, which is the source to all the equivalences we mentioned above.

References


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