REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 50, Número 1, 2009, Páginas 75–85

EXPONENTIAL FAMILIES OF MINIMALLY NON-COORDINATED GRAPHS

FRANCISCO SOULIGNAC* AND GABRIEL SUEIRO

Abstract. A graph G is *coordinated* if, for every induced subgraph H of G, the minimum number of colors that can be assigned to the cliques of H in such a way that no two cliques with non-empty intersection receive the same color is equal to the maximum number of cliques of H with a common vertex. In a previous work, coordinated graphs were characterized by minimal forbidden induced subgraphs within some classes of graphs. In this note, we present families of minimally non-coordinated graphs whose cardinality grows exponentially on the number of vertices and edges. Furthermore, we describe some ideas to generate similar families. Based on these results, it seems difficult to find a general characterization of coordinated graphs by minimal forbidden induced subgraphs.

1. INTRODUCTION

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $\overline{G}$ the complement of G. Given two graphs G and G' we say that G *contains* G' if G' is isomorphic to an induced subgraph of G. If $X \subseteq V(G)$, we denote by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$.

A *complete set* or just a *complete* of G is a subset of vertices pairwise adjacent. A *clique* is a complete set not properly contained in any other. We may also use the term clique to refer to the corresponding complete subgraph. Let X and Y be two sets of vertices of G. We say that X is *complete to* Y if every vertex in X is adjacent to every vertex in Y, and that X is *anticomplete to* Y if no vertex of X is adjacent to a vertex of Y. A complete of three vertices is called a *triangle*.

The *neighborhood* of a vertex v is the set $N(v)$ consisting of all the vertices which are adjacent to v. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A vertex of G is *simplicial* if $N(v)$ is a complete. Equivalently, a vertex is simplicial if it belongs to only one clique.

*Partially supported by UBACyT Grant X184, Argentina and CNPq under PROSUL project Proc. 490333/2004-4, Brazil.

Given a graph G, we will denote by $\mathcal{Q}(G)$ the set of cliques of G. Also, for every $v \in V(G)$, we will denote by $\mathcal{Q}(v)$ the set of cliques containing v. Finally, define $m(v) = |\mathcal{Q}(v)|$.

The *chromatic number* of a graph G is the smallest number of colors that can be assigned to the vertices of G in such a way that no two adjacent vertices receive the same color, and is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of G, the *clique number* of G, denoted by $\omega(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. Perfect graphs were defined by Berge in 1960 [1] and are interesting from an algorithmic point of view: while determining the chromatic number and the clique number of a graph are NP-hard problems, they are solvable in polynomial time for perfect graphs [8].

A *hole* is a chordless cycle of length at least 4. An *antihole* is the complement of a hole. A hole or antihole is said to be *odd* if it consists of an odd number of vertices.

Given a graph G, the *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. A *K-coloring* of a graph G is an assignment of colors to the cliques of G such that no two cliques with non-empty intersection receive the same color (equivalently, a K-coloring is a coloring of $K(G)$). A *Helly K-complete* of a graph G is a collection of cliques of G with common intersection. A *Helly K-clique* is a maximal Helly K-complete. The *K-chromatic number* and *Helly K-clique number* of G, denoted by $F(G)$ and $M(G)$, are the sizes of a minimum K-coloring and a maximum Helly K-clique of G, respectively. It is easy to see by definition that $F(G) = \chi(K(G))$ and that $M(G) = max_{v \in V(G)} m(v)$. Also, $F(G) \geq M(G)$ for any graph G. A graph G is *coordinated* if $F(H) = M(H)$ for every induced subgraph H of G. Coordinated graphs were defined and studied in [3]. There are three main open problems concerning this class of graphs:

(i) find all minimal forbidden induced subgraphs for the class of coordinated graphs,
(ii) determine the computational complexity of finding the parameters M and F for coordinated graphs and/or some of their subclasses, and
(iii) is there a polynomial time recognition algorithm for the class of coordinated graphs?

Recently, in [2] and [4], questions (i) and (ii) were answered partially. For question (iii), it is shown in [9] that the problem is NP-hard and it is NP-complete even when restricted to a subclass of graphs with $M = 3$. In this note, we answer a question related to these problems, which is: how many minimally non-coordinated graphs with n vertices and $M = 3$ are there? In particular, we show (algorithmic) operations for generating a family $\mathcal{G}_k$ of minimally non-coordinated graphs, of size 2^k, such that every graph of the family has $16k + 9$ vertices and $24k + 14$ edges, for every k. It is not difficult to see that the operations we give are not enough for generating every minimally non-coordinated graph with $M = 3$, so the question of how to generate every minimally non-coordinated graph is still open.

2. GENERATING NON-COORDINATED GRAPHS

It has been proved recently that perfect graphs can be characterized by two families of minimal forbidden induced subgraphs [7] and recognized in polynomial time [6].

Theorem 1 (Strong Perfect Graph Theorem [7])**.** *Let G be a graph. Then the following are equivalent:*

(i) *no induced subgraph of G is an odd hole or an odd antihole.*
(ii) *G is perfect.*

Coordinated graphs are a subclass of perfect graphs [3]. Moreover, $\overline{C_7}$, $\overline{C_8}$, $\overline{C_9}$, $\overline{2P_4}$ are minimally non-coordinated [3, 5]. Therefore, $\overline{C_k}$ is not coordinated for all $k \geq 7$.

In [2, 4] and manuscript [5], partial characterizations of coordinated graphs by minimal forbidden induced subgraphs were found. In these partial characterizations, the families of minimal forbidden induced subgraphs with k vertices have $O(1)$ size for every k. Another partial characterization which is not difficult to prove (see [9] for a sharper result) is the following.

Theorem 2. *Let G be a graph such that $M(G) \leq 2$. Then the following are equivalent:*

(i) *G is perfect.*
(ii) *G does not contain odd holes.*
(iii) *G is coordinated.*

Corollary 3. *Let G be a graph with $M(G) \leq 3$. Then G is coordinated if and only if G does not contain odd holes and $F(G) \leq 3$.*

The aim of this note is to show, for every k, a family of minimally non-coordinated graph of size 2^k such that every graph has $16k + 9$ vertices and $24k + 14$ edges. In order to define our families, we are going to use exchanger and preserver graphs which were defined in [9].

A graph G *exchanges colors* between two different vertices v_1, v_2, or simply G is an *exchanger* between v_1, v_2, if G satisfies the following conditions:

(i) No induced subgraph of G is an odd hole.
(ii) $F(G) = M(G) = 3$.
(iii) $m(v_1) = m(v_2) = 2$.
(iv) Every induced path between v_1, v_2 has odd length.
(v) In any 3-K-coloring of G the cliques of $\mathcal{Q}(v_1) \cup \mathcal{Q}(v_2)$ are colored with the three colors.

Vertices v_1 and v_2 are called *connectors*. Call *redundant* to every simplicial vertex $z \in V(G)$ such that $K(G \setminus \{z\}) = K(G)$. We say that an exchanger G is a *minimal exchanger* if and only if $G \setminus \{z\}$ does not satisfy condition (v) for every non-redundant vertex z (although this is not the standard way to define minimality, this minimality is useful for "joining" G with other graphs, because redundant vertices of G may be needed so that the cliques of G are also cliques of the joined graph). Please note that conditions (i) and (iv) are hereditary and that if G satisfies (i) and (ii) but $G \setminus \{z\}$ does not satisfy (ii) then, by Theorem 2, $G \setminus \{z\}$ does not satisfy (v).

A graph G *preserves colors* between a set of distinct vertices $\{v_1, \ldots, v_k\}$, or simply G is a *preserver* between $v_1, \ldots, v_k$, if G satisfies the following conditions:

- (i) No induced subgraph of G is an odd hole.
- (ii) $F(G) = M(G) \leq 3$.
- (iii) $m(v_i) = 1$ for every $i \in \{1, \ldots, k\}$.
- (iv) Every induced path between v_i, v_j has odd length for all $1 \leq i < j \leq k$.
- (v) In any 3-K-coloring of G the cliques of $\bigcup_{i=1}^{k} \mathcal{Q}(v_i)$ are colored with only one color.

Vertices $v_1, \ldots, v_k$ are called *connectors*. We say that a preserver G is a *minimal preserver* if and only if $G \setminus \{z\}$ does not satisfy condition (v) for every non-redundant vertex z. Please note that conditions (i), (ii) and (iv) are hereditary and that condition (v) is satisfied only if condition (iii) is also satisfied.

Let G_1, G_2 be two graphs ($V(G_1) \cap V(G_2)$ may be non-empty). The graph $G = G_1 \cup G_2$ has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

We say that graphs $G_1, G_2, \ldots, G_k$ are *compatible* when $\{\mathcal{Q}(G_1), \mathcal{Q}(G_2), \ldots, \mathcal{Q}(G_k)\}$ is a partition of $\mathcal{Q}(\bigcup_{i=1}^{k} G_i)$. We call them *minimally compatible* if they are compatible and $\bigcup_{i=1}^{k} G_i$ contains no redundant vertices.

Theorem 4. *Let GX be an exchanger between v, w and GP be a preserver between v, w such that GX, GP are compatible and $V(GX) \cap V(GP) = \{v, w\}$. Then $G = GX \cup GP$ is non-coordinated. Moreover, if both GX and GP are minimal and minimally compatible then G is minimally non-coordinated.*

Proof. Since $\{\mathcal{Q}(GP), \mathcal{Q}(GX)\}$ is a partition of $\mathcal{Q}(G)$ and $m_{GX}(v) + m_{GP}(v) = m_{GX}(w) + m_{GP}(w) = 3$ by definition, then it follows that $m_G(v) = m_G(w) = 3$, $m_G(u) = m_{GX}(u)$ for every $u \in V(GX) \setminus \{v, w\}$ and $m_G(u) = m_{GP}(u)$ for every $u \in V(GP) \setminus \{v, w\}$. Consequently, $M(G) = 3$.

Suppose, contrary to our claim, that G is coordinated. Then, there exists a 3-K-coloring c_G of G. Since the cliques of GX are also cliques of G, then the K-coloring c_{GX} obtained by restricting the domain of c_G to the cliques of GX is a 3-K-coloring of GX. Analogously, define c_{GP} which is a 3-K-coloring of GP. Since GX is an exchanger then there exist cliques Q_1, Q_2, Q_3 such that $c_{GX}(Q_i) = i$ for $i \in \{1, 2, 3\}$ where, w.l.o.g., $v \in Q_1 \cap Q_2$ and $w \in Q_3$. Also, since GP is a preserver, there exist cliques R_1, R_2 such that $c_{GP}(R_1) = c_{GP}(R_2)$ where $v \in R_1$ and $w \in R_2$. Therefore, $c(Q_i) = i$ for $i \in \{1, 2, 3\}$ and $1 \leq c(R_1) = c(R_2) \leq 3$ which is a contradiction because Q_1, Q_2, Q_3, R_1 and R_2 are pairwise different, $v \in Q_1 \cap Q_2 \cap R_1$ and $w \in Q_3 \cap R_2$. Consequently, G is a non-coordinated graph.

From now on, suppose that GP and GX are both minimal and minimally compatible. Let us see that G contains no odd hole. On the contrary, suppose G contains an odd hole C. Since neither GP nor GX contains odd holes and $V(GP) \setminus \{v, w\}$ is anticomplete to $V(GX) \setminus \{v, w\}$, then it follows that C can be partitioned into two paths P_1, P_2 from v to w with disjoint interior and such that P_1 is an induced path of GP and P_2 is an induced path of GX. But this is impossible, because every induced path between v and w has odd length in both GP and GX, by definition.

Let H be any proper induced subgraph of G; we have to prove that $F(H) = M(H)$. Since G contains no odd hole then, by Corollary 3, if $M(H) \leq 2$ it follows that H is coordinated. Thus, it suffices to prove that if $M(H) = 3$ then $F(H) = 3$ which is equivalent to prove that $F(G \setminus \{u\}) \leq 3$ for every vertex $u \in V(G)$ (because $F(H) \leq F(G)$ for every induced subgraph H of G). We divide the proof into three cases:

Case 1: $u = v$ ($u = w$ is analogous). If $G \setminus \{v\} = GX \setminus \{v\}$ then $F(G \setminus \{v\}) \leq 3$. Otherwise, let c_{GX} be a 3-K-coloring of $GX \setminus \{v\}$ where the cliques of $\mathcal{Q}(w)$ are colored using colors from the set $\{1, 2\}$, and let c_{GP} be a 3-K-coloring of $GP \setminus \{v\}$ where the clique to which w belongs has color 3. Since $G \setminus \{v\} \neq GX \setminus \{v\}$ and $GP \setminus \{v\}$ is connected it follows that the clique to which w belongs in GP has at least one more vertex, thus it is still a clique in $G \setminus \{v\}$. Therefore $c_{GP} \cup c_{GX}$ is a valid 3-K-coloring of $G \setminus \{v\}$.

Case 2: $u \in V(GP) \setminus \{v, w\}$. Suppose first that u is a redundant vertex of GP, that is, u is simplicial and $K(GP \setminus \{u\}) = K(GP)$. Since $K(G \setminus \{u\}) \neq K(G)$ (because GP and GX are minimally compatible) and u is simplicial in G, then it follows that $N(u)$ is a complete but not a clique of $G \setminus \{u\}$. Since $\mathcal{Q}(G)$ is the disjoint union of $\mathcal{Q}(GP)$ and $\mathcal{Q}(GX)$ and $N(u)$ is a clique of GP, it follows that $N(u) \subseteq V(GX)$ and $N(u)$ is complete in GX. Hence $N(u) \subseteq \{v, w\}$. Since $m_{GP}(v) = m_{GP}(w) = m_{GP}(u) = 1$ and GP is connected, it follows that GP is a triangle, consequently $G \setminus \{u\} = GX$ and $F(G \setminus \{u\}) \leq 3$. Now, suppose that u is not simplicial or $K(GP \setminus \{u\}) \neq K(GP)$. Then, $GP \setminus \{u\}$ has a 3-K-coloring c_{GP} where the clique containing v has color 1 and the clique containing w has color 2 (because GP is minimal). Let c_{GX} be a 3-K-coloring of GX where the cliques containing v have colors $1, 2$ and the cliques containing w have colors $1, 3$. Then $c_{GP} \cup c_{GX}$ is a valid 3-K-coloring of $G \setminus \{u\}$.

Case 3: $u \in V(GX) \setminus \{v, w\}$. By minimality, if u is not redundant then $GX \setminus \{u\}$ has a 3-K-coloring c_{GX} where the cliques of $\mathcal{Q}_{GX}(v) \cup \mathcal{Q}_{GX}(w)$ are colored with at most two colors, say $2, 3$. Let c_{GP} be a 3-K-coloring of GP where the cliques containing v and w all have color 1. Then $c_{GP} \cup c_{GX}$ is a 3-K-coloring of $G \setminus \{u\}$. If u is redundant then, as before, it follows that $N[u]$ is the clique $\{u, v, w\}$ in GX and v, w belong to a clique Q in GP. Let c_{GX} be a 3-K-coloring of GX where $\{u, v, w\}$ has color 1 and c_{GP} be a 3-K-coloring of GP where Q has color 1. Then $(c_{GX} \cup c_{GP}) \setminus \{\{u, v, w\} \mapsto 1\}$ is a 3-K-coloring of $G \setminus \{u\}$. □

3. THE EXPONENTIAL FAMILIES

In this section we define a recursive operator for constructing exponentially many non-isomorphic exchangers. Then we use these exchangers and one preserver to generate the exponential families, as in Section 2. More operators for constructing exchangers and preservers are shown in the next section.

Let $u \neq v$ be vertices of a graph G_1 and $w \neq z$ be vertices of G_2. The graph $G = Join(G_1, G_2, uw, vz)$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uw, vz\}$. In particular, when G_1 is an exchanger between u, v, G_2 is a preserver between w, z, and $G_1 \cap G_2 = \emptyset$, we denote $Join(G_1, G_2) = Join(G_2, G_1) = Join(G_1, G_2, uw, vz)$. Figure 2 shows examples of this operation, using graphs in Figure 1.

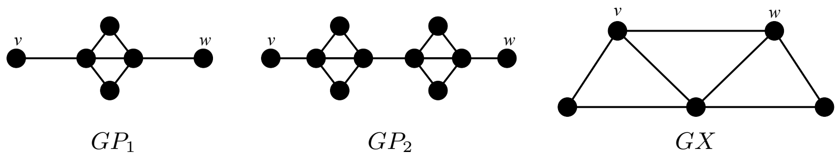

Figure 1. Preservers GP_1 and GP_2 and exchanger GX. In every graph the connectors are v and w.

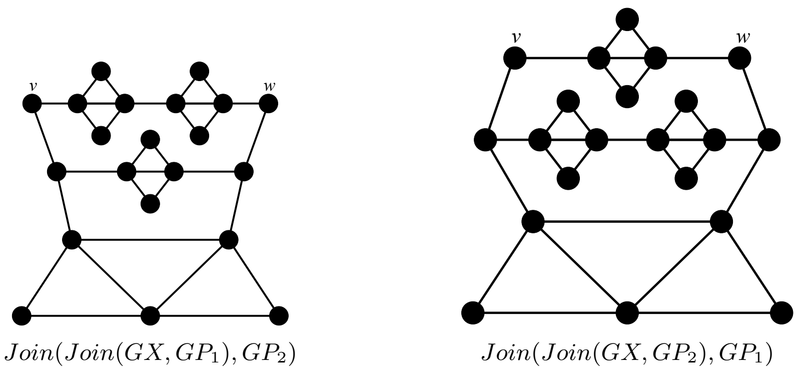

Figure 2. $Join(Join(GX, GP_1), GP_2)$ and $Join(Join(GX, GP_2), GP_1)$ are shown. Both graphs are minimal exchangers with connectors v and w.

Lemma 5. *Let GX be an exchanger between u, v and GP be a preserver between w, z where w is not adjacent to z. Suppose also that $V(GX) \cap V(GP) = \emptyset$. Then $G = Join(GX, GP)$ exchanges colors between w, z. Moreover, if both GX and GP are minimal then G is also minimal.*

Proof. Arguments similar to those in Theorem 4 show that G contains no odd hole and that $M(G) = 3$. Also, since $m_{GP}(w) = m_{GP}(z) = 1$ and $\{\{u, w\}, \{v, z\}\}$ are cliques, it follows that $m_G(w) = m_G(z) = 2$.

Let P be an induced path between w, z. If P is also a path of GP then it must have odd length. If P is not a path of GX, then there must exists a subpath P' which is a path of GX between u, v such that $P = wuP'vz$. Since P' has odd length it follows that P also has odd length.

Suppose for a moment that G is 3-K-colorable and let c_G be a 3-K-coloring of G. Let $Q_1(u)$, $Q_2(u)$ and $Q_1(v)$, $Q_2(v)$ be the cliques of u and v in GX, respectively, and $Q(w)$ and $Q(z)$ be the cliques of w and z in GP, respectively. Since $\mathcal{Q}(G) = \mathcal{Q}(GX) \cup \mathcal{Q}(GP) \cup \{\{u, w\}, \{v, z\}\}$, and this union is disjoint, then it follows that the coloring c_{GX} obtained by restricting the domain of c_G to the cliques of GX is a 3-K-coloring of GX. Analogously, define c_{GP}. Therefore, we may assume without loss of generality that $c_G(Q_1(u)) = 1$, $c_G(Q_2(u)) = 2$, $c_G(Q_1(v)) = 1$, $c_G(Q_1(v)) = 3$. Then $c_G(\{u, w\}) = 3$, $c_G(\{v, z\}) = 2$ and by definition of GP, $c_G(Q_1(w)) = c_G(Q_1(z)) = 1$. Hence in every 3-K-coloring of G the cliques of $\mathcal{Q}(w) \cup \mathcal{Q}(z)$ are colored with the three colors. By using the conditions derived for c_G, it is easy to see that G has at least one 3-K-coloring as supposed, thus $F(G) = M(G) = 3$.

From now on, suppose GP and GX are both minimal. Let $x \in V(GX)$. If $x = u$ then w belongs to only one clique of G. Since GP is a preserver, the clique of w has the same color as one of the cliques of z, therefore, $G \setminus \{x\}$ does not satisfy condition (v). The case $x = v$ is analogous. If $x \notin \{u, v\}$ is redundant in $V(GX)$, then it is also redundant in G. If $x \notin \{u, v\}$ is a non-redundant vertex of GX then, by minimality of GX, there exists a K-coloring c_{GX} which contradicts condition (v) for $GX \setminus \{x\}$. As in Theorem 4, it is easy to combine c_{GX} with a K-coloring of GP such that the coloring obtained is a K-coloring of G not satisfying condition (v). Similar arguments can be used to conclude the proof when $x \in V(GP)$. □

We are now ready to show the exponential families of minimally non-coordinated graphs. Let GP_1 and GP_2 be the preserver graphs and GX be the exchanger graph shown in Figure 1. Let $\mathcal{F}_k$ $(k \geq 0)$ denote the family of minimal exchangers defined by:

$$\begin{aligned} \mathcal{F}_0 &= \{GX\}. \\ \mathcal{F}_{k+1} &= \{Join(Join(X, GP_1), GP_2), Join(Join(X, GP_2), GP_1)\}_{X \in \mathcal{F}_k} \end{aligned}$$

Both graphs of $\mathcal{F}_1$ are shown in Figure 2. It is easy to see that graphs in $\mathcal{F}_k$ are pairwise non-isomorphic and that $|\mathcal{F}_k| = 2^k$. Also, by construction, $|V(G)| = |V(G')| = 16k + 5$ and $|E(G)| = |E(G')| = 24k + 7$ for every $G, G' \in \mathcal{F}_k$. Finally, every exchanger in $\mathcal{F}_k$ is a minimal exchanger by Lemma 5, and is minimally compatible with GP_1 (where connectors of GP_1 are identified with the connectors of the exchanger). Now, define $\mathcal{G}_k = \{X \cup GP_1\}_{X \in \mathcal{F}_k}$ for every $k \geq 0$. By Theorem 4 every graph in $\mathcal{G}_k$ is minimally non-coordinated, that is, $\mathcal{G}_k$ is one of the exponential families.

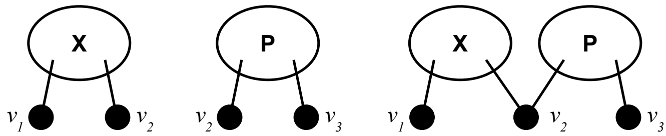

Figure 3. Examples of sketches. On the left there is a component for an exchanger X, in the middle there is a sketch component for a preserver P between v_2, v_3, and on the right there is a sketch for $X \cup P$.

4. OTHER OPERATIONS

In this section we show other recursive operations for constructing exchangers and preservers. The motivation is to show how exchangers and preservers can be "joined" in a recursive manner. The proofs that these operations generate exchangers and preservers are left to the reader, and can be done in a similar way as the one in the previous section. Instead, we are going to draw sketches showing how vertices of the input graphs are tied together. The components of these sketches are shown in Figure 3. To represent a preserver P between v_1, v_2 we are going to draw an oval labeled with **P** together with two points labeled v_1, v_2 each one joined to the oval by a line. The oval represents the graph $P \setminus \{v_1, v_2\}$, the points represent v_1 and v_2 and the line between v_1 (v_2) and the oval represents the clique of v_1 (v_2). In a similar manner, to represent an exchanger X between v_1, v_2 we are going to draw an oval labeled with **X** together with two points labeled v_1, v_2 each one joined to the oval by two lines. Again, the oval represents $X \setminus \{v_1, v_2\}$, the points represent vertices $\{v_1, v_2\}$ and the lines represent their cliques (in this case, one line of v_1 and one of v_2 may represent the same clique). Finally, a clique with n vertices is represented by an oval labeled with $\mathbf{K_n}$ and n points outside the oval, representing each of the n vertices. A line from one point to the oval means that the vertex belongs to the clique. Recall that a clique is a special kind of preserver, thus their vertices are also called connectors. Sometimes we also decorate the lines of the sketches with colors that represent a valid K-coloring.

Let G_1, G_2 be graphs which are preservers or exchangers between V_1 and V_2, respectively, where $G_1 \cap G_2 \subseteq V_1 \cap V_2$, and let S_1 and S_2 be sketches representing G_1 and G_2, respectively. We are going to represent the graph $G_1 \cup G_2$ with a sketch formed by S_1 and S_2, where for every vertex v of $G_1 \cap G_2$, the corresponding points of S_1 and S_2 are drawn as a single point. One of such sketches is shown in Figure 3.

The sketches in Figure 4 represent preservers between two vertices v_1, v_2, the sketches in Figure 5 represent exchangers between v_1, v_2 and the sketch in Figure 6 represents a preserver between v_1, v_2, v_3, v_4. If the set of graphs of a sketch S are minimally compatible and minimal (as preservers or exchangers) then the graph represented by S is also minimal.

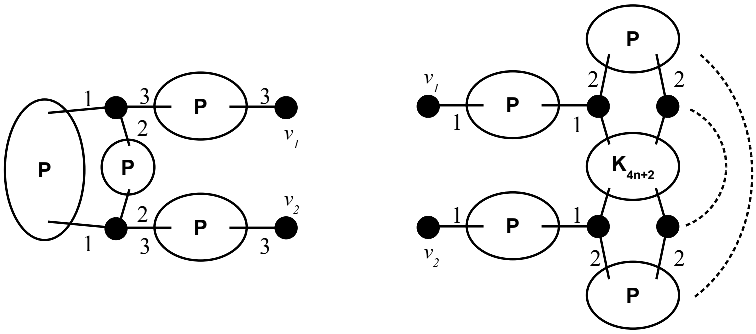

Figure 4. Two sketches of preservers between vertices v_1 and v_2.

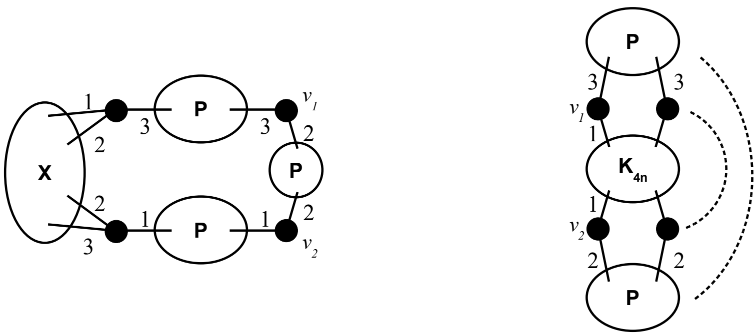

Figure 5. Two sketches of exchangers between v_1, v_2.

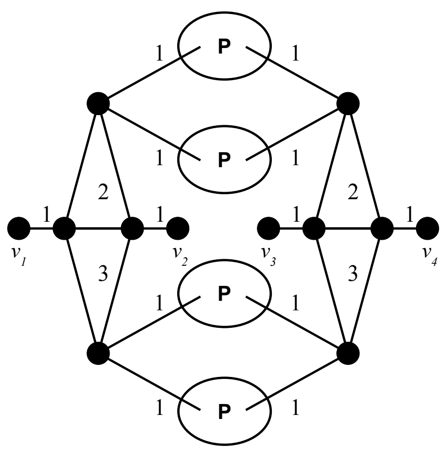

Figure 6. A sketch of a preserver between v_1, v_2, v_3, v_4.

5. CONCLUSIONS AND FURTHER REMARKS

In this note we have shown one exponential-size family of minimally non-coordinated graphs for every natural number, and several operations for building preservers and exchangers. It is not difficult to define other operations for constructing minimal preservers or exchangers in order to generate different families of minimally non-coordinated graphs. Also, adding edges to some minimally non-coordinated graph may result into another minimally non-coordinated graph. Moreover, it is not clear that preservers and exchangers are enough to define every minimally non-coordinated graph. Perhaps a set of basic graphs together with operations for generating minimally non-coordinated graphs can be defined in a more convenient way.

With all these observations it seems difficult to find a characterization of coordinated graphs by minimal forbidden induced subgraphs, even when we restrict our attention to the class of graphs with $M = 3$. This is in turn a complementary result to that one in [9], which states that the problem of determining whether a graph in a very restricted subclass of graphs with $M = 3$ is coordinated is NP-complete.

References

[1] Claude Berge. Les problèmes de coloration en théorie des graphes. *Publ. Inst. Statist. Univ. Paris*, 9:123–160, 1960.

[2] Flavia Bonomo, Maria Chudnovsky, and Guillermo Durán. Partial characterizations of clique-perfect graphs. I. Subclasses of claw-free graphs. *Discrete Appl. Math.*, 156(7):1058–1082, 2008.

[3] Flavia Bonomo, Guillermo Durán, and Marina Groshaus. Coordinated graphs and clique graphs of clique-Helly perfect graphs. *Util. Math.*, 72:175–191, 2007.

[4] Flavia Bonomo, Guillermo Durán, Francisco Soulignac, and Gabriel Sueiro. Partial characterizations of clique-perfect and coordinated graphs: superclasses of triangle-free graphs. *Discrete Appl. Math.*, 2009. In press.

[5] Flavia Bonomo, Guillermo Durán, Francisco Soulignac, and Gabriel Sueiro. Partial characterizations of coordinated graphs: line graphs and complements of forests. *Math. Methods Oper. Res.*, 69(2):251–270, 2009.

[6] Maria Chudnovsky, Gérard Cornuéjols, Xinming Liu, Paul Seymour, and Kristina Vušković. Recognizing Berge graphs. *Combinatorica*, 25(2):143–186, 2005.

[7] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. The strong perfect graph theorem. *Ann. Math. (2)*, 164(1):51–229, 2006.

[8] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[9] Francisco Soulignac and Gabriel Sueiro. Np-hardness of the recognition of coordinated graphs. *Ann. Oper. Res.*, 169(1):17–34, 2009.

Francisco Soulignac
Departamento de Computación
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad de Buenos Aires, Argentina
fsoulign@dc.uba.ar

Gabriel Sueiro
Departamento de Computación
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad de Buenos Aires, Argentina
gsueiro@dc.uba.ar

Recibido: 6 de agosto de 2007
Aceptado: 10 de julio de 2008