FORMULAS FOR THE EULER-MASCHERONI CONSTANT

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ABSTRACT. We give several integral representations for the Euler-Mascheroni constant using a combinatorial identity for \( \sum_{n=1}^{N} (n+x)(n+y) \). The derivation of this combinatorial identity is done in an elemental way.

Introduction. There exist many formulas for Euler-Mascheroni constant \( \gamma = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i} - \ln(n) \), see for example [7], [6]. Indeed the irrationality of \( \gamma \) would follow from criteria given in [3] (see also [5]).

The purpose of this note is to give integral representations for \( \gamma \) which seem to be new. As usual we write \((x)_{n} = (x + n - 1)(x + n - 2) \ldots x\).

Theorem. If \( f(x, n) := \frac{3x}{2n} + 2 + \frac{n + x + \frac{3}{2}}{2n-1} \), \( g(y, n) := \frac{2n}{n+y} - \frac{y}{2n} + \frac{n + \frac{3}{2}}{2n-1} \) then

i) \( \gamma = \sum_{n=1}^{\infty} (-1)^{n} \int_{0}^{1} \frac{(-x)_{n}(x)_{n}}{(x)_{2n+1}} f(x, n)dx \).

ii) \( \gamma = \sum_{n=1}^{\infty} (-1)^{n} \int_{0}^{1} \frac{(-y)_{n}(y)_{n}}{(2n)!y} g(y, n)dy \)

Remark 1. The formulae stated converge more rapidly than the usual definition. For example, notice that for \( 1 \leq n \)

\[
| \int_{0}^{1} \frac{(-x)_{n}(x)_{n}}{(x)_{2n+1}} f(x, n)dx | \leq \frac{6}{n^{2}(2n)}
\]

Indeed this follows from the fact that for \( 0 \leq x \leq 1 \) one has \( \frac{|(-x)_{n}(x)_{n}|}{(x)_{2n+1}} \leq \frac{1}{n^{2}(2n)} \) and \( f(x, n) \leq f(1, n) \leq 6 \) if \( 1 \leq n \).

Proof. We use the following formula: if \( f_{1}(n, x, y) := \frac{(2n+x)}{(n+y)} + \frac{1}{(2n-1)} (n + x + \frac{3}{2}) + \frac{(x-y)}{2n} \) and \( f_{2}(n, N, x, y) := \frac{1}{2(1-2n)} + \frac{N + x}{(1-2n)} - \frac{(x-y)}{2n} \), then

\[
\sum_{n=1}^{N} \frac{1}{(n+x)(n+y)} = \sum_{n=1}^{N} (-1)^{n-1} \frac{(1^{2} - (x - y)^{2}) \ldots ((n-1)^{2} - (x - y)^{2})}{(2n+x)(2n-1+x) \ldots (x+1)} f_{1}(n, x, y) + \ldots
\]

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where we set \((1^2 - (x - y)^2) \ldots ((n - 1)^2 - (x - y)^2) = 1\) if \(n = 1\).

Recall the well-known representation

\[
rac{\Gamma(x + 1)}{\Gamma(x + 1)} = -\gamma + x \sum_{n=1}^{\infty} \frac{1}{n(n+x)}.
\]

(2)

Notice that in (1), \(B_N(x, y) \to 0\) as \(N \to \infty\) if \(x\) and \(y\) are bounded. We prove this in a moment.

Now i) follows from integrating (2) from 0 to 1 and using (1) with \(y = 0\), letting \(N \to \infty\). The first formula of ii) is proved in the same way putting \(x = 0\) in (1).

Now we prove (1): set \(C_k := C_k(n, x, y) = \frac{b_1 \ldots b_k}{(n+k+1) \ldots (n+k+x) \ldots (n+y)}\); \(C_0 := C_0(n, x, y) = \frac{1}{(n+x)(n+y)}\) where \(b_k := b_k(x, y) = (x-y)^2-k^2\), and define \(b_1 \ldots b_{i-1} = 1\) if \(i = 1\).

Add from \(n = 1\) to \(N\) the trivial identity \(C_0 - C_{n-1} = (C_0 - C_1) + \ldots + (C_{n-2} - C_{n-1})\) to get

\[
\sum_{n=1}^{N} \frac{1}{(n+x)(n+y)} - \sum_{n=1}^{N} \frac{b_1 \ldots b_{n-1}}{(n+y)((2n-1+x) \ldots (x+1))} =
\]

\[
\sum_{n=1}^{N} \sum_{k=1}^{n-1} (C_{k-1} - C_k) =
\sum_{n=1}^{N} \sum_{k=1}^{n-1} \frac{b_1 \ldots b_{k-1}}{(n+k+x) \ldots (n+2x-y)}(n+2x-y) =
\]

\[
= \sum_{n=1}^{N} \sum_{k=1}^{n-1} \epsilon_{n,k}(x, y) - \epsilon_{n-1,k}(x, y) = \sum_{k=1}^{N} \epsilon_{N,k}(x, y) - \sum_{k=1}^{N} \epsilon_{k,k}(x, y),
\]

with \(\epsilon_{n,k}(x, y) := b_1 \ldots b_{k-1} \frac{1}{(n+k+1) \ldots (n+k+x)} \frac{(x-y)(-\frac{1}{2n})}{(n+k+x) \ldots (n-k+1+x)}\).

From the equality of the first expression in (3) and the last one, we obtain (1).

We now prove that if \(0 \leq x \leq 1, 0 \leq y \leq 1\) then \(B_N(x, y) \to 0\) as \(N \to \infty\) (the proof for \(x, y\) bounded is similar). Indeed in this range of \(x\) and \(y\) one has

\[
|B_N(x, y)| = O\left(\sum_{n=1}^{N} \frac{1}{(N+n)(2n)} \frac{N}{n^3}\right) = O\left(\sum_{1 \leq n \leq N/4} \frac{1}{(N+n)(2n)} \frac{N}{n^3}\right) = O(1/N)
\]

But \(O\left(\sum_{1 \leq n \leq N/4} \frac{1}{(2n)} \frac{N}{n^3}\right) = O\left(\sum_{1 \leq n \leq N/4} \frac{N}{n^3}\right) = O(1/N)\), where we have used \(N(N+1)/2 \leq \binom{N+n}{2n}\) for \(1 \leq n \leq N/4, N \geq 4\).

This finishes the proof of the theorem. ■

A corollary of formula (1) is the following
Corollary. Let $h(n, M, y) := \frac{1/2 + M + n}{2n - 1} - \frac{y}{2n} + \frac{M + 2n}{M + n + y}$. Set

$$D_M := (\sum_{n=1}^{M} \frac{1}{n}) - \ln(M+1) + \int_{0}^{1} y \sum_{1 \leq n \leq M/2} (-1)^{n-1} \frac{(1^2 - y^2) \ldots ((n-1)^2 - y^2)}{(2n)! \left(\frac{2n+M}{2n}\right)^2} h(n, M, y) dy$$

Then

$$\gamma - D_M = O(1/8^M)$$

Proof. From (1), letting $N \to \infty$ one gets

$$\sum_{n=1}^{\infty} \frac{1}{(n+x)(n+y)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1^2 - (x-y)^2) \ldots ((n-1)^2 - (x-y)^2)}{(2n+x)(2n-1+x) \ldots (x+1)} f_1(n, x, y).$$

Now substitute $y$ by $M + y$ and $x$ by $M$ to get for $0 \leq y \leq 1$

$$\sum_{n=M+1}^{\infty} \frac{1}{n(n+y)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(1^2 - y^2) \ldots ((n-1)^2 - y^2)}{(2n)! \left(\frac{2n+M}{2n}\right)^2} f_1(n, M, M + y) = \sum_{1 \leq n \leq M/2} + \sum_{M/2 < n < \infty} = \sum_{1 \leq n \leq M/2} + O(1/8^M)$$

Now the corollary follows from this last formula inserted in (recall formula (2))

$$\gamma = \int_{0}^{1} y \left(\sum_{n=1}^{M} \frac{1}{n(n+y)} + \sum_{n=M+1}^{\infty} \frac{1}{n(n+y)}\right) dy = \sum_{n=1}^{M} \frac{1}{n} - \ln(M+1) + \int_{0}^{1} y \sum_{n=M+1}^{\infty} \frac{1}{n(n+y)} dy,$$

observing that $h(n, M, y) = f_1(n, M, M + y)$. ■

Remark 2. The corollary stated seems to give clean approximation formulas. Indeed

$$D_1 = 1 - \ln 2, D_2 = \frac{283}{144} - \ln 4, D_3 = \frac{35}{16} - \ln 5,$$

$$D_4 = \frac{169553}{67200} - \ln 7, D_5 = \frac{192809}{72576} - \ln 8$$

Numerically we have checked that $D_M$ is always of the form $r - \ln n$, with $r$ a rational number and $2 \leq n \leq 2M$, $n \in \mathbb{Z}$.

Notice that (1) or derivates of (1) give formulae for Hurwitz-Riemann’s zeta function $\sum_{n=1}^{\infty} \frac{1}{(n+x)}$ for $s = 2, 3, 4, \ldots$.

We mention without proof that formula ii) of theorem 1 is equivalent to

$$\gamma = \int_{0}^{1} \{y \; _3F_2[1/2, 1-y, 1+y; 3/2, 3/2; -1/4] + \frac{y}{1+y} \; _3F_2[1-y, 1+y, 1+y; 3/2, 2+y; -1/4]\}$$
\[-\frac{y^2}{4} \quad _4F_3[1, 1, 1 - y, 1 + y; 3/2, 2, 2; -1/4] - \frac{1}{y} \quad \text{Sinh}^2(y, \text{ArcSinh}(1/2)) \, dy.\]

Here \( _pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} \) is the general hypergeometric function.

**References**


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