ATOMIC DECOMPOSITIONS AND OPERATORS ON HARDY SPACES

STEFANO MEDA, PETER SJÖGREN AND MARIA VALLARINO

Abstract. This paper is essentially the second author’s lecture at the CIMPA–UNESCO Argentina School 2008, Real Analysis and its Applications. It summarises large parts of the three authors’ paper [MSV]. Only one proof is given. In the setting of a Euclidean space, we consider operators defined and uniformly bounded on atoms of a Hardy space $H^p$. The question discussed is whether such an operator must be bounded on $H^p$. This leads to a study of the difference between countable and finite atomic decompositions in Hardy spaces.

1. Introduction and definitions

We start by introducing the Hardy spaces in $\mathbb{R}^n$. In this paper, we shall have $0 < p \leq 1$, except when otherwise explicitly stated. We write as usual

$$
\|f\|_p = \left( \int |f(x)|^p \, dx \right)^{1/p}
$$

for $f \in L^p(\mathbb{R}^n)$. Observe that for $0 < p < 1$, this is only a quasinorm, in the sense that there is a factor $C = C(p) > 1$ in the right-hand side of the triangle inequality.

The Hardy spaces $H^p = H^p(\mathbb{R}^n)$ have several equivalent definitions. The oldest is for $n = 1$; the space $H^p(\mathbb{R})$ can be defined as the boundary values on the real axis of the real parts of those analytic functions $F$ in the upper half-plane which satisfy

$$
\sup_{y > 0} \int |F(x + iy)|^p \, dx < \infty.
$$

The definition by means of maximal functions works in all dimensions, in the following way. Let $\varphi \in \mathcal{S}$ be any Schwartz function with $\int \varphi \neq 0$ and write $\varphi_t(x) = t^{-n}\varphi(x/t)$ for $t > 0$. The associated maximal operator $M_\varphi$ is then defined by

$$
M_\varphi f(x) = \sup_{t > 0} |\varphi_t * f(x)|,
$$

where $f$ can be any distribution in $\mathcal{S}'$. To define the so-called grand maximal function, one takes the supremum of this expression when $\varphi$ varies over a suitable...
subset of $S$. More precisely, for a fixed, large natural number $N$ we set

$$B_N = \{ \varphi \in S : \sup_x (1 + |x|)^N |\partial^\alpha \varphi(x)| \leq 1 \text{ for } \alpha \in \mathbb{N}^n, \ |\alpha| \leq N \}.$$ 

Then

$$Mf(x) = \sup_{\varphi \in B_N} M\varphi f(x).$$

Choose again a $\varphi \in S$ such that $\int \varphi \neq 0$. Given $f$ in $S'$, one can prove that $M\varphi f$ is in $L^p$ if and only if $Mf$ is in $L^p$, and their $L^p$ norms are equivalent. Here $N = N(n,p)$ must be taken large enough; actually $N > 1 + n/p$ always works. One now defines $H^p$ as the space of distributions $f$ in $S'$ such that $Mf$ is in $L^p$. For further details, we refer to [St, Ch. III].

**Example.** The function $f = 1_B$, where $B$ is the unit ball, is not in $H^p$ for any $0 < p \leq 1$. Indeed, $M\varphi f$ decays like $|x|^{-n}$ at infinity. But if one splits $B$ into halves, say $B_+$ and $B_-$, and considers instead $g = 1_{B_+} - 1_{B_-}$, the decay of $M\varphi g$ will be better, and one verifies that $g$ is in $H^p$ for $n/(n+1) < p \leq 1$. What matters here is the vanishing of the integral, i.e., the moment of order 0, of $g$. This is actually an example of (a multiple of) an atom.

**Definition.** Given $q \in [1, \infty]$ with $p < q$, a $(p,q)$-atom is a function $a \in L^q$ supported in a ball $B$, verifying $\|a\|_q \leq |B|^{1/q-1/p}$ and having vanishing moments up to order $[n(1/p - 1)]$, i.e.,

$$\int f(x) x^\alpha \, dx = 0 \quad \text{for } \alpha \in \mathbb{N}^n, \ |\alpha| \leq [n(1/p - 1)].$$

Observe that each $(p, \infty)$-atom is a $(p,q)$-atom, and also that for $n/(n+1) < p \leq 1$ only the moment of order 0 is required to vanish.

It is easy to verify that any $(p,q)$-atom is in $H^p$. More remarkably, this has a converse, if one passes to linear combinations of atoms. Indeed, let $q$ be as above. Then a distribution $f \in S'$ is in $H^p$ if and only if it can be written as $f = \sum_{j=1}^\infty \lambda_j a_j$, where the $a_j$ are $(p,q)$-atoms and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. The sum representing $f$ here converges in $S'$, and for $p = 1$ also in $L^1$.

To define the quasinorm $\|f\|_{H^p}$, one can use either of the three equivalent expressions

$$\|M\varphi f\|_p, \quad \|Mf\|_p \quad \text{or} \quad \inf \left\{ \left( \sum_{j=1}^\infty |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^\infty \lambda_j a_j, \ a_j \ (p,q)-\text{atoms} \right\}.$$

Different admissible choices of $\varphi$ and $q$ will lead to equivalent quasinorms. In the sequel, we shall choose the third expression here. If $p = 1$, this quasinorm is a norm, otherwise not.

In Section 2 we state the results. Section 3 contains a proof of one of the results in the case $0 < p < 1$.

In the sequel, $C$ will denote several different positive constants. The open ball of centre $x$ and radius $R$ is written $B(x, R)$. 

*Rev. Un. Mat. Argentina, Vol 50-2*
2. Statement of results

Hardy spaces are often used to state so-called endpoint results for $L^p$ boundedness of operators. In particular, many linear and sublinear operators are bounded on $L^p$ for $1 < p < \infty$ but not on $L^1$. However, they often turn out to be bounded from $H^1$ to $L^1$, and this is useful, since one can in many cases start by proving this last property and then obtain the $L^p$ boundedness by interpolation.

To prove the boundedness from $H^1$ to $L^1$, or from $H^p$ to $L^p$, of an operator, a common method is to take one atom at a time. It is usually not hard to verify that $(p,q)$-atoms are mapped into $L^p$, uniformly. From this one wants to conclude the boundedness from $H^p$ to $L^p$, by summing over the atomic decomposition. We shall see that without extra assumptions this conclusion is correct in some cases, but not always.

With $p$ and $q$ as above, we consider a linear operator $T$, defined on the space $H_{\text{fin}}^{p,q}$ of all finite linear combinations of $(p,q)$-atoms. This space is endowed with the natural quasinorm

$$
\|f\|_{H_{\text{fin}}^{p,q}} = \inf \left\{ \left( \sum_{j=1}^{N} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=1}^{N} \lambda_j a_j, \quad a_j \text{ (p,q)-atoms. } N \in \mathbb{N} \right\}.
$$

Notice that $H_{\text{fin}}^{p,q}$ coincides with the space of $L^q$ functions with compact support and vanishing moments up to order $[n(1/p - 1)]$. It is dense in $H^p$.

For linear operators and $q < \infty$, the above conclusion about the boundedness of operators is always correct. More precisely, one has the following extension theorem.

**Theorem 1.** Let $q \in [1, \infty)$ with $p < q$. Assume that $T : H_{\text{fin}}^{p,q} \to L^p$ is a linear operator such that

$$
\sup \{ \|Ta\|_p : a \text{ is a (p,q)-atom} \} < \infty.
$$

Then $T$ has a (necessarily unique) extension to a bounded linear operator from $H^p$ to $L^p$.

The case $q = 2$ of this theorem was obtained independently by Yang and Zhou [YZ1]. Theorem 1 is an immediate consequence of the following quasinorm equivalence result and the density of $H_{\text{fin}}^{p,q}$ in $H^p$.

**Proposition 2.** Let $q$ be as in Theorem 1. The two quasinorms $\| \cdot \|_{H_{\text{fin}}^{p,q}}$ and $\| \cdot \|_{H^p}$ are equivalent on $H_{\text{fin}}^{p,q}$.

It is a remarkable fact that this proposition does not hold for $q = \infty$. An explicit construction of a counterexample for $q = \infty$ and $p = 1$ can be found in Meyer, Taibleson and Weiss [MTW, p. 513]. See also García-Cuerva and Rubio de Francia [GR, III.8.3]. Bownik [B] used this construction to show that Theorem 1 fails for $q = \infty$, if $p = 1$. He also extended the counterexample to the case $0 < p < 1$.

The construction in [MTW] is based on the discontinuities of the function considered. This is crucial; the two results above hold also in the “bad” case $q = \infty$,
if only continuous atoms are considered. Indeed, let $H_{\text{fin,cont}}^{p,\infty}$ be the space of finite linear combinations of continuous $(p, \infty)$-atoms, endowed with the quasinorm $\|f\|_{H_{\text{fin,cont}}^{p,\infty}}$ defined as the infimum of the $\ell^p$ quasinorm of the coefficients, taken over all such atomic decompositions. Then one has the following results.

**Theorem 3.** Assume that $T : H_{\text{fin,cont}}^{p,\infty} \to L^p$ is a linear operator and
\[
\sup \{ \|Ta\|_p : a \text{ is a continuous } (p, \infty)\text{-atom} \} < \infty.
\]
Then $T$ has a (necessarily unique) extension to a bounded linear operator from $H^p$ to $L^p$.

**Proposition 4.** The two quasinorms $\| \cdot \|_{H_{\text{fin,cont}}^{p,\infty}}$ and $\| \cdot \|_{H^p}$ are equivalent on $H_{\text{fin,cont}}^{p,\infty}$.

Observe that if $T$ is a linear operator defined on $H_{\text{fin}}^{p,\infty}$ and uniformly bounded on $(p, \infty)$-atoms, then by Theorem 3 its restriction to $H_{\text{fin,cont}}^{p,\infty}$ has an extension $\tilde{T}$ to a bounded operator from $H^p$ to $L^p$. But $\tilde{T}$ will in general not be an extension of $T$, since these two operators may differ on discontinuous atoms.

In [MSV], Theorem 1 was proved for $p = 1$ also in the setting of a space of homogeneous type with infinite measure. Grafakos, Liu and Yang [GLY] then dealt with a space of homogeneous type which locally has a well-defined dimension, and there proved Theorems 1 and 3 and the propositions, for $p$ close to 1. See also Yang and Zhou [YZ2].

These results were extended to the weighted case by Bownik, Li, Yang and Zhou [BLYZ]. It should be pointed out that most of the papers mentioned consider operators from $H^p$ into general (quasi-)Banach spaces rather than into $L^p$. Recently, Ricci and Verdera [RV] found a description of the completion of the space $H_{\text{fin}}^{p,\infty}$ and of its dual space, for $0 < p \leq 1$.

## 3. A proof

In [MSV], the proofs of the results stated above were given for the case $p = 1$; in the case $p < 1$, the proof of Proposition 2 was only sketched and that of Proposition 4 omitted. Therefore, we shall prove the case $p < 1$ of Proposition 4 here.

Assuming thus $p < 1$, we first observe that the estimate $\|f\|_{H^p} \leq \|f\|_{H_{\text{fin,cont}}^{p,\infty}}$ is obvious. For the converse inequality, let $f \in H_{\text{fin,cont}}^{p,\infty}$ be given. This means that $f$ is a continuous function with compact support and vanishing moments up to order $[n(1/p - 1)]$. We must find a finite atomic decomposition of $f$, using continuous atoms and with control of the coefficients.

As in the first part of the proof of [MSV, Theorem 3.1], we assume that the support of $f$ is contained in a ball $B = B(0, R)$, and introduce the dyadic level sets of the grand maximal function $\Omega_k = \{ x : Mf(x) > 2^k \}, k \in \mathbb{Z}$. Now $f$ is bounded, and so is $Mf$, since the operator $M$ is bounded on $L^\infty$. Thus $\Omega_k$ will be empty if $k > \kappa$, for some $\kappa \in \mathbb{Z}$. Following [MSV], we cover each $\Omega_k$, $k \leq \kappa$, with Whitney cubes $Q_k^i$, $i = 1, 2, \ldots$. We then invoke the proof of the atomic decomposition in

---

*Rev. Un. Mat. Argentina, Vol 50-2*
$H^p$ given in [St, Theorem III.2, p. 107] or [SJ, Thm 3.5, p. 12]. This produces a countable decomposition
\[ f = \sum_{k \leq \kappa} \sum_{i=1}^{\infty} \lambda_i^k a_i^k, \]
where the $a_i^k$ are $(p, \infty)$-atoms and $(\sum_k \sum_i |\lambda_i^k|^p)^{1/p} \leq C\|f\|_{H^p}$. The sum converges in the distribution sense. Moreover, the $a_i^k$ are supported in balls $B_i^k$ concentric with the $Q_i^k$, and contained in $\Omega_k$. As pointed out in [MSV, p. 2924], the balls $B_i^k$, $i = 1, 2, \ldots$, will have bounded overlap, uniformly in $k$, if the Whitney cubes are chosen in a suitable way. Further, one will have the bound
\[ |\lambda_i^k a_i^k| \leq C 2^k \] (1)
in the support of $a_i^k$. A few more properties can also be seen from the construction in [St] or [SJ]. In particular, the continuity of $f$ will imply that of each $a_i^k$. Moreover, each term $\lambda_i^k a_i^k$ will depend only on the restriction of $f$ to a ball $\tilde{B}_i^k = CB_i^k$, a concentric enlargement of $B_i^k$. It will also depend continuously on this restriction, in the sense that
\[ \sup_{\tilde{B}_i^k} |\lambda_i^k a_i^k| \leq C \sup_{\tilde{B}_i^k} |f|. \] (2)
In this estimate, one can replace $f$ by $f - c$ for any constant $c$; in particular $a_i^k$ will vanish when $f$ is constant in $\tilde{B}_i^k$.

We next consider $Mf$ in the set $\overline{B(0,2R)}^c$, which is far from the support of $f$.

**Lemma 5.** Assume that $|x|, |y| > 2R$ and $1/2 < |z|/|y| < 2$. Then there exists a positive constant $C$ depending only on $N$ such that
\[ \frac{1}{C} Mf(y) \leq Mf(x) \leq CMf(y). \]

**Proof.** We shall estimate
\[ |\varphi_t \ast f(x)| = t^{-n} \left| \left\langle f, \varphi \left( \frac{x - \cdot}{t} \right) \right\rangle \right|, \] (3)
where $\varphi \in B_N$ and $t > 0$, with an analogous expression containing $y$ instead of $x$. Here the brackets $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2$.

Consider first the case $t \geq |x|$. Defining $\psi(z) = \varphi(z + (x - y)/t)$, we get
\[ \varphi \left( \frac{x - \cdot}{t} \right) = \psi \left( \frac{y - \cdot}{t} \right). \]
Since by our assumptions $|x - y| \leq C|x|$ and so $|x - y|/t \leq C$, we see that $\psi \in CB_N$, i.e., $\psi/C \in B_N$. Thus the expression in (3) is no larger than $CMf(y)$.

In the case $t < |x|$, we choose a function $\beta \in C_0^\infty$ supported in the ball $B(0, 3/2)$ and such that $\beta = 1$ in $B(0, 1)$. Since supp $f \subset B(0, R)$ and $|x| > 2R$, the expression in (3) will not change if we replace the right-hand side of the scalar product by
\[ \varphi \left( \frac{x - \cdot}{t} \right) \beta \left( \frac{\cdot}{|x|/2} \right). \]
We therefore define
\[ \tilde{\varphi}(z) = \varphi(z) \beta \left( \frac{x-tz}{|x|/2} \right) \] (4)
and conclude that \( \varphi_t * f(x) = t^{-n} \langle f, \tilde{\varphi}((x-\cdot)/t) \rangle \). Since \( t/|x| < 1 \), the last factor in (4) and all its derivatives with respect to \( z \) are bounded, uniformly in \( t \) and \( x \). It follows that \( \tilde{\varphi} \) is a Schwartz function and that \( \tilde{\varphi} \in CB_N \). A point \( z \) in the support of \( \tilde{\varphi} \) must verify \( x-tz \in B(0,3|x|/4) \), so that \( t|z| > |x|/4 \). Thus supp \( \tilde{\varphi} \) is contained in the set \( \{ z : |z| \geq |x|/(4t) \} \). Define now \( \psi(z) = \tilde{\varphi}(z+(x-y)/t) \).

We claim that \( \psi \in CB_N \), which would make it possible to repeat the translation argument above and again dominate the expression in (3) by \( CMf(y) \).

This would end the proof of the lemma, since in both the cases considered we could conclude that \( Mf(x) \leq CMf(y) \). Symmetry then gives the converse inequality.

It thus only remains to verify the claim \( \psi \in CB_N \). This results from the next lemma, with \( r = |x|/(4t) \) and \( x_0 = (x-y)/t \).

**Lemma 6.** If \( \eta \in B_N \) is supported in \( B(0,r)^c \) for some \( r > 0 \) and \( |x_0| < Ar \) with \( A > 0 \), then the translate \( \eta(\cdot + x_0) \) is in \( CB_N \), for some \( C = C(A,N) \).

**Proof.** We must verify that
\[ (1 + |x|)^N |\partial^\alpha \eta(x + x_0)| \leq C \] (5)
for \( \alpha \leq N \). When \( |x| > 2Ar \), we have \( (1 + |x|)^N \leq C(1 + |x + x_0|)^N \), and (5) follows from the fact that \( \eta \in B_N \). For \( |x| \leq 2Ar \), one can use the estimate \( (1 + r)^N |\partial^\alpha \eta(x)| \leq 1 \), which is for all \( x \) a consequence of the assumed properties of \( \eta \). This ends the proof of the two lemmata.

Lemma 5 implies that \( Mf(x) \leq CR^{-n/p} \| f \|_{H^p} \) for \( |x| > 2R \). Indeed, if this were false for some \( x \), the conclusion of the lemma would force \( Mf \) to be large in the ring \( \{ y : |x| < |y| < 2|x| \} \), and this would contradict the fact that \( \| Mf \|_p \leq C \| f \|_{H^p} \). It follows that \( \Omega_k \subset \overline{B(0,2R)} \) if \( k > k' \), for some \( k' \) satisfying \( C^{-1}R^{-n/p} \| f \|_{H^p} \leq 2k' \leq CR^{-n/p} \| f \|_{H^p} \).

Now define
\[ h = \sum_{k \leq k'} \sum_i \lambda_i^k a_i^k \quad \text{and} \quad \ell = \sum_{k' < k \leq \kappa} \sum_i \lambda_i^k a_i^k, \] (6)
so that \( f = h + \ell \). Since both \( f \) and \( \ell \) are supported in \( \overline{B(0,2R)} \), so is \( h \).

We first consider \( \ell \). Let \( \varepsilon > 0 \). The uniform continuity of \( f \) implies that there exists a \( \delta > 0 \) such that
\[ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon. \]

We split \( \ell \) as \( \ell = \ell_1^\varepsilon + \ell_2^\varepsilon \), where \( \ell_2^\varepsilon \) is that part of the sum defining \( \ell \) involving only those atoms \( a_i^k \) for which \( \text{diam} \overline{B_i^k} < \delta \). Notice that the remaining part \( \ell_1^\varepsilon \) will then be a finite sum. Since the atoms are continuous, \( \ell_1^\varepsilon \) will be a continuous function. The estimate (2), with \( f \) replaced by \( f(y) \) for some point \( y \in \overline{B_i^k} \), implies that each term occurring in the sum defining \( \ell_2^\varepsilon \) satisfies \( |\lambda_i^k a_i^k| < C\varepsilon \). As \( i \) varies, the supports of the \( a_i^k \) have bounded overlap, and the sum defining \( \ell_2^\varepsilon \) involves only
κ − k′ values of k. It follows that \(|\ell_2^c| \leq C(\kappa - k')\varepsilon\). This means that one can write \(\ell\) as the sum of one continuous term and one which is uniformly arbitrarily small. Hence, \(\ell\) is continuous, and so is \(h = f - \ell\).

To find the required finite atomic decomposition of \(f\), we use again the splitting \(\ell = \ell_1^c + \ell_2^c\). The part \(\ell_1^c\) is a finite sum of multiples of \((p, \infty)\)-atoms, and the \(\ell^p\) quasinorm of the corresponding coefficients is no larger than that of all the coefficients \(\lambda_k^c\), which is controlled by \(\|f\|_{H^p}\). The part \(\ell_2^c\) is supported in \(B(0, 2R)\). It satisfies the moment conditions, since it is given by a sum which converges in \(S'\) and whose terms are supported in a fixed ball and verify the moment conditions. By choosing \(\varepsilon\) small, we can make its \(L^\infty\) norm small and thus make \(\ell_2^c/\|f\|_{H^p}\) into a \((p, \infty)\)-atom.

As for \(h\), we know that it is a continuous function supported in \(B(0, 2R)\). Since \(h = f - \ell_1^c - \ell_2^c\), we see that it also satisfies the moment conditions. The bound (1) and the bounded overlap for varying \(i\) imply that

\[
|h| \leq C \sum_{k \leq k'} 2^k \leq C 2^{k'} \leq C R^{-n/p}\|f\|_{H^p}
\]

at all points. This means that \(h\) is a multiple of a \((p, \infty)\)-atom, with a coefficient no larger than \(C\|f\|_{H^p}\). Together with the splitting of \(\ell\) just discussed, this gives the desired finite atomic decomposition of \(f\), and the norm of \(f\) in \(H^p_{\text{fin, cont}}\) is controlled by \(C\|f\|_{H^p}\). The proof is complete.

**References**


S. Meda  
Dipartimento di Matematica e Applicazioni  
Università di Milano-Bicocca  
via R. Cozzi 53  
20125 Milano, Italy  
stefano.meda@unimib.it

P. Sjögren  
Mathematical Sciences  
University of Gothenburg and  
Chalmers University of Technology  
SE-412 96 Göteborg, Sweden  
peters@chalmers.se

M. Vallarino  
Dipartimento di Matematica e Applicazioni  
Università di Milano-Bicocca  
via R. Cozzi 53  
20125 Milano, Italy  
maria.vallarino@unimib.it

Recibido: 14 de noviembre de 2008  
Aceptado: 28 de octubre de 2009