

FLATNESS AND FREENESS PROPERTIES OF THE GENERIC HOPF GALOIS EXTENSIONS

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ABSTRACT. In previous work, to each Hopf algebra H and each invertible two-cocycle α , Eli Aljadeff and the first-named author attached a subalgebra \mathcal{B}_H^α of the free commutative Hopf algebra $S(t_H)_\Theta$ generated by the coalgebra underlying H ; the algebra \mathcal{B}_H^α is the subalgebra of coinvariants of a generic Hopf Galois extension. In this paper we give conditions under which $S(t_H)_\Theta$ is faithfully flat, or even free, as a \mathcal{B}_H^α -module. We also show that \mathcal{B}_H^α is generated as an algebra by certain elements arising from the theory of polynomial identities for comodule algebras developed jointly with Aljadeff.

INTRODUCTION

The aim of the present article is to answer two questions left unanswered in [1] and in the follow-up paper [5]. Eli Aljadeff and the first-named author associated to any Hopf algebra H and any invertible two-cocycle α a subalgebra \mathcal{B}_H^α of the free commutative Hopf algebra $S(t_H)_\Theta$ generated by the coalgebra underlying H ; the commutative algebra \mathcal{B}_H^α is the subalgebra of coinvariants of the *generic Hopf Galois extension* \mathcal{A}_H^α introduced in [1].

In [1, Sect. 7] it was proved that if $S(t_H)_\Theta$ is integral over the subalgebra \mathcal{B}_H^α , then for any field extension K/k , there is a surjective map

$$\mathrm{Alg}(\mathcal{B}_H^\alpha, K) \rightarrow \mathrm{Forms}_K({}^\alpha H) \quad (0.1)$$

from the set of algebra morphisms $\mathcal{B}_H^\alpha \rightarrow K$ to the set of isomorphism classes of K -forms of the twisted algebra ${}^\alpha H$ (for details, see *loc. cit.* or Section 1 below).

The first question raised in [5] (Question 6.1) was: under which condition on the pair (H, α) is $S(t_H)_\Theta$ integral over \mathcal{B}_H^α ? Now, the existence and the surjectivity of the map (0.1) hold under the less restrictive condition that $\mathcal{B}_H^\alpha \hookrightarrow S(t_H)_\Theta$ be *faithfully flat*. In the present paper we shall give sufficient conditions for the latter to hold.

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After having defined in Section 1 the objects of interest to us, in particular the algebras $S(t_H)_\Theta$ and \mathcal{B}_H^α , we reduce the question to the case where $\alpha = \varepsilon$ is the trivial two-cocycle and to the corresponding subalgebra $\mathcal{B}_H = \mathcal{B}_H^\varepsilon$ (see Section 2).

In Section 3 we show that the \mathcal{B}_H -module $S(t_H)_\Theta$ is

- (i) a *finitely generated projective generator* if H is finite-dimensional (see Theorem 3.6),
- (ii) *faithfully flat* if H is cocommutative (see Theorem 3.8), and
- (iii) *free* if
 - (a) H is a pointed Hopf algebra whose grouplike elements satisfy a certain finiteness condition (see Theorem 3.9), or
 - (b) H is commutative or pointed cocommutative (see Theorem 3.13).

In all these cases, $\mathcal{B}_H \hookrightarrow S(t_H)_\Theta$ is faithfully flat. These results extend by far Theorem 6.2 of [5], which in particular implies that the above integrality condition is satisfied by any finite-dimensional Hopf algebra generated by grouplike and skew-primitive elements.

The algebra \mathcal{B}_H is generated by the values of the so-called generic cocycle and of its inverse (see Section 1.3). In [1] a theory of polynomial identities for H -comodule algebras was worked out. Such identities live in a tensor algebra $T(X_H)$. Certain coinvariant elements of $T(X_H)$ give rise to natural elements $p_x, p'_x, q_{x,y}, q'_{x,y}$ of \mathcal{B}_H . The second question we address here is the following: do these elements generate \mathcal{B}_H ? In Section 4 we give a positive answer for any Hopf algebra (the answer had been known so far only for group algebras and the four-dimensional Sweedler algebra).

Throughout the paper we fix a field k over which all our constructions are defined. In particular, all linear maps are supposed to be k -linear and unadorned tensor products mean tensor products over k .

For us an algebra is an associative unital k -algebra and a coalgebra is a coassociative counital k -coalgebra. We denote the coproduct of a coalgebra by Δ and its counit by ε . We shall also make use of a Heyneman-Sweedler-type notation for the image

$$\Delta(x) = x_1 \otimes x_2$$

of an element x of a coalgebra C under the coproduct, and we write

$$\Delta^{(2)}(x) = x_1 \otimes x_2 \otimes x_3$$

for the iterated coproduct $\Delta^{(2)} = (\Delta \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \Delta) \circ \Delta$, and so on.

For any Hopf algebra H we denote its antipode by S and its group of grouplike elements by $G(H)$.

1. THE BASIC OBJECTS

We now define the basic objects investigated in this paper.

1.1. The free commutative Hopf algebra generated by a coalgebra. Let C be a coalgebra. Pick another copy t_C of the underlying vector space of C and denote the identity map from C to t_C by $x \mapsto t_x$ ($x \in C$).

Let $S(t_C)$ be the symmetric algebra over the vector space t_C . If $\{x_i\}_{i \in I}$ is a linear basis of C , then $S(t_C)$ is isomorphic to the polynomial algebra over the indeterminates $\{t_{x_i}\}_{i \in I}$. The commutative algebra $S(t_C)$ has a bialgebra structure with coproduct $\Delta : S(t_C) \rightarrow S(t_C) \otimes S(t_C)$ and counit $\varepsilon : S(t_C) \rightarrow k$ induced by the coproduct and the counit of C , respectively, namely

$$\Delta(t_x) = t_{x_1} \otimes t_{x_2} \quad \text{and} \quad \varepsilon(t_x) = \varepsilon(x) \tag{1.1}$$

for all $x \in C$. The bialgebra $S(t_C)$ is the *free commutative bialgebra* generated by C : this means that for any coalgebra morphism $f : C \rightarrow B$ from C to a commutative bialgebra B , there is a unique bialgebra morphism $\bar{f} : S(t_C) \rightarrow B$ such that $\bar{f}(t_x) = f(x)$ for all $x \in C$.

By [1, Lemma A.1] there is a unique linear map $x \mapsto t_x^{-1}$ from C to the field of fractions $\text{Frac } S(t_C)$ of $S(t_C)$ such that for all $x \in C$,

$$t_{x_1} t_{x_2}^{-1} = t_{x_1}^{-1} t_{x_2} = \varepsilon(x) 1.$$

As in [1, App. B] we denote by $S(t_C)_\Theta$ the subalgebra of $\text{Frac } S(t_C)$ generated by all elements t_x and t_x^{-1} . The commutative algebra $S(t_C)_\Theta$ has a Hopf algebra structure with coproduct $\Delta : S(t_C)_\Theta \rightarrow S(t_C)_\Theta \otimes S(t_C)_\Theta$, counit $\varepsilon : S(t_C)_\Theta \rightarrow k$, and antipode $S : S(t_C)_\Theta \rightarrow S(t_C)_\Theta$ determined by (1.1), by

$$\Delta(t_x^{-1}) = t_{x_2}^{-1} \otimes t_{x_1}^{-1} \quad \text{and} \quad \varepsilon(t_x^{-1}) = \varepsilon(x), \tag{1.2}$$

and by

$$S(t_x) = t_x^{-1} \quad \text{and} \quad S(t_x^{-1}) = t_x \tag{1.3}$$

for all $x \in C$. The Hopf algebra $S(t_C)_\Theta$ is the *free commutative Hopf algebra* generated by C , as constructed by Takeuchi in [12]: more precisely, for any coalgebra morphism $f : C \rightarrow H$ from C to a commutative Hopf algebra H , there is a unique Hopf algebra morphism $\tilde{f} : S(t_C)_\Theta \rightarrow H$ such that $\tilde{f}(t_x) = f(x)$ and $\tilde{f}(t_x^{-1}) = S(f(x))$ for all $x \in C$.

As an example, consider the coalgebra $C = kX$ freely spanned by a set X of grouplike elements. The resulting commutative Hopf algebra $S(t_C)_\Theta$ is isomorphic to the group algebra $k\mathbb{Z}^{(X)}$, where $\mathbb{Z}^{(X)}$ is the free abelian group with basis X . Note that $k\mathbb{Z}^{(X)}$ is isomorphic to the Laurent polynomial algebra $k[t_x^{\pm 1} \mid x \in X]$.

1.2. Cocycles. Let H be a Hopf algebra. Recall that a *two-cocycle* α on H is a bilinear form $\alpha : H \times H \rightarrow k$ such that

$$\alpha(x_1, y_1) \alpha(x_2 y_2, z) = \alpha(y_1, z_1) \alpha(x, y_2 z_2)$$

for all $x, y, z \in H$. We assume that α is convolution-invertible and write α^{-1} for its inverse.

We denote by ${}^\alpha H$ the *twisted H -comodule algebra* defined as follows: as a right H -comodule, ${}^\alpha H = H$; as an algebra it is equipped with the product

$$(x, y) \mapsto \alpha(x_1, y_1) x_2 y_2,$$

where $x, y \in H$. The algebra ${}^\alpha H$ is a right H -comodule algebra whose subalgebra of H -coinvariants consists of the multiple scalars of the unit:

$$({}^\alpha H)^{\text{co-}H} = k 1.$$

Two-cocycles $\alpha, \beta : H \times H \rightarrow k$ are said to be *cohomologous* if there is a convolution-invertible linear form $\lambda : H \rightarrow k$ with inverse λ^{-1} such that

$$\beta(x, y) = \lambda(x_1) \lambda(y_1) \alpha(x_2, y_2) \lambda^{-1}(x_3 y_3) \tag{1.4}$$

for all $x, y \in H$. If α, β, λ are related as in the previous equation, then the map

$${}^\alpha H \longrightarrow {}^\beta H; \quad x \longmapsto \lambda^{-1}(x_1) x_2$$

is an isomorphism of H -comodule algebras.

1.3. The generic cocycle and the generic base algebra. Now, to a pair (H, α) consisting of a Hopf algebra H and a convolution-invertible two-cocycle α , we attach a bilinear map $\sigma : H \times H \rightarrow S(t_H)_\Theta$ with values in the previously defined algebra $S(t_H)_\Theta$. The map σ is given for all $x, y \in H$ by

$$\sigma(x, y) = t_{x_1} t_{y_1} \alpha(x_2, y_2) t_{x_3 y_3}^{-1}.$$

The map σ is a two-cocycle of H with values in $S(t_H)_\Theta$; by construction, σ is cohomologous to α over $S(t_H)_\Theta$. We call σ the *generic cocycle attached to α* .

The cocycle α being invertible, so is σ , with inverse σ^{-1} given for all $x, y \in H$ by

$$\sigma^{-1}(x, y) = t_{x_1 y_1} \alpha^{-1}(x_2, y_2) t_{x_3}^{-1} t_{y_3}^{-1}.$$

Following [1, Sect. 5] and [5, Sect. 3], we define the *generic base algebra* as the subalgebra \mathcal{B}_H^α of $S(t_H)_\Theta$ generated by the values of the generic cocycle σ and of its inverse σ^{-1} .

Since \mathcal{B}_H^α is a subalgebra of $S(t_H)_\Theta$, the latter is a module over the former. In Section 3 we shall discuss when such a module is free, projective, or flat.

1.4. The generic Galois extension. Since the generic cocycle σ takes its values in \mathcal{B}_H^α , we can consider the twisted H -comodule algebra

$$\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes {}^\sigma H,$$

which is equal to $\mathcal{B}_H^\alpha \otimes H$ as a right H -comodule, and whose product is given for all $b, c \in \mathcal{B}_H^\alpha$ and $x, y \in H$ by

$$(b \otimes x)(c \otimes y) = bc \sigma(x_1, y_1) \otimes x_2 y_2.$$

The H -comodule algebra \mathcal{A}_H^α is a H -Galois extension with $\mathcal{B}_H^\alpha = \mathcal{B}_H^\alpha \otimes 1$ as subalgebra of coinvariants. We call \mathcal{A}_H^α the *generic Galois extension* attached to the cocycle α .

By [1, Prop. 5.3], there is an algebra morphism $\chi_0 : \mathcal{B}_H^\alpha \rightarrow k$ such that

$$\chi_0(\sigma(x, y)) = \alpha(x, y) \quad \text{and} \quad \chi_0(\sigma^{-1}(x, y)) = \alpha^{-1}(x, y)$$

for all $x, y \in H$. Consider the maximal ideal $\mathfrak{m}_0 = \text{Ker}(\chi_0 : \mathcal{B}_H^\alpha \rightarrow k)$ of \mathcal{B}_H^α . According to [1, Prop. 6.2], there is an isomorphism of H -comodule algebras

$$\mathcal{A}_H^\alpha / \mathfrak{m}_0 \mathcal{A}_H^\alpha \cong {}^\alpha H.$$

Thus, ${}^\alpha H$ is a central specialization of \mathcal{A}_H^α , and \mathcal{A}_H^α is a flat deformation of ${}^\alpha H$ over the commutative algebra \mathcal{B}_H^α .

What about the other central specializations of \mathcal{A}_H^α , that is the quotient-algebras $\mathcal{A}_H^\alpha/\mathfrak{m}\mathcal{A}_H^\alpha$, where \mathfrak{m} is an arbitrary maximal ideal of \mathcal{B}_H^α ? To answer this question, we need the following terminology.

Let $\beta : H \times H \rightarrow K$ be an invertible right two-cocycle with values in a field K containing the ground field k . We say that the twisted comodule algebra $K \otimes_k {}^\beta H$ is a K -form of ${}^\alpha H$ if there is a field L containing K and an L -linear isomorphism of comodule algebras

$$L \otimes_K (K \otimes_k {}^\beta H) \cong L \otimes_k {}^\alpha H.$$

By [1, Sect. 7], for any K -form $K \otimes_k {}^\beta H$ of ${}^\alpha H$, there exist an algebra morphism $\chi : \mathcal{B}_H^\alpha \rightarrow K$ and a K -linear isomorphism of H -comodule algebras

$$K_\chi \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \cong K \otimes_k {}^\beta H.$$

Here K_χ stands for K equipped with the \mathcal{B}_H^α -module structure induced by the algebra morphism $\chi : \mathcal{B}_H^\alpha \rightarrow K$. We have

$$K_\chi \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha \cong \mathcal{A}_H^\alpha/\mathfrak{m}_\chi \mathcal{A}_H^\alpha,$$

where $\mathfrak{m}_\chi = \text{Ker}(\chi : \mathcal{B}_H^\alpha \rightarrow K)$.

The converse is the following statement: for any field K containing k and any algebra morphism $\chi : \mathcal{B}_H^\alpha \rightarrow K$, the H -comodule K -algebra

$$K_\chi \otimes_{\mathcal{B}_H^\alpha} \mathcal{A}_H^\alpha = \mathcal{A}_H^\alpha/\mathfrak{m}_\chi \mathcal{A}_H^\alpha$$

is a K -form of ${}^\alpha H$.

Theorem 7.2 of [1] asserts that this converse holds if the algebra $S(t_H)_\Theta$ is integral over the subalgebra \mathcal{B}_H^α . Using [10, Th. 3 in Sect. (4.D)] or [14, Th. on p. 105], we see that the proof of [1, Th. 7.2] works in the more general case where $S(t_H)_\Theta$ is a *faithfully flat* \mathcal{B}_H^α -module (in particular, if it is a free module).

2. REDUCTION TO THE TRIVIAL COCYCLE

The aim of this section is to reduce the previous situation to the case of the trivial cocycle.

Let us start again with a Hopf algebra H and an invertible two-cocycle $\alpha : H \times H \rightarrow k$. Let $L = {}^\alpha H^{\alpha^{-1}}$ be the cocycle deformation of H by α : as a coalgebra, $L = H$, and the product $*$ of L is given for $x, y \in H$ by

$$x * y = \alpha(x_1, y_1) x_2 y_2 \alpha^{-1}(x_3, y_3). \tag{2.1}$$

Let $\varepsilon : L \times L \rightarrow k$ be the *trivial cocycle* on L given for all $x, y \in L$ by

$$\varepsilon(x, y) = \varepsilon(x)\varepsilon(y).$$

On the level of generic base algebras we have the following result.

Proposition 2.1. *We have $\mathcal{B}_H^\alpha = \mathcal{B}_L^\varepsilon$ inside $S(t_H)_\Theta$.*

Proof. Let us denote the generic cocycle attached to α by σ_α and the generic cocycle attached to ε by σ_ε . Observe that for $x, y \in H$, we have

$$\sigma_\varepsilon(x, y) = t_{x_1} t_{y_1} \varepsilon(x_2) \varepsilon(y_2) t_{x_3 * y_3}^{-1} = t_{x_1} t_{y_1} t_{x_2 * y_2}^{-1}.$$

We then have

$$\begin{aligned} \sigma_\alpha(x, y) &= t_{x_1} t_{y_1} \alpha(x_2, y_2) t_{x_3 y_3}^{-1} = t_{x_1} t_{y_1} t_{\alpha(x_2, y_2) x_3 y_3}^{-1} \\ &= t_{x_1} t_{y_1} t_{\alpha(x_2, y_2) x_3 y_3}^{-1} \alpha^{-1}(x_4, y_4) \alpha(x_5, y_5) \\ &= t_{x_1} t_{y_1} t_{x_2 * y_2}^{-1} \alpha(x_3, y_3) \\ &= \sigma_\varepsilon(x_1, y_1) \alpha(x_2, y_2). \end{aligned}$$

Similarly, $\sigma_\alpha^{-1}(x, y) = \alpha^{-1}(x_1, y_1) \sigma_\varepsilon^{-1}(x_2, y_2)$. This proves that the subalgebras \mathcal{B}_H^α and $\mathcal{B}_L^\varepsilon$ coincide. \square

Corollary 2.2. *For any cocommutative Hopf algebra H , we have $\mathcal{B}_H^\alpha = \mathcal{B}_H^\varepsilon$.*

Proof. Under the hypothesis, the product (2.1) of L reduces to the product on H : $x * y = xy$ for all $x, y \in H$. \square

Recall that there is a one-to-one correspondence $A \mapsto A^\alpha$ between the right L -comodule algebras and the right H -comodule algebras; it is defined as follows: if A is a right L -comodule algebras with coaction $a \mapsto a_0 \otimes a_1$ ($a \in A$), then A^α is A as an H -comodule (which is the same as an L -comodule) with the twisted product

$$a *_\alpha b = a_0 b_0 \alpha(a_1, b_1).$$

The following result for generic Galois extensions is a companion to Proposition 2.1.

Proposition 2.3. (a) *We have $\mathcal{A}_H^\alpha = (\mathcal{A}_L^\varepsilon)^\alpha$.*

(b) *For any field extension K/k there is a one-to-one correspondence between the K -forms of the H -comodule algebra ${}^\alpha H$ and the K -forms of the L -comodule algebra L .*

Proof. (a) By definition, $\mathcal{A}_H^\alpha = \mathcal{B}_H^\alpha \otimes^{\sigma_\alpha} H$ and $\mathcal{A}_L^\varepsilon = \mathcal{B}_L^\varepsilon \otimes^{\sigma_\varepsilon} H$. Let $b, c \in \mathcal{B}_L^\varepsilon = \mathcal{B}_H^\alpha$ and $x, y \in H = L$. Let us compute the product $(b \otimes x)(c \otimes y)$ in $(\mathcal{A}_L^\varepsilon)^\alpha$. We have

$$\begin{aligned} (b \otimes x) *_\alpha (c \otimes y) &= bc \sigma_\varepsilon(x_1, y_1) \otimes x_2 * y_2 \alpha(x_3, y_3) \\ &= bc \sigma_\varepsilon(x_1, y_1) \otimes \alpha(x_2, y_2) x_3 y_3 \alpha^{-1}(x_4, y_4) \alpha(x_5, y_5) \\ &= bc \sigma_\varepsilon(x_1, y_1) \alpha(x_2, y_2) \otimes x_3 y_3 \\ &= bc \sigma_\alpha(x_1, y_1) \otimes x_2 y_2, \end{aligned}$$

which is the product in \mathcal{A}_H^α . For the last equality, we have used the formula expressing σ_α in terms of σ_ε established in the proof of Proposition 2.1.

(b) The above equivalence is compatible with base extension. \square

Remarks 2.4. (a) Corollary 2.2 holds under the following weaker condition. Recall that a two-cocycle α on H is *lazy* in the sense of [2] if for all $x, y \in H$,

$$\alpha(x_1, y_1) x_2 y_2 = \alpha(x_2, y_2) x_1 y_1.$$

If α is lazy, then obviously the Hopf algebra ${}^\alpha H^{\alpha^{-1}}$ coincides with H . This includes the case where H is cocommutative since all two-cocycles on a cocommutative Hopf algebra are obviously lazy.

(b) In the case of a general Hopf algebra H , one can check that if α and β are cohomologous two-cocycles, then the twisted Hopf algebras ${}^\alpha H^{\alpha^{-1}}$ and ${}^\beta H^{\beta^{-1}}$ are isomorphic, the isomorphism $f : {}^\alpha H^{\alpha^{-1}} \rightarrow {}^\beta H^{\beta^{-1}}$ being given by

$$f(x) = \lambda^{-1}(x_1) x_2 \lambda(x_3),$$

where λ, λ^{-1} are the linear forms appearing in (1.4); see also [1, Prop. 5.4].

(c) It follows from the previous discussion that if H is such that any two-cocycle on H is cohomologous to a lazy one, then ${}^\alpha H^{\alpha^{-1}} \cong H$ for any α . This holds for instance for Sweedler’s four-dimensional Hopf algebra.

(d) Examples of cocycles α for which the Hopf algebras H and ${}^\alpha H^{\alpha^{-1}}$ are not isomorphic can be drawn from the proofs of [6, Th. 4.5] or [8, Th. 4.3]. These examples include the quantized enveloping algebras and the naturally associated graded pointed Hopf algebras.

3. FLATNESS AND FREENESS RESULTS

From now on we consider a Hopf algebra H together with the trivial cocycle ε . As we have seen in the proof of Proposition 2.1, the generic cocycle σ attached to ε and its inverse σ^{-1} are given for all $x, y \in H$ by

$$\sigma(x, y) = t_{x_1} t_{y_1} t_{x_2 y_2}^{-1} \quad \text{and} \quad \sigma^{-1}(x, y) = t_{x_1 y_1} t_{x_2}^{-1} t_{y_2}^{-1}. \tag{3.1}$$

To simplify notation, we write \mathcal{B}_H for $\mathcal{B}_H^\varepsilon$, and \mathcal{A}_H for $\mathcal{A}_H^\varepsilon$.

3.1. On the subalgebra \mathcal{B}_H . We give some general properties of \mathcal{B}_H as a subalgebra of $S(t_H)_\Theta$, including flatness.

Let us start by analyzing the difference between the two algebras. To this end we consider the ideal (\mathcal{B}_H^+) of $S(t_H)_\Theta$ generated by the augmentation ideal $\mathcal{B}_H^+ = \mathcal{B}_H \cap \text{Ker}(\varepsilon)$. Let

$$S(t_H)_\Theta / (\mathcal{B}_H^+)$$

be the quotient algebra.

In order to identify this quotient, we consider the largest commutative quotient algebra H_{ab} of H : this is the quotient of H by the two-sided ideal generated by all commutators $xy - yx$ ($x, y \in H$). It is easily checked that H_{ab} is a quotient Hopf algebra of H ; another way to see this is to observe that H_{ab} represents the group-valued functor

$$R \longmapsto \text{Alg}_k(H, R)$$

on the category of commutative algebras R ; see [14].

We can now state the following.

Proposition 3.1. *The canonical coalgebra morphism $t_H \rightarrow H$; $t_x \mapsto x$, composed with the projection $H \rightarrow H_{\text{ab}}$, induces a Hopf algebra epimorphism $S(t_H)_\Theta \rightarrow H_{\text{ab}}$. This epimorphism has $(\mathcal{B}_H)^+$ as kernel. Therefore, $(\mathcal{B}_H)^+$ is a Hopf ideal of $S(t_H)_\Theta$, and we have an isomorphism*

$$S(t_H)_\Theta / (\mathcal{B}_H^+) \cong H_{\text{ab}}$$

of commutative Hopf algebras.

Proof. For $x \in H$ let t'_x be the image of t_x in $Q = S(t_H)_\Theta / (\mathcal{B}_H^+)$. Since $t_{x_1} t_{y_1} t_{x_2 y_2}^{-1} - \varepsilon(x)\varepsilon(y)$ belongs to \mathcal{B}_H^+ , we have

$$t'_{xy} = t'_x t'_y \in Q$$

for all $x, y \in H$. The algebra morphism $H \rightarrow Q; x \mapsto t'_x$ takes values in a commutative algebra; therefore, it factors through H_{ab} . It thus induces an algebra morphism $H_{\text{ab}} \rightarrow Q$.

On the other hand, by the universal property of $S(t_H)_\Theta$, the canonical Hopf algebra morphism $H \rightarrow H_{\text{ab}}; x \mapsto \bar{x}$ gives rise to a Hopf algebra morphism $S(t_H)_\Theta \rightarrow H_{\text{ab}}$ induced by $t_x \mapsto \bar{x}$. Since $\overline{x_1 y_1} S(\overline{x_2 y_2}) = \varepsilon(x)\varepsilon(y)$ in H_{ab} , the morphism $t_x \mapsto \bar{x}$ vanishes on \mathcal{B}_H^+ , yielding a Hopf algebra morphism $Q \rightarrow H_{\text{ab}}$. It is easy to check that the latter is the inverse of the morphism $H_{\text{ab}} \rightarrow Q$ constructed above. \square

We next identify a generating set for $S(t_H)_\Theta$ as a \mathcal{B}_H -module.

Lemma 3.2. *As a \mathcal{B}_H -module, $S(t_H)_\Theta$ is generated by the elements t_x , where x runs over H .*

Proof. Let M be the \mathcal{B}_H -submodule of $S(t_H)_\Theta$ generated by the elements t_x ($x \in H$). We wish to prove that $M = S(t_H)_\Theta$. Since $S(t_H)_\Theta$ is generated by the elements $t_x^{\pm 1}$ as an algebra, it suffices to check that each product $t_x t_y$ and each inverse t_x^{-1} are linear combinations of elements t_z ($z \in H$) with coefficients in \mathcal{B}_H . For the product $t_x t_y$, we have

$$\sigma(x_1, y_1) t_{x_2 y_2} = t_{x_1} t_{y_1} t_{x_2 y_2}^{-1} t_{x_3 y_3} = t_{x_1} t_{y_1} \varepsilon(x_2) \varepsilon(y_2) = t_x t_y.$$

This shows that $t_x t_y$ belongs to M .

Let us now check that t_x^{-1} belongs to M for all $x \in H$. By [1, Lemma 5.1] we have $t_1^{-1} = \sigma^{-1}(1, 1) \in \mathcal{B}_H$. Next, we claim that $t_{S(x_1)}^{-1} t_{x_2}^{-1}$ belongs to \mathcal{B}_H . Indeed, the element $\sigma^{-1}(S(x_1), x_2) t_1^{-1}$ of \mathcal{B}_H can be rewritten as

$$\begin{aligned} \sigma^{-1}(S(x_1), x_2) t_1^{-1} &= t_{S(x_2)x_3} t_{S(x_1)}^{-1} t_{x_4}^{-1} t_1^{-1} \\ &= \varepsilon(x_2) t_1 t_{S(x_1)}^{-1} t_{x_3}^{-1} t_1^{-1} \\ &= t_{S(x_1)}^{-1} t_{x_2}^{-1}, \end{aligned}$$

which proves the claim. Using the latter, we obtain

$$t_x^{-1} = \varepsilon(x_1) t_{x_2}^{-1} = \varepsilon(S(x_1)) t_{x_2}^{-1} = t_{S(x_1)} (t_{S(x_2)}^{-1} t_{x_3}^{-1}) \in \sum_{y \in H} \mathcal{B}_H t_y. \tag{3.2}$$

This shows that t_x^{-1} belongs to M . \square

Let us compute the coproduct and the counit on the values of σ .

Lemma 3.3. *In the commutative Hopf algebra $S(t_H)_\Theta$ we have*

$$\begin{aligned} \Delta(\sigma(x, y)) &= t_{x_1} t_{y_1} t_{x_3 y_3}^{-1} \otimes \sigma(x_2, y_2), \\ \Delta(\sigma^{-1}(x, y)) &= t_{x_1 y_1} t_{x_3}^{-1} t_{y_3}^{-1} \otimes \sigma^{-1}(x_2, y_2), \end{aligned}$$

and

$$\varepsilon(\sigma(x, y)) = \varepsilon(\sigma^{-1}(x, y)) = \varepsilon(x) \varepsilon(y)$$

for all $x, y \in H$.

Proof. Let us check the first formula. By (1.1) and (1.2),

$$\begin{aligned} \Delta(\sigma(x, y)) &= \Delta(t_{x_1}) \Delta(t_{y_1}) \Delta(t_{x_3 y_3}^{-1}) \\ &= (t_{x_1} \otimes t_{x_2}) (t_{y_1} \otimes t_{y_2}) (t_{x_4 y_4}^{-1} \otimes t_{x_3 y_3}^{-1}) \\ &= t_{x_1} t_{y_1} t_{x_4 y_4}^{-1} \otimes t_{x_2} t_{y_2} t_{x_3 y_3}^{-1} \\ &= t_{x_1} t_{y_1} t_{x_3 y_3}^{-1} \otimes \sigma(x_2, y_2). \end{aligned}$$

A similar computation yields the remaining formulas. □

As a consequence, we obtain the following.

Proposition 3.4. *For any Hopf algebra H , the commutative algebra \mathcal{B}_H is a left coideal subalgebra of $S(t_H)_\Theta$.*

If in addition H is cocommutative, then \mathcal{B}_H is a Hopf subalgebra of $S(t_H)_\Theta$.

Proof. Since \mathcal{B}_H is generated by the values of σ and σ^{-1} , the first assertion is a consequence of Lemma 3.3.

When H is cocommutative, then so is $S(t_H)_\Theta$, and $\Delta(\sigma(x, y))$ can be rewritten as

$$\Delta(\sigma(x, y)) = t_{x_1} t_{y_1} t_{x_2 y_2}^{-1} \otimes \sigma(x_3, y_3) = \sigma(x_1, y_1) \otimes \sigma(x_2, y_2).$$

Similarly, we have

$$\Delta(\sigma^{-1}(x, y)) = \sigma^{-1}(x_1, y_1) \otimes \sigma^{-1}(x_2, y_2).$$

This implies that \mathcal{B}_H is a sub-bialgebra of $S(t_H)_\Theta$.

It remains to show that \mathcal{B}_H is stable under the antipode S of $S(t_H)_\Theta$. By (1.3) and (3.1) we have

$$\begin{aligned} S(\sigma(x, y)) &= S(t_{x_2 y_2}^{-1}) S(t_{x_1}) S(t_{y_1}) = t_{x_2 y_2} t_{x_1}^{-1} t_{y_1}^{-1} \\ &= t_{x_1 y_1} t_{x_2}^{-1} t_{y_2}^{-1} = \sigma^{-1}(x, y). \end{aligned}$$

Since $S(t_H)_\Theta$ is cocommutative, the antipode is involutive and, as a consequence of the previous computation, we obtain

$$S(\sigma^{-1}(x, y)) = \sigma(x, y).$$

This proves that \mathcal{B}_H is a Hopf subalgebra of $S(t_H)_\Theta$. □

In general, a commutative Hopf algebra A is flat over any left (or right) coideal subalgebra B (see [9, Th. 3.4]). In view of Proposition 3.4, we can apply this to $A = S(t_H)_\Theta$ and $B = \mathcal{B}_H$, to obtain the following.

Theorem 3.5. *For any Hopf algebra H , the \mathcal{B}_H -module $S(t_H)_\Theta$ is flat.*

3.2. Faithful flatness. We give sufficient conditions for $S(t_H)_\Theta$ to be faithfully flat over \mathcal{B}_H . This faithful flatness is equivalent to the condition that $S(t_H)_\Theta$ is a projective generator as a \mathcal{B}_H -module, since in general, a commutative Hopf algebra is faithfully flat over a left (or right) coideal subalgebra if and only if it is a projective generator (see [9, Cor. 3.5] and also [9, Th. 2.1]).

Firstly, we show that the algebras have stronger properties in the finite-dimensional case.

Theorem 3.6. *Let H be a finite-dimensional Hopf algebra. Then \mathcal{B}_H is Noetherian and $S(t_H)_\Theta$ is a finitely generated projective \mathcal{B}_H -module. Moreover, it is a generator as a \mathcal{B}_H -module.*

It follows that $S(t_H)_\Theta$ is integral over \mathcal{B}_H . This answers Question 6.1 in [5] when H is finite-dimensional. Consequently, the results of [1, Sect. 7] hold unconditionally for any finite-dimensional Hopf algebra.

Proof. Since the algebra \mathcal{B}_H is generated by the finitely many elements $\sigma^{\pm 1}(x_i, x_j)$, where $(x_i)_i$ is an arbitrarily chosen linear basis of H , it is Noetherian.

Next, it follows from Lemma 3.2 that $S(t_H)_\Theta$ is generated as a \mathcal{B}_H -module by the elements t_{x_i} , where x_i runs through the chosen finite basis. Therefore, $S(t_H)_\Theta$ is finitely generated as a \mathcal{B}_H -module. Since \mathcal{B}_H is Noetherian, it follows that the \mathcal{B}_H -module $S(t_H)_\Theta$ is finitely presented.

We know from Theorem 3.5 that $S(t_H)_\Theta$ is a flat \mathcal{B}_H -module. This together with the finite presentation assertion implies that the module is finitely generated projective. The last assertion in the theorem is a consequence of the previous one. \square

Corollary 3.7. *If H is a finite-dimensional Hopf algebra, then the Krull dimension of \mathcal{B}_H is equal to $\dim_k H$.*

In other words, the affine algebraic variety $\text{Spec}(\mathcal{B}_H)$ has dimension $\dim_k H$.

Proof. The Krull dimension of $S(t_H)_\Theta$ is clearly equal to $\dim_k H$. Since $S(t_H)_\Theta$ is integral over \mathcal{B}_H , it follows from [10, Th. 20 in Sect. (13.C)] that \mathcal{B}_H and $S(t_H)_\Theta$ have the same Krull dimension. \square

Secondly, we have the following.

Theorem 3.8. *If H is a cocommutative Hopf algebra, then $S(t_H)_\Theta$ is faithfully flat over \mathcal{B}_H .*

Proof. Takeuchi proved that a cocommutative Hopf algebra A is a projective generator over any Hopf subalgebra B (see [9, Remark 3.6]). In view of Proposition 3.4, we may apply this to $A = S(t_H)_\Theta$ and $B = \mathcal{B}_H$, to obtain the desired result. \square

3.3. Freeness. We now give sufficient conditions for $S(t_H)_\Theta$ to be free over \mathcal{B}_H . Let us start with the following.

Theorem 3.9. *If H is a pointed Hopf algebra such that each element of the kernel of the natural epimorphism*

$$G(H)_{\text{ab}} \rightarrow G(H_{\text{ab}})$$

is of finite order, then the algebra $S(t_H)_\Theta$ is a free \mathcal{B}_H -module.

Here $G(H)_{\text{ab}}$ denotes the abelianization of the group $G(H)$. The finiteness condition in the theorem is satisfied for instance if all elements of $G(H)_{\text{ab}}$ are of finite order, or if $G(H)_{\text{ab}}$ is trivial. Theorem 3.9 generalizes [5, Th. 6.2].

In order to prove the theorem, we need the following preliminaries. If H is pointed, then so is $S(t_H)_\Theta$, and its grouplike elements form the free abelian group on the grouplike elements of H :

$$G(S(t_H)_\Theta) = \langle t_g \mid g \in G(H) \rangle_{\text{ab}}.$$

Let $Q = S(t_H)_\Theta / (\mathcal{B}_H^+)$ be the quotient Hopf algebra considered in Proposition 3.1. We denote by

$$C = S(t_H)_\Theta^{\text{co-}Q}$$

the left coideal subalgebra of $S(t_H)_\Theta$ consisting of all Q -coinvariants. Then C contains \mathcal{B}_H . Let

$$G(C) = C \cap G(S(t_H)_\Theta) \quad \text{and} \quad G(\mathcal{B}_H) = \mathcal{B}_H \cap G(S(t_H)_\Theta)$$

be the submonoids of $G(S(t_H)_\Theta)$ consisting of the grouplike elements contained in C and in \mathcal{B}_H , respectively. By definition of C ,

$$G(C) = \text{Ker}(G(S(t_H)_\Theta) \rightarrow G(Q)), \tag{3.3}$$

which is the kernel of the group homomorphism (indeed, epimorphism) induced by the projection $S(t_H)_\Theta \rightarrow Q \cong H_{\text{ab}}$. In particular, $G(C)$ is a group.

Lemma 3.10. *The group $G(C)$ is the subgroup of $G(S(t_H)_\Theta)$ generated by all elements of the form $t_g t_h t_{gh}^{-1}$, where $g, h \in G(H)$, and of the form t_f , where f belongs to the kernel of $G(H) \rightarrow G(H_{\text{ab}})$.*

Proof. Let G_0 be the subgroup of $G(S(t_H)_\Theta)$ generated by the elements $t_g t_h t_{gh}^{-1}$, where $g, h \in G(H)$. Since

$$t_{gh} \equiv t_g t_h, \quad t_1 \equiv 1, \quad t_{g^{-1}} \equiv t_g^{-1}, \quad t_{ghg^{-1}h^{-1}} \equiv 1 \tag{3.4}$$

modulo G_0 , we have the isomorphism

$$G(S(t_H)_\Theta) / G_0 \cong G(H)_{\text{ab}}.$$

Now let G_1 be the smallest subgroup containing G_0 and the elements t_f described in the lemma. We have the isomorphism

$$G(S(t_H)_\Theta) / G_1 \cong G(H_{\text{ab}}).$$

The lemma follows from this isomorphism and from (3.3). □

The following is a consequence of [7, Th. 1.3].

Lemma 3.11. *If H is a pointed Hopf algebra, then the following are equivalent:*

- (a) $\mathcal{B}_H \hookrightarrow S(t_H)_\Theta$ is faithfully flat;
- (b) $S(t_H)_\Theta$ is a free \mathcal{B}_H -module;
- (c) $G(\mathcal{B}_H)$ is closed under inverse;
- (d) $G(\mathcal{B}_H) = G(C)$;
- (e) $\mathcal{B}_H = C$.

Proof of Theorem 3.9. We will verify Condition (c) of Lemma 3.11. Let G_0 be as in the proof of Lemma 3.10. Pick any element $\gamma \in G(\mathcal{B}_H)$; we will prove that it has an inverse in $G(\mathcal{B}_H)$. Since $G(\mathcal{B}_H) \subset G(C)$, it follows from Lemma 3.10 and from (3.4) that

$$\gamma = \beta t_f$$

for some $\beta \in G_0$ and $f \in \text{Ker}(G(H) \rightarrow G(H_{\text{ab}}))$. We must prove $t_f^{-1} \in \mathcal{B}_H$. If the image of f in the kernel of $G(H)_{\text{ab}} \rightarrow G(H_{\text{ab}})$ has order $N < \infty$, then $f^N \equiv 1$ modulo $[G(H), G(H)]$. By (3.4), this implies that $t_f^N \equiv 1$, and so $t_f^{-1} \equiv t_f^{N-1}$ modulo G_0 . Since t_f^{N-1} belongs to \mathcal{B}_H , so does t_f^{-1} , as desired. \square

Question 3.12. Suppose that H is pointed. If $t_g \in \mathcal{B}_H$ for some $g \in G(H)$, does t_g^{-1} belong to \mathcal{B}_H , or equivalently, does $t_{g^{-1}}$ belong to \mathcal{B}_H ? If this is true, then following the proof of Theorem 3.9, we can conclude that $S(t_H)_\Theta$ is a free \mathcal{B}_H -module for any pointed Hopf algebra H .

We now give more sufficient conditions for $S(t_H)_\Theta$ to be free over \mathcal{B}_H .

Theorem 3.13. *Let H be a Hopf algebra. If H is commutative or pointed cocommutative, then the algebra $S(t_H)_\Theta$ is a free \mathcal{B}_H -module.*

Thus the conclusions of Theorem 7.2 and Corollary 7.3 of [1] hold true whenever H satisfies one of the hypotheses of Theorems 3.6, 3.8, 3.9, and 3.13.

Proof. (a) Assume that H is commutative. We shall prove that $\mathcal{A}_H = S(t_H)_\Theta$; since $\mathcal{A}_H = \mathcal{B}_H \otimes H$ as a \mathcal{B}_H -module, this implies the desired freeness. By the commutativity of H , the identity map of H induces a Hopf algebra epimorphism $S(t_H)_\Theta \rightarrow H$, by which we regard $S(t_H)_\Theta$ as a right H -comodule algebra. Since the natural map $H \rightarrow S(t_H)_\Theta$ is a coalgebra section of the epimorphism above, it follows from the clefness theorem of Doi and Takeuchi [3, Th. 9] that the section induces an isomorphism

$$S(t_H)_\Theta^{\text{co-}H} \otimes {}^\sigma H \xrightarrow{\cong} S(t_H)_\Theta$$

of H -comodule algebras over $S(t_H)_\Theta^{\text{co-}H}$. Here, the left-hand side denotes the twisted product associated to the generic two-cocycle σ . Therefore, $S(t_H)_\Theta^{\text{co-}H}$ contains \mathcal{B}_H , and $S(t_H)_\Theta$ naturally contains $\mathcal{A}_H = \mathcal{B}_H \otimes {}^\sigma H$ as an H -comodule subalgebra. But \mathcal{A}_H contains the generators of $S(t_H)_\Theta$ (note from (3.2) that t_x^{-1} is contained in \mathcal{A}_H). Therefore, $\mathcal{A}_H = S(t_H)_\Theta$, as desired.

(b) Assume that H is pointed cocommutative. Then by Kostant's theorem (see [11, Sect. 8.1]) the inclusion $kG(H) \hookrightarrow H$ splits as a Hopf algebra morphism.

It follows that

$$k[G(H)_{\text{ab}}] = (kG(H))_{\text{ab}} \rightarrow H_{\text{ab}}$$

is a split monomorphism; hence the natural epimorphism $G(H)_{\text{ab}} \rightarrow G(H_{\text{ab}})$ is an isomorphism. We can then apply Theorem 3.9. \square

Remarks 3.14. (a) Theorem 3.13 also holds true if H is a pointed Hopf algebra such that the inclusion $kG(H) \hookrightarrow H$ splits as an algebra morphism. In this case the proof above works as well.

(b) If $\mathcal{B}_H \hookrightarrow S(t_H)_\Theta$ is *faithfully flat*, then by [13, Th. 3] we have

$$\mathcal{B}_H = S(t_H)_\Theta^{\text{co-}Q}, \tag{3.5}$$

where $Q = S(t_H)_\Theta / (\mathcal{B}_H^+)$ as above. It follows that if $H_{\text{ab}} \cong Q$ is the trivial one-dimensional Hopf algebra, then $\mathcal{B}_H = S(t_H)_\Theta$. This applies for instance to the case where G is a group with *trivial abelianization* (e.g., to a non-abelian simple group), so that in this case,

$$\mathcal{B}_{kG} = S(t_{kG})_\Theta = k[t_g^{\pm 1} \mid g \in G],$$

the Laurent polynomial algebra in the indeterminates t_g ($g \in G$).

(c) Let G be a *finite* group and $H = O_k(G)$ the dual Hopf algebra of the group algebra kG . The elements of $O_k(G)$ can be seen as k -valued functions on G . In this case,

$$S(t_H)_\Theta = k[t_g \mid g \in G] \left[\frac{1}{\Theta_G} \right],$$

where $\Theta_G = \det(t_{gh^{-1}})_{g,h \in G}$ is *Dedekind's group determinant* (see [1, Ex. B.5]). Since $Q \cong H_{\text{ab}} = H$ and since H -coinvariants identify with G -invariants, it follows from (3.5) above that \mathcal{B}_H is the subalgebra of G -invariant elements of $S(t_H)_\Theta$:

$$\mathcal{B}_{kG} = (S(t_H)_\Theta)^G = k[t_g \mid g \in G] \left[\frac{1}{\Theta_G^2} \right],$$

where G acts on the elements t_g by translation. Incidentally, the lazy two-cocycles have been classified in this case by [4].

4. ANOTHER SET OF GENERATORS FOR \mathcal{B}_H

In [5, Sect. 7] a systematic method to exhibit elements of \mathcal{B}_H was given. Let us recall it.

Let H be a Hopf algebra and X_H a copy of the underlying vector space of H ; we denote the identity map from H to X_H by $x \mapsto X_x$ for all $x \in H$. Consider the tensor algebra $T(X_H)$. If $\{x_i\}_{i \in I}$ is a linear basis of H , then $T(X_H)$ is the free non-commutative algebra over the indeterminates $\{X_{x_i}\}_{i \in I}$. The natural right (resp. left) H -comodule structure on H extends to a right (resp. left) H -comodule algebra structure on $T(X_H)$.

Now, for $x, y \in H$ consider the following elements of $T(X_H)$:

$$P_x = X_{x_1} X_{S(x_2)}, \quad P'_x = X_{S(x_1)} X_{x_2}, \tag{4.1}$$

$$Q_{x,y} = X_{x_1} X_{y_1} X_{S(x_2 y_2)}, \quad Q'_{x,y} = X_{S(x_1 y_1)} X_{x_2} X_{y_2}. \tag{4.2}$$

It is easy to check that P_x and $Q_{x,y}$ are right coinvariant elements of $T(X_H)$, and that P'_x and $Q'_{x,y}$ are left coinvariant elements; see [1, Lemma 2.1] or [5, Prop. 7.1].

The non-commutative algebra $T(X_H)$ can be related to the commutative algebra $S(t_H)$ of Section 1 by the right H -comodule algebra morphism

$$\mu : T(X_H) \longrightarrow S(t_H) \otimes H ; X_x \longmapsto t_{x_1} \otimes x_2 .$$

By [1, Lemma 4.2] or [5, Prop. 7.2], the map μ is universal in the sense that any right H -comodule algebra morphism $T(X_H) \rightarrow H$ factors through μ .

Similarly, we can define an anti-algebra morphism

$$\mu' : T(X_H) \longrightarrow S(t_H)_\Theta \otimes H$$

by $\mu'(X_x) = t_{x_2}^{-1} \otimes S(x_1)$ for all $x \in H$. It is easy to check that μ' is the convolution inverse of μ .

Consider the following elements of $S(t_H)_\Theta \otimes H$:

$$p_x = \mu(P_x), \quad p'_x = \mu'(P'_x), \quad q_{x,y} = \mu(Q_{x,y}), \quad q'_{x,y} = \mu(Q'_{x,y}).$$

Lemma 4.1. *For all $x, y \in H$, we have*

$$\begin{aligned} p_x &= t_{x_1} t_{S(x_2)} \otimes 1, & p'_x &= t_{S(x_1)}^{-1} t_{x_2}^{-1} \otimes 1, \\ q_{x,y} &= t_{x_1} t_{y_1} t_{S(x_2 y_2)} \otimes 1, & q'_{x,y} &= t_{S(x_1 y_1)}^{-1} t_{x_2}^{-1} t_{y_2}^{-1} \otimes 1. \end{aligned}$$

Proof. We have

$$\begin{aligned} p_x &= \mu(X_{x_1}) \mu(X_{S(x_2)}) = t_{x_1} t_{S(x_4)} \otimes x_2 S(x_3) \\ &= t_{x_1} t_{S(x_3)} \otimes \varepsilon(x_2) 1 = t_{x_1} t_{S(x_2)} \otimes 1. \end{aligned}$$

Similar computations work for the other elements. □

It follows that the elements $p_x, p'_x, q_{x,y}, q'_{x,y}$ belong to the subalgebra $S(t_H)_\Theta \otimes 1$ of $S(t_H)_\Theta \otimes H$, which we identify with $S(t_H)_\Theta$. We can therefore drop the suffix $\otimes 1$ in the previous formulas.

Note that p_x and $q_{x,y}$ belong to the polynomial algebra $S(t_H)$ generated by the elements t_x , whereas p'_x and $q'_{x,y}$ belong to the subalgebra generated by the elements t_x^{-1} .

It follows from (1.3) and Lemma 4.1 that if H is cocommutative, then p'_x and $q'_{x,y}$ are the antipodes of p_x and $q_{x,y}$, respectively:

$$S(p'_x) = t_{S(x_1)} t_{x_2} = p_x \quad \text{and} \quad S(q'_{x,y}) = t_{S(x_1 y_1)} t_{x_2} t_{y_2} = q_{x,y} .$$

The following holds for any general Hopf algebra H .

Proposition 4.2. *For all $x, y \in H$, the elements $p_x, p'_x, q_{x,y}, q'_{x,y}$ belong to \mathcal{B}_H .*

This had been observed for the elements p_x and $q_{x,y}$ in [5, Prop. 7.3].

Proof. It suffices to express these elements in terms of the generators $\sigma^{\pm 1}(x, y)$ of \mathcal{B}_H . Using (3.1), it is easily checked that

$$\begin{aligned} p_x &= \sigma(x_1, S(x_2)) \sigma(1, 1), \\ p'_x &= \sigma^{-1}(S(x_1), x_2) \sigma^{-1}(1, 1), \\ q_{x,y} &= \sigma(x_1, y_1) \sigma(x_2 y_2, S(x_3 y_3)) \sigma(1, 1), \\ q'_{x,y} &= \sigma^{-1}(S(x_1 y_1), x_2 y_2) \sigma^{-1}(x_3, y_3) \sigma^{-1}(1, 1). \end{aligned}$$

(The second equality has already been established in the proof of Lemma 3.2.) The conclusion follows. \square

We now express the values of σ and of σ^{-1} in terms of the previous elements.

Proposition 4.3. *For all $x, y \in H$ we have*

$$\sigma(x, y) = q_{x_1, y_1} p'_{x_2 y_2} \quad \text{and} \quad \sigma^{-1}(x, y) = p_{x_1 y_1} q'_{x_2, y_2}.$$

Proof. In view of (3.1) again, we have

$$\begin{aligned} q_{x_1, y_1} p'_{x_2 y_2} &= t_{x_1} t_{y_1} t_{S(x_2 y_2)} t_{S(x_3 y_3)}^{-1} t_{x_4 y_4}^{-1} \\ &= t_{x_1} t_{y_1} t_{x_2 y_2}^{-1} = \sigma(x, y). \end{aligned}$$

Similarly,

$$\begin{aligned} p_{x_1 y_1} q'_{x_2, y_2} &= t_{x_1 y_1} t_{S(x_2 y_2)} t_{S(x_3 y_3)}^{-1} t_{x_4}^{-1} t_{y_4}^{-1} \\ &= t_{x_1 y_1} t_{x_2}^{-1} t_{y_2}^{-1} = \sigma^{-1}(x, y). \end{aligned}$$

\square

Corollary 4.4. *For any Hopf algebra H , the algebra \mathcal{B}_H is generated by the elements $p_x, p'_x, q_{x,y}, q'_{x,y}$ ($x, y \in H$).*

The theory of polynomial identities worked out in [1] and the specific computations in the case of the Sweedler algebra (see [1, Sect. 10] and [5, Sect. 4 & Ex. 7.4]) suggest that the generators $p_x, p'_x, q_{x,y}, q'_{x,y}$ are more natural than the values of $\sigma^{\pm 1}$.

Corollary 4.5. *If H is a cocommutative Hopf algebra, then \mathcal{B}_H is generated by the elements $p_x, q_{x,y}$ and their antipodes.*

Proof. In this case we have already observed that p'_x and $q'_{x,y}$ are the antipodes of p_x and $q_{x,y}$, respectively. \square

Remark 4.6. It follows from Corollaries 2.2 and 4.5 that any non-degenerate invertible two-cocycle on a cocommutative Hopf algebra is *nice* in the sense of [1, Sect. 9].

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