HANKEL OPERATORS BETWEEN HARDY-ORLICZ SPACES
AND PRODUCTS OF HOLOMORPHIC FUNCTIONS

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Abstract. For $\mathbb{B}_n$, the unit ball of $\mathbb{C}^n$, we consider Hardy-Orlicz spaces of holomorphic functions $H^\Phi$, which are preduals of spaces of $BMOA$ type with weight. We characterize the symbols of Hankel operators that extend into bounded operators from the Hardy-Orlicz $H^\Phi_1$ into $H^\Phi_2$. We also consider the closely related question of integrability properties of the product of two functions, one in $H^\Phi_1$ and the other one in the dual of $H^\Phi_2$.

1. Introduction and statement of results

In the whole paper, $\Phi$ is a continuous and increasing function from $[0, \infty)$ onto itself, which is of strictly positive lower type, that is, there exists $p > 0$ and $c > 0$ such that, for $t > 0$ and $0 < s \leq 1$,

$$\Phi(st) \leq cs^p \Phi(t).$$

When this inequality is satisfied, we say also that $\Phi$ is of lower type $p$. Such a function $\Phi$ will be called a growth function.

We note $\sigma$ the Lebesgue measure on the unit sphere $\mathbb{S}^n$. The Orlicz space $L^\Phi(\mathbb{S}^n)$ is the space of measurable functions $f$ such that

$$\|f\|_{L^\Phi}^{\text{lux}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{S}^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) d\sigma(x) \leq 1 \right\} < \infty.$$

The quantity $\|f\|_{L^\Phi}^{\text{lux}}$ is called the Luxembourg (quasi)-norm of $f$ in the Orlicz space $L^\Phi(\mathbb{S}^n)$. It generalizes $L^p$ norms, which correspond to the case $\Phi(t) = t^p$. Two growth functions $\Phi_1$ and $\Phi_2$ are said equivalent if there exists some constant $c$ such that

$$c\Phi_1(ct) \leq \Phi_2(t) \leq c^{-1}\Phi_1(c^{-1}t).$$

Such equivalent growth functions define the same Orlicz space.

The following is proved in [VT]:

Proposition 1.1 (Volberg, Tolokonnikov). For $\Phi_1$ and $\Phi_2$ two growth functions, the bilinear map $(f, g) \mapsto fg$ sends $L^{\Phi_1} \times L^{\Phi_2}$ onto $L^\Phi$, with the inverse mappings of $\Phi_1$, $\Phi_2$ and $\Phi$ related by

$$\Phi^{-1} = \Phi_1^{-1} \times \Phi_2^{-1}.$$
Moreover, there exists some constant $c$ such that
\[ \|fg\|^{\text{lux}}_{L^g} \leq c\|f\|^{\text{lux}}_{L^f} \times \|g\|^{\text{lux}}_{L^g}. \]

Next, we define the Hardy-Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$.

**Definition 1.2.** $\mathcal{H}^\Phi(\mathbb{B}^n)$ is the space of holomorphic functions $f$ in the unit ball $\mathbb{B}^n$ such that the functions $f_r$, defined by $f_r(w) = f(rw)$, are uniformly in $L^\Phi(\mathbb{S}^n)$. Moreover,
\[ \|f\|^{\text{lux}}_{\mathcal{H}^\Phi} := \sup_{r < 1} \|f_r\|_{L^\Phi}. \]

It follows from the last proposition that the product of two functions, one in $\mathcal{H}^\Phi_1(\mathbb{B}^n)$ and the other in $\mathcal{H}^\Phi_2(\mathbb{B}^n)$, belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$, with $\Phi$ given by (2). The converse, that is, the problem of factorizing a function of $\mathcal{H}^\Phi(\mathbb{B}^n)$ as a product $f_1 f_2$ with $f_i \in \mathcal{H}^\Phi_i(\mathbb{B}^n)$, is called the Factorization Problem. It is well known that it can be solved for $n = 1$ when the functions $\Phi$ are power functions. For $n \geq 2$, even in this case one only knows that there is weak factorization (see [CRW]) when $\Phi(t) = t^p$ with $p \leq 1$.

We will consider a particular case of this problem. Moreover, instead of dealing with two Hardy-Orlicz spaces, we will replace one of them by a space of $BMOA$ type with weight. More precisely, we assume that $\varrho$ is a continuous increasing function from $[0, \infty)$ onto itself, which is of upper type $\alpha$ on $[0, 1]$, that is,
\[ \varrho(st) \leq s^\alpha \varrho(t) \quad (3) \]
for $s > 1$, with $st \leq 1$. In the next definition, $B$ stands for a ball on $\mathbb{S}^n$ for the Koranyi metrics, given by $d(z, w) := |1 - \overline{z}w|$ (see [R]). We first define
\[ BMO(\varrho) = \left\{ f \in L^2(\mathbb{S}^n); \sup_B \inf_{\sigma \in \mathcal{P}_N(B)} \frac{1}{(\varrho(\sigma(B)))^2 \sigma(B)} \int_B |f - R|^2 \, d\sigma < \infty \right\}, \]
where, for $B$ a ball centered at $\zeta_0$, the space $\mathcal{P}_N(B)$ is the space of polynomials of order $\leq N$ in the $(2n - 1)$ last coordinates related to an orthonormal basis whose first element is $\zeta_0$ and second element $\xi_\zeta_0$. Here $N$ is taken larger than $2n\alpha - 1$.

We call $\|f\|_{BMO(\varrho)} := \|f\|_2 + C$, where $C$ is the quantity appearing in the definition of $BMO(\varrho)$. Next, we define $BMOA(\varrho)$ as
\[ BMOA(\varrho) = \left\{ f \in \mathcal{H}^2(\mathbb{B}^n); \sup_{r < 1} \|f_r\|_{BMO(\varrho)} < \infty \right\}. \]

It follows from classical arguments that it coincides with the space of holomorphic functions in $\mathcal{H}^2(\mathbb{B}^n)$ such that their boundary values on $\mathbb{S}^n$ lie in $BMO(\varrho)$. The space $BMO(\varrho)$ is the dual of a so-called “real” Hardy-Orlicz space that we shall define later on (see Janson [J2] for the corresponding duality in the Euclidean setting, or Viviani [V] for spaces of homogeneous type). As a consequence, $BMOA(\varrho)$ spaces appear as duals of particular Hardy-Orlicz spaces, see also [BG] in this context.

**Theorem 1.3.** The dual space of $\mathcal{H}^\Phi(\mathbb{B}^n)$ is the space $BMOA(\varrho)$ when $\Phi$ and $\varrho$ are related by
\[ \varrho(t) := \varrho_{\Phi}(t) = \frac{1}{t^\Phi - 1(1/t)} \quad (4) \]
The duality is given by the limit, as \( r < 1 \) tends to 1, of scalar products on spheres of radius \( r \).

Such Hardy-Orlicz spaces are defined by using growth functions \( \Phi \) of lower type \((1 + \alpha)^{-1}\). Functions \( \Phi \) have moreover the additional property that the function \( t \mapsto \frac{\Phi(t)}{t} \) is non-increasing. Without loss of generality, eventually replacing \( \Phi \) by the equivalent growth function \( \int_0^t \frac{\Phi(s)}{s} \, ds \) (which is well defined because of the positive lower type), we can assume that \( \Phi \) is concave and \( \Phi \) is a \( C^1 \) function, with derivative \( \Phi'(t) \simeq \frac{\Phi(t)}{t} \). This leads us to the following definition.

**Definition 1.4.** We call \( T_p \) the set of concave growth functions of lower type \( p \), which are \( C^1 \) functions and satisfy \( \Phi'(t) \simeq \frac{\Phi(t)}{t} \). We assume also, without loss of generality, that \( \Phi(1) = 1 \).

Before stating our main result, let us define an auxiliary function.

**Definition 1.5.** For \( \Phi \in T_p \), we call \( \lambda_\Phi \) the function defined for \( t > 0 \) by

\[
\lambda_\Phi(t) := \int_0^t \min \left( 1, \frac{\Phi(s)}{s^2} \right) \, ds.
\]

(5)

When \( \Phi(t) = t \), or \( \varrho = 1 \), so that we are dealing with \( \mathcal{H}^1 \) and \( BMOA \), this function is equivalent to a logarithm for \( t > 1 \).

A variation of this function has been introduced in [BM] in relation to Hankel operators. It can be identified with the following function, introduced by [J1] in relation to multipliers of \( BMO(\varrho) \),

\[
\psi_\varrho(t) := \int_t^\infty \min \left( \frac{1}{s^2}, \frac{\varrho(s)}{s} \right) \, ds.
\]

The two following lemmas are elementary. Their proofs are given in the next section.

**Lemma 1.6.** For \( \Phi \in T_p \) and \( \varrho \) given by (4), then we have

\[
\lambda_\Phi(t) \simeq \psi_\varrho(1/t) \quad \text{for } t > 1.
\]

Remark in particular that the Dini condition \( \int_0^1 \varrho(s)ds/s < \infty \) is equivalent to the condition \( \int_1^\infty \frac{\Phi(s)}{s} ds < \infty \). When this condition holds, the function \( \psi_\varrho \) (resp. \( \lambda_\Phi \)) is equivalent to a constant for \( t < 1 \) (resp. \( t > 1 \)).

**Lemma 1.7.** Let \( \Phi_1 \) and \( \Phi_2 \) in \( T_p \), and let \( \varrho_i(t) = \frac{1}{t\Phi_i^{-1}(1/t)} \). Then, if

\[
\varrho_\Phi := \frac{\varrho_1}{\psi_{\varrho_2}},
\]

we have also

\[
\Phi(t) \simeq \Phi_1 \left( \frac{t}{\lambda_{\varrho_2}(t)} \right) \quad \text{for } t > 1.
\]
Theorem 1.8. Let $\Phi_1$ and $\Phi_2$ in $\mathcal{T}_p$, and $\varrho_i(t) = \frac{1}{t\Phi_i'(1/t)}$. Then the product of two functions, one in $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and the other one in $\text{BMOA}(\varrho_2)$, is in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$, with $\Phi$ such that

$$\varrho_\Phi := \varrho_1 \psi_{\varrho_2},$$

or, equivalently,

$$\Phi(t) := \Phi_1 \left( \frac{t}{\lambda_{\Phi_2}(t)} \right).$$

Moreover, functions in $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ can be weakly factorized in terms of products of functions of $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ and $\text{BMOA}(\varrho_2)$.

We postpone to Section 4 the description of what we mean by a weak factorization.

Next, recall that, for $b \in H^2(\mathbb{B}^n)$, the (small) Hankel operator $h_b$ of symbol $b$ is given, for functions $f \in H^2(\mathbb{B}^n)$, by $h_b(f) = P(b\bar{f})$. Here $P$ is the Szegö projection, that is the orthogonal projection from $L^2(S^n)$ onto $H^2(\mathbb{B}^n)$, considered as its closed subspace (see [R]). We can now state our second theorem.

Theorem 1.9. Let $\Phi_1$ and $\Phi_2$ in $\mathcal{T}_p$, and $\varrho_i(t) = \frac{1}{t\Phi_i'(1/t)}$. Assume that $b \in (\mathcal{H}^{\Phi}(\mathbb{B}^n))' = \text{BMOA}(\varrho_\Phi)$, with

$$\varrho_\Phi := \varrho_1 \psi_{\varrho_2}.$$

Then the Hankel operator $h_b$ extends into a continuous operator from $\mathcal{H}^{\Phi_1}(\mathbb{B}^n)$ to $\mathcal{H}^{\Phi_2}(\mathbb{B}^n)$. Moreover the condition is necessary under one of the following conditions: $\Phi_2$ is of lower type $\frac{n}{n+1}$, or $\Phi_2$ satisfies the Dini condition $\int_1^\infty \varrho_2(t) t^2 dt < \infty$.

Let us comment these results. For the unit disc, Theorem 1.9 is due to Janson, Peetre and Semmes [JPS] when both Hardy spaces are $H^1$. In whole generality (for the unit disc), it has been obtained by Bonami and Madan [BM] by a different method, based on balayage of generalized Carleson measures. See also Tolokonnikov [T] for $H^p$ spaces. We have generalized the $H^1$ case to the unit ball in $\mathbb{C}^n$ in [BGS]. The link between products and weak factorization for Hardy-Orlicz spaces has been considered by Bonami and Grellier in [BG], where both theorems are proved in the particular case $\Phi_2(t) = t$. Moreover, we will use the atomic decomposition of Hardy-Orlicz spaces given in [BG], so that this paper can be seen as a continuation of [BG]. Hardy-Orlicz spaces appear as a natural context for these questions of products of holomorphic functions and Hankel operators, which have been first considered in [BIJZ]. The main tool for weak factorization is the use of an explicit function in the weighted $\text{BMO}$ space, which is due to Spanne [Sp] and replaces the logarithm in the usual case.

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2. Technical lemmas on growth functions

We start by the following lemma.

Lemma 2.1. Let $\Phi \in \mathfrak{T}_p$. The function $\lambda_\Phi$ satisfies the three following estimates. For $t > 1$,
\[
\lambda_\Phi(t) = 1 + \int_1^t \frac{\Phi(s)}{s^2} \, ds. \tag{8}
\]
For $\varepsilon > 0$ there exists a constant $C$ such that, for all $s, t > 1$,
\[
\lambda_\Phi(st) \leq Cs^\varepsilon \lambda_\Phi(t). \tag{9}
\]
For $q > 1$ there exists a constant $C$ such that, for $t > 1$,
\[
\lambda_\Phi(t^q) \leq C\lambda_\Phi(t). \tag{10}
\]

Proof. The first equality is direct (recall that $\Phi(1) = 1$). For the second inequality we use the elementary inequalities
\[
\lambda_\Phi(s) \leq 1 + \log s,
\]
\[
\lambda_\Phi(st) \leq \lambda_\Phi(s) + \lambda_\Phi(t),
\]
where, for the second one, we have cut the integral into the integral from 0 to $t$ and the integral from $t$ to $st$, changed of variable in the second one and used the decrease of $\Phi(u)/u$. To prove (10), it is sufficient to show that, for $t > 1$,
\[
\int_1^t \frac{\Phi(s)}{s} \frac{ds}{s} \leq \int_1^{t^q} \frac{\Phi(s)}{s} \frac{ds}{s}.
\]
By change of variables the left hand side is, up to a constant, equal to $\int_1^t \frac{\Phi(s)}{s^q} \frac{ds}{s}$. We conclude by using the fact that $\Phi(s)/s$ lies between two powers of $s$.

Proof of Lemma 1.6.
We have
\[
\int_1^t \frac{q(s)}{s} \frac{ds}{s} = \int_1^t \frac{ds}{\Phi^{-1}(s)} = \int_{\Phi^{-1}(1)}^{\Phi^{-1}(t)} \frac{\Phi'(s)}{s} \, ds.
\]
We conclude by using the fact that $\Phi'(s) \sim \Phi(s)/s$ and (10), since $\Phi(s)$ lies between two powers of $s$.

Proof of Lemma 1.7. Let us pose $\lambda := \lambda_{\Phi_2}$ for simplification. By assumption, we have
\[
\Phi^{-1}(t) = \Phi_1^{-1}(t) \psi_{\phi_2}(1/t) \sim \Phi_1^{-1}(t) \lambda(t).
\]
Let us pose $\theta(t) = \frac{t}{\lambda(t)}$. We want to prove that $\Phi \sim \Phi_1 \circ \theta$, or, equivalently, $\Phi^{-1} \sim \theta^{-1} \circ \Phi_1^{-1}$. Remark that, since $\Phi_1$ is bounded below and above by two power functions, the same is valid for $\Phi_1^{-1}$, and by (10) we have the equivalence $\lambda \circ \Phi_1^{-1} \sim \lambda$. So it is sufficient to prove that $\theta^{-1}(t) \simeq t\lambda(t)$ to conclude. Again, this last inequality is a consequence of the fact that $\lambda(t\lambda(t)) \simeq \lambda(t)$, since $t\lambda(t)$ lives between two powers of $t$. \qed
In the sequel, we will assume the following condition,

\[ \int_1^{\infty} \frac{\Phi(t)}{t^2} \, dt = \infty, \tag{11} \]

so that \( \lambda_\Phi \) defines an homeomorphism of \([0, \infty)\). Under this condition, we define, for \( \Phi \in \mathcal{T}_p \), a pair of Young convex functions, namely

\[ \Phi^*(t) = \int_0^t \lambda_\Phi(s) \, ds \tag{12} \]
\[ \Phi^{**}(t) = \int_0^t \lambda^{-1}_\Phi(s) \, ds. \tag{13} \]

Moreover \( \Phi^* \) is doubling (see also [BM]). So we have the following duality.

Lemma 2.2. The space \( L^{\Phi^{**}} \) is the dual of the space \( L^{\Phi^*} \).

Remark 2.3. When the condition \((11)\) is not satisfied, or, equivalently, if the Dini condition

\[ \int_1^{\infty} \frac{\phi(t)}{t^2} \, dt < \infty \]

is satisfied, then \( \Phi^*(t) \) is equivalent to \( t \) for \( t > 1 \), and the space \( L^{\Phi^*} \) is the space \( L^1 \), with dual \( L^\infty \). Remark that, in this case, the space \( L^1_{\text{weak}} \) is contained in \( L^p \). For the necessity, we conclude as in [BG], writing that, for \( f \) in a dense subset,

\[ |\langle b, f \rangle| = |h_b(f)(0)| \lesssim \|f\|_{H^p}, \]

so that \( b \) defines a continuous linear form on the space \( H^p \). We have used the fact that \( H^{\Phi_2} \) is continuously contained in \( H^p \), and the evaluation at 0 is bounded on this space.

In the next sections we will concentrate on growth functions that satisfy \((11)\). We will need the following elementary property.

Lemma 2.4. For \( \Phi \in \mathcal{T}_p \) satisfying the condition \((11)\), we have the two estimates, for \( t > 1 \),

\[ \Phi^*(t) \simeq t \lambda_\Phi(t), \tag{14} \]
\[ (\Phi^{**})^{-1}(t) \simeq \lambda_\Phi(t). \tag{15} \]
Proof. For the first one, a simple integration gives
\[
\Phi^*(t) = t + \int_1^t (t-s) \frac{\Phi(s)}{s^2} ds \quad \text{for } t > 1.
\]
For proving that \( \int_1^t \frac{\Phi(s)}{s^2} ds \lesssim \Phi^*(t) \), we first prove that the quantity is bounded by \( 2\Phi^*(2t) \), then use the doubling property of \( \Phi^* \). For the second one, we first remark that, from (10), it follows that, for \( p > 1 \) and \( t > 1 \),
\[
(\lambda^{-1}_\Phi(t))^p \leq \lambda^{-1}_\Phi(Ct).
\]
Also, from the inequality \( \Phi(t) \leq t \) it follows that \( \lambda_\Phi(t) \leq 1 + \log t \) for \( t > 1 \), so that \( \lambda_\Phi^{-1}(t) \geq e^{t-1} \). The required inequality on \( (\Phi^{**})^{-1} \) is equivalent to the double inequality
\[
\lambda_\Phi^{-1}(t/C) \leq \Phi^{**}(t) \leq \lambda_\Phi^{-1}(Ct).
\]
The right hand side inequality follows from the fact that
\[
\Phi^{**}(t) \leq t \lambda_\Phi^{-1}(t) \leq e^{t-1} \lambda_\Phi^{-1}(t) \leq (\lambda_\Phi^{-1}(t))^2 \leq \lambda_\Phi^{-1}(Ct).
\]
For the left hand side inequality, we use that, for \( t > 2 \), we have \( \Phi^{**}(t) \geq \lambda_\Phi^{-1}(t/2) \).

□

3. Sufficient conditions

We will use the following proposition (see [BM]).

Proposition 3.1. The Hardy-Littlewood maximal operator \( \mathcal{M}_{HL} \) maps continuously \( L^{\Phi^*}(\mathbb{S}^n) \) into \( L^{\Phi}(\mathbb{S}^n) \), while the Szegő projection \( P \) maps continuously \( L^{\Phi^*}(\mathbb{S}^n) \) into \( \mathcal{H}^{\Phi}(\mathbb{B}^n) \).

Proof. For \( f \in L^{\Phi^*}(\mathbb{S}^n) \) with norm \( \leq 1 \), we use the maximal inequality and the equivalence \( \Phi'(t) \simeq \frac{\Phi(t)}{t} \) to obtain
\[
\int_{\mathbb{S}^n} \Phi(\mathcal{M}_{HL}(f)) d\sigma = \int_0^\infty \Phi'(t) \sigma(\mathcal{M}_{HL}(f) > t) dt
\leq 1 + \int_1^\infty \Phi'(t) \sigma(\mathcal{M}_{HL}(f) > t) dt
\leq 1 + \int_1^\infty \frac{\Phi(t)}{t^2} \int_{\{ |f| > \frac{t}{2} \}} |f| d\sigma dt
\leq 1 + \int_{\mathbb{S}^n} |f| \left( \int_1^{2|f|} \frac{\Phi(t)}{t^2} dt \right) d\sigma \lesssim 1.
\]

We have used Lemma 2.4 for the last inequality. This allows to conclude for the maximal function. To conclude for the Szegő projection, we use the classical Calderón-Zygmund theory in view of the following inequality, since \( \Phi \) is doubling,
\[
\int \Phi(|Pf|) d\sigma \lesssim \int \Phi(|\mathcal{M}_{HL}f|) d\sigma.
\]

As a consequence, we have the following, which could as well be obtained directly, by the classical method of John and Nirenberg, see [Sp].
Proposition 3.2. The space $\text{BMO}(p_\Phi)$ is contained in $L^{\Phi^*}(S^n)$.

Proof. For $f$ a distribution on $S^n$, its Poisson-Szeg"o maximal function is defined by

$$Mf(\xi) = \sup_{r<1} \left| \int_{S^n} P(r\xi, w)f(w) d\sigma(w) \right|,$$

where $P(z, w)$ is the Poisson-Szeg"o kernel. Then, by analogy with the spaces $\mathcal{H}^p(S^n)$, let us define the Hardy-Orlicz "real" space $\mathcal{H}^\Phi(S^n)$ as the space of distributions $f$ on $S^n$ such that

$$\int_{S^n} \Phi(Mf)d\sigma < \infty.$$ 

A direct generalization of the methods of Janson in \cite{J2} allows to prove that the dual of $\mathcal{H}^\Phi(S^n)$ is $\text{BMO}(p_\Phi)$. It follows from the previous lemma and the well-known inequality $Mf \lesssim M_{HL}f$ that $L^{\Phi^*}$ is continuously contained in $\mathcal{H}^\Phi(S^n)$. We conclude for the proposition by duality. \hfill \Box

Proof of Theorem 1.8, sufficient condition. By Proposition 3.2, it is sufficient to prove that the product of two functions, one in $L^{\Phi_1}$ and one in $L^{\Phi_2^*}$, is in $L^{\Phi}$. One verifies that the link between the three growth functions coincides with the one of Theorem 1.1. \hfill \Box

The sufficient condition in Theorem 1.9 is a consequence of the following proposition, which is stronger, since the Szeg"o projection maps $L^{\Phi_2^*}$ into $\mathcal{H}^\Phi$.

Proposition 3.3. With the notations of Theorem 1.9, for $b \in \text{BMOA}(p_\Phi)$ the operator $h_b$ maps $\mathcal{H}^{\Phi_1}$ into $P(L^{\Phi_2^*}(S^n))$.

Proof. We use the fact that $P(L^{\Phi_2^*}(S^n))$ is a Banach space, with dual $\mathcal{H}^{\Phi_2^*}(\mathbb{B}^n)$. The norm of $h_b(f)$ is obtained by duality, testing on $g \in \mathcal{H}^{\Phi_2^*}(\mathbb{B}^n)$. So,

$$\|h_b(f)\|_{P(L^{\Phi_2^*}(S^n))} = \sup |\langle h_b(f), g \rangle| = \sup |\langle b, fg \rangle| \leq \sup \|b\|_{\text{BMO}(p_\Phi)} \|fg\|_{\mathcal{H}^{\Phi_2^*}},$$

where the sup is taken on all functions $g \in \mathcal{H}^{\Phi_2^*}$ with (Luxembourg) norm less than 1. We use again Proposition 1.1 for $fg$ to conclude. \hfill \Box

4. Weak factorization and necessary conditions

We start from the following definition (see \cite{BG} and the references therein).

Definition 4.1. A holomorphic function $A \in \mathcal{H}^2(\mathbb{B}^n)$ is called a molecule of order $L$, associated to the ball $B := B(z_0, r_0) \subset S^n$, if it satisfies

$$\|A\|_{\text{mol}(B,L)} := \left( \sup_{r<1} \int_{S^n} \left( 1 + \frac{d(z_0, r\xi)}{r_0} \right)^{L+n} |A(r\xi)|^2 \frac{d\sigma(\xi)}{\sigma(B)} \right)^{1/2} < \infty. \quad (16)$$
We have used the notation $d(z,w) := |1 - z, w|$ for $z, w \in \mathbb{B}^n$.

Every molecule $A$ of order $L$ so that $L > L_p := 2n(1/p - 1)$ belongs to $\mathcal{H}^\Phi(\mathbb{B}^n)$ with

$$\|A\|_{\mathcal{H}^\Phi} \lesssim \Phi(\|A\|_{\text{mol}(B,L)}) \sigma(B),$$

see [BG]. Every function has a molecular decomposition.

**Theorem 4.2** ([BG]). For any $f \in \mathcal{H}^\Phi(\mathbb{B}^n)$, there exists molecules $A_j$ of order $L > L_p$, associated to the balls $B_j$, so that $f$ may be written as

$$f = \sum_j A_j$$

with $\|f\|_{\mathcal{H}^\Phi(\mathbb{B}^n)} \simeq \sum_j \Phi(\|A_j\|_{\text{mol}(B_j,L)}) \sigma(B_j)$.

What we describe now is a factorization of each molecule. More precisely, we have the following theorem.

**Theorem 4.3.** Let $\Phi_1$ and $\Phi_2$ in $\mathcal{T}_p$, and $q_1(t) = \frac{1}{t \phi_1^{-1}(1/t)}$. Let $\Phi(t) := \Phi_1\left(\frac{t}{\lambda_{q_2}(t)}\right)$. We assume moreover that $q_2$ is of upper type $1/n$ on $[0,1]$ (or that $p \geq \frac{n}{n+1}$). Then a molecule $A$ associated to the ball $B$ may be written as $fg$, where $f$ is a molecule and $g$ is in $\text{BMOA}(q_2)$. Moreover, for $L, L'$ given with $L' < L$, then $f$ and $g$ may be chosen such that

$$\|g\|_{\text{BMOA}(q_2)} \leq 1, \quad \|f\|_{\text{mol}(B,L')} \lesssim \frac{\|A\|_{\text{mol}(B,L)}}{\psi_{q_2}(\sigma(B))},$$

or such that

$$\|g\|_{\text{BMOA}(q_2)} \leq 1, \quad \|f\|_{\mathcal{H}^{\Phi_1}} \lesssim \|A\|_{\text{mol}(B,L)} \sigma(B) \varrho(\sigma(B)).$$

**Proof.** The main point is the following proposition.

**Proposition 4.4.** Let $q$ be a continuous increasing function of upper type $1/n$ on $[0,1]$. Then, for each ball $B := B(z_0,r)$, there exists a function $g$ such that $\|g\|_{\text{BMOA}(q)} \leq 1$ and $|g(z)| \gtrsim \psi_2(r + d(z,z_0))$ for $z \in \mathbb{B}^n$.

Let us take for granted this proposition and prove the theorem. For simplification we use the notation $\psi := \psi_{q_2}$. We pose $f = A/g$, where $g$ is given by the proposition, and prove that

$$\|f\|_{\text{mol}(B,L')} \lesssim \frac{\|A\|_{\text{mol}(B,L)}}{\psi(\sigma(B))}.$$  

We cut the integral into two parts, one in $B(z_0,2r)$, the other one outside. For the first one, we prove that

$$\|f\|_{\mathcal{H}^2} \lesssim \frac{\|A\|_{\text{mol}(B,L)}}{\psi(\sigma(B))}.$$  

The $\mathcal{H}^2$ norm is obtained as the $L^2$ norm at the boundary $\mathbb{S}^n$, which we cut again into two parts: the integral inside $B(z_0,2r)$ and the integral outside. Inside the ball $B(z_0,2r)$, we use the fact that $|g(z)| \gtrsim \psi(3r) \simeq \psi(\sigma(B))$. The remaining integral.
can be treated as the second term appearing in the computation of $\|f\|_{\text{mol}(B,L')}$. One sees easily that it is sufficient to prove that, for $\varepsilon > 0$

$$\frac{\psi(r)}{\psi(r + d(z_0, w))} \lesssim \left(\frac{r}{d(z_0, w)}\right)^\varepsilon.$$  

This is a consequence of (3), using Lemma 1.6. We conclude from this for (17).

It remains to prove (18). By homogeneity, it is sufficient to prove that, for $\|A\|_{\text{mol}(B,L)} \sigma(B) \varrho(\sigma(B)) = 1$, the function $f$ that has been chosen is such that $\int \Phi(|f|) d\sigma \lesssim 1$, or, which is stronger, $\Phi(\|f\|_{\text{mol}(B,L')}) \sigma(B) \lesssim 1$, which in turn is equivalent to $\|f\|_{\text{mol}(B,L')} \lesssim \Phi^{-1}(\sigma(B))$. We use the definition of $\varrho$ to conclude.

It remains to prove Proposition 4.4. Without loss of generality we may assume that the ball $B(\zeta_0, r)$ is such that $\zeta_0 = (1, 0, \cdots, 0)$. The general case is then obtained by the action of $\text{SU}(n)$.

We start with the dimension 1 with the following lemma, which has been essentially proved by Spanne in [Sp], Theorem 2.

**Lemma 4.5.** For $n = 1$, the function $\phi(e^{iu}) := \psi(|1-re^{iu}|)$ belongs to $\text{BMO}(\varrho)(\mathbb{T})$, with norm bounded independently of $r < 1$.

Next we choose for function $g$ the holomorphic function $\gamma$ that has $\phi$ as real part on the boundary of $S^1 = \mathbb{T}$. The fact that $g$ is in $\text{BMOA}(\varrho)$ is a consequence of the fact that the Hilbert transform maps continuously $\text{BMO}(\varrho)$ into itself, which has been proved by Janson in [J1]. Finally, to conclude for Proposition 4.4 in dimension 1, we use the following lemma.

**Lemma 4.6.** Let $\phi_s := \phi \ast P_s$, with $P_s$ the Poisson kernel in the unit disc. Then $\phi_s(e^{iu}) \gtrsim \psi(r + |1-se^{iu}|)$.

**Proof.** We shall prove that $\phi_s(e^{iu}) \gtrsim \psi(r + 3|1-se^{iu}|)$, which is equivalent by (3). Coming back to the definition of $\phi$ and changing the order of integration, we have to prove that

$$1 + \int_r^1 \left[ \int_{|1-e^{i(u-\theta)}| \leq v-r} P_s(e^{i\theta}) d\theta \right] \frac{\varrho(v)}{v} dv \geq c \int_{r+3|1-se^{iu}|}^1 \frac{\varrho(v)}{v} dv.$$

It is sufficient to prove that, for $v \geq r + 3|1-se^{iu}|$, the quantity in the bracket is bounded below independently of $s$. Since the integral of the Poisson kernel $P_s$ on the interval $|e^{i\theta} - 1| \leq 1 - s$ is uniformly bounded below, it is sufficient to prove that this interval is contained in the set of integration when $v \geq r + 3|1-se^{iu}|$. In other words, we use the fact that the two inequalities $v \geq r + 3|1-se^{iu}|$ and $|e^{i\theta} - 1| \leq 1 - s$ imply the third one, $|1-e^{i(u-\theta)}| \leq v-r$. This is an easy consequence of the triangular inequality, since $|e^{i\theta} - e^{iu}| \leq |e^{i\theta} - 1| + |1-s| + |s-e^{iu}| \leq 3|1-se^{iu}|$.

This allows to conclude for the lemma.

This concludes for the proof of Proposition 4.4 for $n = 1$. We consider now the general case. Assuming again that $\zeta_0 = (1, 0, \cdots, 0)$, we choose as function $g$ the function $g(z) := \gamma(z_1)$, with $\gamma$ the function used in dimension 1 when $g$ is replaced.

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by \( g(t) := g(t^n) \) (so that \( g \) has upper type 1 on \([0,1]\)). More precisely, \( \gamma \) is the function that has \( \psi_\tilde{g}([1-re^{iu}]) \) as real part on the boundary of \( S^1 = T \). The bound below of \( g \) is straightforward, using (10) to see that \( \psi_\tilde{g} \simeq \psi_\tilde{g} \). The only point that we have to prove is the fact that \( g \) is in \( BMOA(\tilde{g}) \). This is a consequence of the Carleson type characterization of \( BMOA(\tilde{g}) \), see [Sm].

**Proposition 4.7.** The function \( h \in \mathcal{H}(\mathbb{B}^n) \) is in \( BMOA(\tilde{g}) \) if and only if the condition

\[
\int_{T(Q)} |\nabla h(z)|^2 (1 - |z|) dV_n(z) \leq C(\sigma(Q)(\tilde{g}(\sigma(Q)))^2,
\]

for all balls \( Q \) on \( S^n \).

Here \( T(Q) \) denotes the tent above \( Q \), and \( dV_n \) the volume measure inside \( \mathbb{C}^n \). Moreover, there is an equivalence between the norm in \( BMOA \) and \( \|h\|_{\mathcal{H}} + \inf C \), where the infimum is taken on all possible constants.

Then, it is sufficient to prove the following, which we will use with \( \tilde{g} \) in place of \( g \) to conclude for the proof of Proposition 4.4.

**Lemma 4.8.** Let \( \tilde{g} \) of upper type 1 on \([0,1]\). Assume that the holomorphic function in one variable \( \gamma \) satisfies in

\[
\int_{T(I)} |\gamma'|^2 (1 - |z|) dV_1(z) \leq C(\sigma(I)(\gamma(\sigma(I)))^2
\]

for each interval of the unit circle. Then \( g(z) = \gamma(z_1) \) satisfies the condition

\[
\int_{T(Q)} |\nabla g(z)|^2 (1 - |z|) dV_n(z) \leq C(\sigma(Q)(\gamma(Q))\gamma(\sigma(Q)))^2,
\]

for all balls \( Q \) on \( S^n \).

**Proof.** Let \( Q \) be the ball \( B(w,s) \). We call \( z' := (z_2, \cdots, z_n) \). Recall that we can take \( T(Q) := \{z \in \mathbb{B}^n : \frac{z}{|z|} \in Q, 1 - |z| > s\} \).

Let us first consider balls \( Q \) such that \( |w_1| = 1 \), and, so \( w' = 0 \). Then \( T(Q) \) is contained in the set defined by \( |1 - z_1 \frac{w}{\overline{w}}| < Cs \) and \( |z'|^2 < Cs \) for some constant \( C \) that depends only on the geometry. Integrating first in \( z' \), we find the required property.

Next, let us consider balls \( Q \) such that \( T(Q) \) contains a point \( z \) such that \( |1 - |z_1| \leq s. \) Then the same is valid for \( Q \). This means that, for some \( z \in Q \), we have \( d(z, (\frac{z_1}{|z_1|}, 0)) \leq s \). So the distance \( d(w, (\frac{z_1}{|z_1|}, 0)) \) is bounded by \( Cs \), for some geometric constant \( C \), and there is a ball \( Q' \), centered at the point \( (\frac{z_1}{|z_1|}, 0) \), which contains \( Q \) and has same measure, up to a geometric constant. So we can rely on the first case to conclude.

Finally, in the remaining cases, the projection on the complex line \( z' = 0 \) of \( T(Q) \) is contained in the disc \( |1 - |z_1| \leq s \). We will use the following lemma.

**Lemma 4.9.** Assume that the holomorphic function in one variable \( \gamma \) satisfies in

\[
\int_{T(I)} ||\gamma'||^2 (1 - |z|) dV_1(z) \leq C(\sigma(I)(\gamma(\sigma(I)))^2
\]
for each interval of the unit circle. Then there exists some $C'$ such that

$$|\gamma'(z)| \leq C' \frac{\varrho(1 - |z|)}{1 - |z|}.$$  

Let us take for granted and finish our proof in this last case. We have $\nabla g$ bounded by $C_\varrho(s)/s$ on $T(Q)$, while

$$\int_{T(Q)} (1 - |z|)dV \simeq s^2\sigma(s^n).$$

Finally, the proof of Lemma 4.9 is classical. We use the mean value property to write that

$$|\gamma'(z)|^2 \leq \frac{C}{(1 - |z|)^2} \int_{|\zeta - z| \leq (1 - |z|)/2} |\gamma'(\zeta)|^2 (1 - |\zeta|)dV_1(\zeta).$$

We conclude by using the assumption and the fact that the disc $|\zeta - z| \leq (1 - |z|)/2$ is contained in $T(I)$, where $I$ is centered at $z/|z|$ and has radius $2(1 - |z|)$.

This finishes the proofs of Lemma 4.9 and 4.8.

The proof of Proposition 4.4 in dimension $n > 1$ follows at once.

**Proof of Theorem 1.8, necessary condition.** We have already considered the case of growth functions satisfying the Dini condition in Remark 2.3. We assume now that $\varrho$ is of upper type $1/n$, so that we have the factorization of molecules. We adapt the proof given in [BGS]. Let $b \in \mathcal{H}^2(\mathbb{B}^n)$. We assume that $h_b$ is bounded from $\mathcal{H}^{a_1}(\mathbb{B}^n)$ to $\mathcal{H}^{a_2}(\mathbb{B}^n)$ and prove that $b$ belongs to $BMOA(\varrho_\Phi)$, with $\varrho_\Phi = \frac{\varrho}{|\varphi|^n}$. For this, it is sufficient to prove that there exists a positive constant $C$ such that, for each ball $B$ we can find a polynomial $R \in \mathcal{P}_N(B)$ such that

$$\int_B |b - R|^2d\sigma \leq C(\varrho(\sigma(B)))^2\sigma(B).$$

We take for $R$ the orthogonal projection of $b$ onto $\mathcal{P}_N(B)$, so that the function $a := \chi_B(b - R)$ on $S^n$ is an atom (see [BG] for a precise definition). One knows that $A := Pa$ is a molecule associated to $B$, with $\|A\|_{\text{mol}(B,L)} \lesssim \|a\|_{2\sigma(B)^{-1/2}}$. From Theorem 4.3, we know that $A$ may be written as $fg$, with

$$\|g\|_{BMOA(\varrho_2)} \leq 1, \quad \|f\|_{L^1(\mathcal{H}_{a_1}^{a_1})} \lesssim \|A\|_{\text{mol}(B,L)}\varrho(\sigma(B))^{1/2}. $$

Next, we write that

$$\int_B |b - R|^2d\sigma = \langle b, a \rangle = \langle b, Pa \rangle = \langle h_b(f), g \rangle,$$

so that

$$\int_B |b - R|^2d\sigma \leq \|h_b\|_1\|f\|_{L^1(\mathcal{H}_{a_1}^{a_1})} \|g\|_{BMOA(\varrho_2)} \lesssim \|h_b\|_1\|a\|_{2\sigma(B)^{1/2}}\varrho(\sigma(B)).$$

We divided both sides by $\|a\|_2 = (\int_B |b - R|^2d\sigma)^{1/2}$ to conclude.

This ends the proof.
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