

## TENSOR CATEGORIES: A SELECTIVE GUIDED TOUR

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ABSTRACT. These are the, somewhat polished and updated, lecture notes for a three hour course on tensor categories, given at the CIRM, Marseille, in April 2008. The coverage in these notes is relatively non-technical, focusing on the essential ideas. They are meant to be accessible for beginners, but it is hoped that also some of the experts will find something interesting in them.

Once the basic definitions are given, the focus is mainly on categories that are linear over a field  $k$  and have finite dimensional hom-spaces. Connections with quantum groups and low dimensional topology are pointed out, but these notes have no pretension to cover the latter subjects to any depth. Essentially, these notes should be considered as annotations to the extensive bibliography. We also recommend the recent review [43], which covers less ground in a deeper way.

### 1. TENSOR CATEGORIES

These informal notes are an outgrowth of the three hours of lectures that I gave at the Centre International de Rencontres Mathématiques, Marseille, in April 2008. The original version of text was projected to the screen and therefore kept maximally concise. For this publication, I have corrected the language where needed, but no serious attempt has been made to make these notes conform with the highest standards of exposition. I still believe that publishing them in this form has a purpose, even if only providing some pointers to the literature.

1.1. **Strict tensor categories.** We begin with strict tensor categories, despite their limited immediate applicability.

- We assume that the reader has a working knowledge of categories, functors and natural transformations. Cf. the standard reference [180]. Instead of  $s \in \text{Hom}(X, Y)$  we will occasionally write  $s : X \rightarrow Y$ .
- We are interested in “categories with multiplication”. (This was the title of a paper [24] by Bénabou 1963, cf. also Mac Lane [178] from the same year). This term was soon replaced by ‘monoidal categories’ or ‘tensor categories’. (We use these synonymously.) It is mysterious to this author why the explicit formalization of tensor categories took twenty years to arrive after that of categories, in particular since monoidal categories appear in protean form, e.g., in Tannaka’s work [255].
- A **strict tensor category** (strict monoidal category) is a triple  $(\mathcal{C}, \otimes, \mathbf{1})$ , where  $\mathcal{C}$  is a category,  $\mathbf{1}$  a distinguished object and  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  is a

functor, satisfying

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \quad \text{and} \quad X \otimes \mathbf{1} = X = \mathbf{1} \otimes X \quad \forall X, Y, Z.$$

If  $(\mathcal{C}, \otimes, \mathbf{1}), (\mathcal{C}', \otimes', \mathbf{1}')$  are strict tensor categories, a **strict tensor functor**  $\mathcal{C} \rightarrow \mathcal{C}'$  is a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$  such that

$$F(X \otimes Y) = F(X) \otimes' F(Y), \quad F(\mathbf{1}) = \mathbf{1}'.$$

If  $F, F' : \mathcal{C} \rightarrow \mathcal{C}'$  are strict tensor functors, a natural transformation  $\alpha : F \rightarrow F'$  is **monoidal** if and only if  $\alpha_{\mathbf{1}} = \text{id}_{\mathbf{1}'}$  and

$$\alpha_{X \otimes Y} = \alpha_X \otimes \alpha_Y \quad \forall X, Y \in \mathcal{C}.$$

(Both sides live in  $\text{Hom}(F(X \otimes Y), F'(X \otimes Y)) = \text{Hom}(F(X) \otimes' F(Y), F'(X) \otimes' F'(Y))$ .)

- **WARNING:** The coherence theorems, to be discussed in a bit more detail in Subsection 1.2, will imply that, in a sense, strict tensor categories are sufficient for all purposes. However, even when dealing with strict tensor categories, one needs non-strict tensor functors!
- **Basic examples:**
  - Let  $\mathcal{C}$  be any category and let  $\text{End } \mathcal{C}$  be the category of functors  $\mathcal{C} \rightarrow \mathcal{C}$  and their natural transformations. Then  $\text{End } \mathcal{C}$  is a strict  $\otimes$ -category, with composition of functors as tensor product. It is also denoted as the ‘center’  $Z_0(\mathcal{C})$ . (The subscript is needed since various other centers will be encountered.)
  - To every group  $G$ , we associate the discrete tensor category  $\mathcal{C}(G)$ :

$$\text{Obj } \mathcal{C}(G) = G, \quad \text{Hom}(g, h) = \begin{cases} \{\text{id}_g\} & g = h \\ \emptyset & g \neq h \end{cases}, \quad g \otimes h = gh.$$

- The **symmetric category**  $\mathbb{S}$ :

$$\text{Obj } \mathbb{S} = \mathbb{Z}_+, \quad \text{Hom}(n, m) = \begin{cases} S_n & n = m \\ \emptyset & n \neq m \end{cases}, \quad n \otimes m = n + m$$

with tensor product of morphisms given by the obvious map  $S_n \times S_m \rightarrow S_{n+m}$ .

Remark: 1.  $\mathbb{S}$  is the free symmetric tensor category on one monoidal generator.

2.  $\mathbb{S}$  is equivalent to the category of finite sets and bijective maps.

2. This construction works with any family  $(G_i)$  of groups with an associative composition  $G_i \times G_j \rightarrow G_{i+j}$ .

- Let  $\mathcal{A}$  be a unital associative algebra with unit over some field. We define  $\text{End } \mathcal{A}$  to have as objects the unital algebra homomorphisms  $\rho : \mathcal{A} \rightarrow \mathcal{A}$ . The morphisms are defined by

$$\text{Hom}(\rho, \sigma) = \{x \in \mathcal{A} \mid x\rho(y) = \sigma(y)x \quad \forall y \in \mathcal{A}\}$$

with  $s \circ t = st$  and  $s \otimes t = s\rho(t) = \rho'(t)s$  for  $s \in \text{Hom}(\rho, \rho'), t \in \text{Hom}(\sigma, \sigma')$ . This construction has important applications in in subfactor theory [169] and (algebraic) quantum field theory [68, 90]. Yamagami [284] proved that every countably generated  $C^*$ -tensor category with conjugates (cf. below) embeds fully into  $\text{End } \mathcal{A}$  for some von Neumann-algebra  $\mathcal{A} = \mathcal{A}(\mathcal{C})$ . (See the final section for a conjecture concerning an algebra that should work for all such categories.)

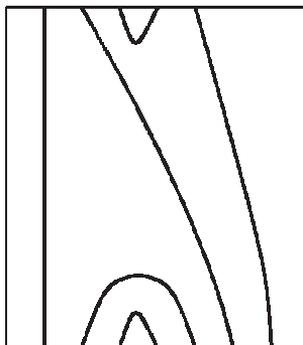
- The **Temperley-Lieb categories**  $\mathcal{TL}(\tau)$ . (Cf. e.g. [107].) Let  $k$  be a field and  $\tau \in k^*$ . We define

$$\text{Obj } \mathcal{TL}(\tau) = \mathbb{Z}_+, \quad n \otimes m = n + m,$$

as for the free symmetric category  $\mathcal{S}$ . But now

$$\text{Hom}(n, m) = \text{span}_k\{\text{Isotopy classes of } (n, m)\text{-TL diagrams}\}.$$

Here, an  $(n, m)$ -diagram is a planar diagram where  $n$  points on a line and  $m$  points on a parallel line are connected by lines without crossings. The following example of a  $(7, 5)$ -TL diagram will explain this sufficiently:

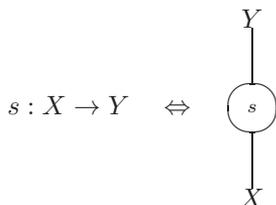


The tensor product of morphisms is given by horizontal juxtaposition, whereas composition of morphisms is defined by vertical juxtaposition, followed by removal all newly formed closed circles and multiplication by a factor  $\tau$  for each circle. (This makes sense since the category is  $k$ -linear.)

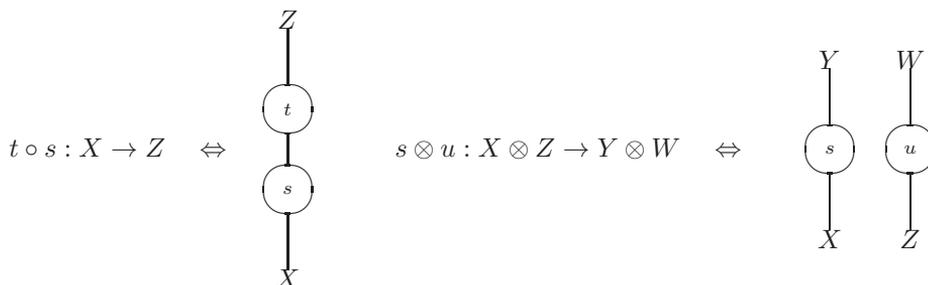
Remark: 1. The Temperley-Lieb algebras  $\text{TL}(n, \tau) = \text{End}_{\mathcal{TL}(\tau)}(n)$  first appeared in the theory of exactly soluble lattice models of statistical mechanics. They, as well as  $\mathcal{TL}(\tau)$  are closely related to the Jones polynomial [127] and the quantum group  $SL_q(2)$ . Cf. [262, Chapter XII].

2. The Temperley-Lieb algebras, as well as the categories  $\mathcal{TL}(\tau)$  can be defined purely algebraically in terms of generators and relations.

- In dealing with (strict) tensor categories, it is often convenient to adopt a graphical notation for morphisms:



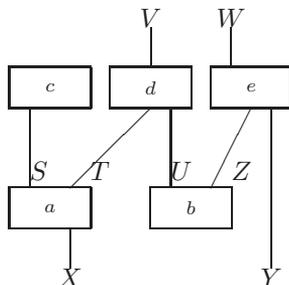
If  $s : X \rightarrow Y, t : Y \rightarrow Z, u : Z \rightarrow W$  then we write



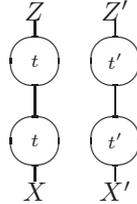
The usefulness of this notation becomes apparent when there are morphisms with ‘different numbers of in- and outputs’: Let, e.g.,  $a : X \rightarrow S \otimes T, b : \mathbf{1} \rightarrow U \otimes Z, c : S \rightarrow \mathbf{1}, d : T \otimes U \rightarrow V, e : Z \otimes Y \rightarrow W$  and consider the composite morphism

$$c \otimes d \otimes e \circ a \otimes b \otimes \text{id}_Y : X \otimes Y \rightarrow V \otimes W. \tag{1.1}$$

This formula is almost unintelligible. (In order to economize on brackets, we follow the majority of authors and declare  $\otimes$  to bind stronger than  $\circ$ , i.e.  $a \circ b \otimes c \equiv a \circ (b \otimes c)$ . Notice that inserting brackets in (1.1) does nothing to render the formula noticeably more intelligible.) It is not even clear whether it represents a morphism in the category. This is immediately obvious from the diagram:



Often, there is more than one way to translate a diagram into a formula, e.g.



can be read as  $t \otimes t' \circ s \otimes s'$  or as  $(t \circ s) \otimes (t' \circ s')$ . But by the interchange law (which is just the functoriality of  $\otimes$ ), these two morphisms coincide. For proofs of consistency of the formalism, cf. [129, 94] or [137].

1.2. Non-strict tensor categories.

- For almost all situations where tensor categories arise, strict tensor categories are not general enough, the main reasons being:
  - Requiring equality of objects as in  $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$  is highly unnatural from a categorical point of view.
  - Many would-be tensor categories are not strict; in particular this is the case for  $\text{Vect}_k$ , as well as for representation categories of groups (irrespective of the class of groups and representations under consideration).
- The obvious minimal modification, namely to require only existence of isomorphisms  $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$  for all  $X, Y, Z$  and  $\mathbf{1} \otimes X \cong X \cong X \otimes \mathbf{1}$  for all  $X$ , turns out to be too weak to be useful.
- The correct definition of not-necessarily-strict **tensor categories** was given in [24]: It is a sextuplet  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ , where  $\mathcal{C}$  is a category,  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  a functor,  $\mathbf{1}$  an object, and  $\alpha : \otimes \circ (\otimes \times \text{id}) \rightarrow \otimes \circ (\text{id} \times \otimes)$ ,  $\lambda : \mathbf{1} \otimes - \rightarrow \text{id}$ ,  $\rho : - \otimes \mathbf{1} \rightarrow \text{id}$  are natural isomorphisms (i.e., for all  $X, Y, Z$  we have isomorphisms  $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$  and  $\lambda_X : \mathbf{1} \otimes X \rightarrow X$ ,  $\rho_X : X \otimes \mathbf{1} \rightarrow X$ ) such that all morphisms between the same pair of objects that can be built from  $\alpha, \lambda, \rho$  coincide. (Examples of what this means are given by the commutativity of the following two diagrams.)
- There are two versions of the coherence theorem for tensor categories: Version I (Mac Lane [178, 180]): All morphisms built from  $\alpha, \lambda, \rho$  are unique provided  $\alpha$  satisfies the pentagon identity, i.e. commutativity of

$$\begin{array}{ccc}
 ((X \otimes Y) \otimes Z) \otimes T & \xrightarrow{\alpha_{X,Y,Z} \otimes \text{id}_T} & (X \otimes (Y \otimes Z)) \otimes T & \xrightarrow{\alpha_{X,Y \otimes Z, T}} & X \otimes ((Y \otimes Z) \otimes T) \\
 \downarrow \alpha_{X \otimes Y, Z, T} & & & & \downarrow \text{id}_X \otimes \alpha_{Y, Z, T} \\
 (X \otimes Y) \otimes (Z \otimes T) & \xrightarrow{\alpha_{X, Y, Z \otimes T}} & & & X \otimes (Y \otimes (Z \otimes T))
 \end{array}$$

and  $\lambda, \rho$  satisfy the unit identity

$$\begin{array}{ccc}
 (X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\alpha_{X, \mathbf{1}, Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 \rho_X \otimes \text{id}_Y \downarrow & & \downarrow \text{id}_X \otimes \lambda_Y \\
 X \otimes Y & \equiv \equiv \equiv \equiv \equiv & X \otimes Y
 \end{array}$$

For modern expositions of the coherence theorem see [180, 137]. (Notice that the original definition of non-strict tensor categories given in [178] was modified in slightly [146, 147].)

- Examples of non-strict tensor categories:
  - Let  $\mathcal{C}$  be a category with products and terminal object  $T$ . Define  $X \otimes Y = X \amalg Y$  (for each pair  $X, Y$  choose a product, non-uniquely) and  $\mathbf{1} = T$ . Then  $(\mathcal{C}, \otimes, \mathbf{1})$  is non-strict tensor category. (Existence of associator and unit isomorphisms follows from the universal properties of product and terminal object). An analogous construction works with coproduct and initial object.
  - $\text{Vect}_k$  with  $\alpha_{U, V, W}$  defined on simple tensors by  $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$ . Note: This trivially satisfies the pentagon identity, but the other choice  $(u \otimes v) \otimes w \mapsto -u \otimes (v \otimes w)$  does not!
  - Let  $G$  be a group,  $A$  an abelian group (written multiplicatively) and  $\omega \in Z^3(G, A)$ , i.e.

$$\omega(h, k, l)\omega(g, hk, l)\omega(g, h, k) = \omega(gh, k, l)\omega(g, h, kl) \quad \forall g, h, k, l \in G.$$

Define  $\mathcal{C}(G, \omega)$  by

$$\text{Obj } \mathcal{C} = G, \quad \text{Hom}(g, h) = \begin{cases} A & g = h \\ \emptyset & g \neq h \end{cases}, \quad g \otimes h = gh.$$

with associator  $\alpha = \omega$ , cf. [245]. If  $k$  is a field,  $A = k^*$ , one has a

$k$ -linear version where  $\text{Hom}(g, h) = \begin{cases} k & g = h \\ \{0\} & g \neq h \end{cases}$ . I denote this by

$\mathcal{C}_k(G, \omega)$ , but also  $\text{Vect}_\omega^G$  appears in the literature.

The importance of this example lies in its showing relations between categories and cohomology, which are reinforced by ‘higher category theory’, cf. e.g. [14]. But also the concrete example is relevant for the classification of fusion categories, at least the large class of ‘group theoretical categories’. (Cf. Ostrik et al. [223, 84].) See Section 3.

- A categorical group is a tensor category that is a groupoid (all morphisms are invertible) and where every object has a tensor-inverse, i.e. for every  $X$  there is an object  $\overline{X}$  such that  $X \otimes \overline{X} \cong \mathbf{1}$ . The categories  $\mathcal{C}(G, \omega)$  are just the skeletal categorical groups.
- Now we can give the general definition of a tensor functor (between non-strict tensor categories or non-strict tensor functors between strict tensor categories): A tensor functor between tensor categories  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ ,

$(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$  consists of a functor  $F : \mathcal{C} \rightarrow \mathcal{C}'$ , an isomorphism  $e^F : F(\mathbf{1}) \rightarrow \mathbf{1}'$  and a family of natural isomorphisms  $d_{X,Y}^F : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$  satisfying commutativity of

$$\begin{array}{ccccc}
 (F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{d_{X,Y} \otimes \text{id}} & F(X \otimes Y) \otimes' F(Z) & \xrightarrow{d_{X \otimes Y, Z}} & F((X \otimes Y) \otimes Z) \\
 \downarrow \alpha'_{F(X), F(Y), F(Z)} & & & & \downarrow F(\alpha_{X,Y,Z}) \\
 F(X) \otimes' (F(Y) \otimes' F(Z)) & \xrightarrow{\text{id} \otimes d_{Y,Z}} & F(X) \otimes' F(Y \otimes Z) & \xrightarrow{d_{X, Y \otimes Z}} & F(X \otimes (Y \otimes Z))
 \end{array}$$

(notice that this is a 2-cocycle condition, in particular when  $\alpha \equiv \text{id}$ ) and

$$\begin{array}{ccc}
 F(X) \otimes F(\mathbf{1}) & \xrightarrow{\text{id} \otimes e^F} & F(X) \otimes \mathbf{1}' \\
 \downarrow d_{X,\mathbf{1}}^F & & \downarrow \rho'_{F(X)} \\
 F(X \otimes \mathbf{1}) & \xrightarrow{F(\rho_X)} & F(X)
 \end{array}
 \quad \text{(and similar for } \lambda_X)$$

Remark: Occasionally, functors as defined above are called **strong** tensor functors in order to distinguish them from the lax variant, where the  $d_{X,Y}^F$  and  $e^F$  are not required to be isomorphisms. (In this case it also makes sense to consider  $d^F, e^F$  with source and target exchanged.)

- Let  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho)$ ,  $(\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$  be tensor categories and  $(F, d, e)$ ,  $(F', d', e') : \mathcal{C} \rightarrow \mathcal{C}'$  tensor functors. Then a natural transformation  $\alpha : F \rightarrow F'$  is monoidal if

$$\begin{array}{ccc}
 F(X) \otimes' F(Y) & \xrightarrow{d_{X,Y}} & F(X \otimes Y) \\
 \alpha_X \otimes \alpha_Y \downarrow & & \downarrow \alpha_{X \otimes Y} \\
 F'(X) \otimes' F'(Y) & \xrightarrow{d'_{X,Y}} & F'(X \otimes Y)
 \end{array}$$

For strict tensor functors, we have  $d \equiv \text{id} \equiv d'$ , and we obtain the earlier condition.

- A tensor functor  $F : (\mathcal{C}, \otimes, \mathbf{1}, \alpha, \lambda, \rho) \rightarrow (\mathcal{C}', \otimes', \mathbf{1}', \alpha', \lambda', \rho')$  is called an equivalence if there exist a tensor functor  $G : \mathcal{C}' \rightarrow \mathcal{C}$  and natural monoidal isomorphisms  $\alpha : G \circ F \rightarrow \text{id}_{\mathcal{C}}$  and  $\beta : F \circ G \rightarrow \text{id}_{\mathcal{C}'}$ . For the existence of such a  $G$  it is necessary and sufficient that  $F$  be full, faithful and essentially surjective (and of course monoidal), cf. [238]. (We follow the practice of not worrying too much about size issues and assuming a sufficiently strong version of the axiom of choice for classes. On this matter, cf. the different discussions of foundational issues given in the two editions of [180].)

- Given a group  $G$  and  $\omega, \omega' \in Z^3(G, A)$ , the identity functor is part of a monoidal equivalence  $\mathcal{C}(G, \omega) \rightarrow \mathcal{C}(G, \omega')$  if and only if  $[\omega] = [\omega']$  in  $H^3(G, A)$ . Cf. e.g. [54, Chapter 2]. Since categorical groups form a 2-category  $\mathcal{CG}$ , they are best classified by providing a 2-equivalence between  $\mathcal{CG}$  and a 2-category  $\mathcal{H}^3$  defined in terms of cohomology groups  $H^3(G, A)$ . The details are too involved to give here; cf. [128]. (Unfortunately, the theory of categorical groups is marred by the fact that important works [245, 128] were never formally published. For a comprehensive recent treatment cf. [12].)
- Version II of the Coherence theorem (equivalent to Version I): Every tensor category is monoidally equivalent to a strict one. [180, 137]. As mentioned earlier, this allows us to pretend that all tensor categories are strict. (But we cannot restrict ourselves to strict tensor functors!)
- One may ask what the strictification of  $\mathcal{C}(G, \omega)$  looks like. The answer is somewhat complicated, cf. [128]: It involves the free group on the set underlying  $G$ . (This shows that sometimes it is actually more convenient to work with non-strict categories!)
- As shown in [241], many non-strict tensor categories can be turned into equivalent strict ones by changing only the tensor functor  $\otimes$ , but *leaving the underlying category unchanged*.
- We recall the “Eckmann-Hilton argument”: If a set has two monoid structures  $\star_1, \star_2$  satisfying  $(a \star_2 b) \star_1 (c \star_2 d) = (a \star_1 c) \star_2 (b \star_1 d)$  with the same unit, the two products coincide and are commutative. If  $\mathcal{C}$  is a tensor category and we consider  $\text{End } \mathbf{1}$  with  $\star_1 = \circ, \star_2 = \otimes$  we find that  $\text{End } \mathbf{1}$  is commutative, cf. [148]. In the Ab- ( $k$ -linear) case, defined in Subsection 1.6,  $\text{End } \mathbf{1}$  is a commutative unital ring ( $k$ -algebra). (Another classical application of the Eckmann-Hilton argument is the abelianness of the higher homotopy groups  $\pi_n(X), n \geq 2$  and of  $\pi_1(M)$  for a topological monoid  $M$ .)

### 1.3. Generalization: 2-categories and bicategories.

- Tensor categories have a very natural and useful generalization. We begin with ‘2-categories’, which generalize strict tensor categories: A 2-category  $\mathcal{E}$  consists of a set (class) of objects and, for every  $X, Y \in \text{Obj } \mathcal{E}$ , a category  $\text{HOM}(X, Y)$ . The objects (morphisms) in  $\text{HOM}(X, Y)$  are called 1-morphisms (2-morphisms) of  $\mathcal{E}$ . For the detailed axioms we refer to the references given below. In particular, we have functors  $\circ : \text{HOM}(\mathfrak{A}, \mathfrak{B}) \times \text{HOM}(\mathfrak{B}, \mathfrak{C}) \rightarrow \text{HOM}(\mathfrak{A}, \mathfrak{C})$ , and  $\circ$  is associative (on the nose).
- The prototypical example of a 2-category is the 2-category  $\mathcal{CAT}$ . Its objects are the small categories, its 1-morphisms are functors and the 2-morphisms are natural transformations.
- We notice that if  $\mathcal{E}$  is a 2-category and  $X \in \text{Obj } \mathcal{E}$ , then  $\text{END}(X) = \text{HOM}(X, X)$  is a strict tensor category. This leads to the non-strict version of 2-categories called bicategories: We replace the associativity of the composition  $\circ$  of 1-morphisms by the existence of invertible 2-morphisms

$(X \circ Y) \circ Z \rightarrow X \circ (Y \circ Z)$  satisfying axioms generalizing those of a tensor category. Now, if  $\mathcal{E}$  is a bicategory and  $X \in \text{Obj } \mathcal{E}$ , then  $\text{END}(X) = \text{HOM}(X, X)$  is a (non-strict) tensor category. Bicategories are a very important generalization of tensor categories, and we'll meet them again. Also the relation between bicategories and tensor categories is prototypical for 'higher category theory'.

References: [150] for 2-categories and [26] for bicategories, as well as the very recent review by Lack [162].

**1.4. Categorification of monoids.** Tensor categories (or monoidal categories) can be considered as the categorification of the notion of a monoid. This has interesting consequences:

- Monoids in monoidal categories: Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be a strict  $\otimes$ -category. A monoid in  $\mathcal{C}$  (Bénabou [25]) is a triple  $(A, m, \eta)$  with  $A \in \mathcal{C}$ ,  $m : A \otimes A \rightarrow A$ ,  $\eta : \mathbf{1} \rightarrow A$  satisfying

$$m \circ m \otimes \text{id}_A = m \circ \text{id}_A \otimes m, \quad m \circ \eta \otimes \text{id}_A = \text{id}_A = m \circ \text{id}_A \otimes \eta.$$

(In the non-strict case, insert an associator at the obvious place.) A monoid in  $\text{Ab}$  ( $\text{Vect}_k$ ) is a ring ( $k$ -algebra). Therefore, in the recent literature monoids are often called 'algebras'.

Monoids in monoidal categories are a prototypical example of the 'microcosm principle' of Baez and Dolan [11] affirming that "certain algebraic structures can be defined in any category equipped with a categorified version of the same structure".

- If  $\mathcal{C}$  is any category, monoids in the tensor category  $\text{End } \mathcal{C}$  are known as 'monads'. As such they are older than tensor categories! Cf. [180].
- If  $(A, m, \eta)$  is a monoid in the strict tensor category  $\mathcal{C}$ , a left  $A$ -module is a pair  $(X, \mu)$ , where  $X \in \mathcal{C}$  and  $\mu : A \otimes X \rightarrow X$  satisfies

$$\mu \circ m \otimes \text{id}_X = \mu \circ \text{id}_A \otimes \mu, \quad \mu \circ \eta \otimes \text{id}_X = \text{id}_X.$$

Together with the obvious notion of  $A$ -module morphism

$$\text{Hom}_{A\text{-Mod}}((X, \mu), (X', \mu')) = \{s \in \text{Hom}_{\mathcal{C}}(X, X') \mid s \circ \mu = \mu' \circ \text{id}_A \otimes s\},$$

$A$ -modules form a category. Right  $A$ -modules and  $A - A$  bimodules are defined analogously.

The free  $A$ -module of rank 1 is just  $(A, m)$ .

- If  $\mathcal{C}$  is abelian, then  $A - \text{Mod}_{\mathcal{C}}$  is abelian under weak assumptions, cf. [6]. (The latter are satisfied when  $A$  has duals, as e.g. when it is a strongly separable Frobenius algebra [98]. All this could also be deduced from [76].)
- Every monoid  $(A, m, \eta)$  in  $\mathcal{C}$  gives rise to a monoid  $\Gamma_A = \text{Hom}(\mathbf{1}, A)$  in the category  $\mathcal{SET}$  of sets. We call it the **elements of  $A$** . ( $\Gamma_A$  is related to the endomorphisms of the unit object in the tensor categories of  $A - A$  bimodules and  $A$ -modules (in the braided case), when the latter exist.)
- Let  $\mathcal{C}$  be abelian and  $(A, m, \eta)$  an algebra in  $\mathcal{C}$ . An **ideal** in  $A$  is an  $A$ -module  $(X, \mu)$  together with a monic morphism  $(X, \mu) \hookrightarrow (A, m)$ . Much as in ordinary algebra, one can define a quotient algebra  $A/I$ . Furthermore,

every ideal is contained in a maximal ideal, and an ideal  $I \subset A$  in a commutative monoid is maximal if and only if the ring  $\Gamma_{A/I}$  is a field. (For the last claim, cf. [197].)

- **Coalgebras** and their comodules are defined analogously. In a tensor category equipped with a symmetry or braiding  $c$  (cf. below), it makes sense to say that an (co)algebra is (co)**commutative**. For an algebra  $(A, m, \eta)$  this means that  $m \circ c_{A,A} = m$ .
- (B) Just as monoids can act on sets, tensor categories can act on categories: Let  $\mathcal{C}$  be a tensor category. A left  **$\mathcal{C}$ -module category** is a pair  $(\mathcal{M}, F)$  where  $\mathcal{M}$  is a category and  $F : \mathcal{C} \rightarrow \text{End } \mathcal{M}$  is a tensor functor. (Here,  $\text{End } \mathcal{M}$  is as in our first example of a tensor category.) This is equivalent to having a functor  $F' : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$  and natural isomorphisms  $\beta_{X,Y,A} : F'(X \otimes Y, A) \rightarrow F(X, F(Y, A))$  satisfying a pentagon-type coherence law, unit constraints, etc. Now one can define indecomposable module categories, etc. (Ostrik [222])
- There is a close connection between module categories and categories of modules:

If  $(A, m, \eta)$  is an algebra in  $\mathcal{C}$ , then there is a natural right  $\mathcal{C}$ -module structure on the category  $A - \text{Mod}_{\mathcal{C}}$  of left  $A$ -modules:

$$F' : A - \text{Mod}_{\mathcal{C}} \times \mathcal{C}, \quad (M, \mu) \times X \mapsto (M \otimes X, \mu \otimes \text{id}_X).$$

(In the case where  $(M, \mu)$  is the free rank-one module  $(A, m)$ , this gives the free  $A$ -modules  $F'((A, m), X) = (A \otimes X, m \otimes \text{id}_X)$ .) For a fusion category (cf. below), one can show that every semisimple indecomposable left  $\mathcal{C}$ -module category arises in this way from an algebra in  $\mathcal{C}$ , cf. [222].

### 1.5. Duality in tensor categories I.

- If  $G$  is a group and  $\pi$  a representation on a finite dimensional vector space  $V$ , we define the ‘dual’ or ‘conjugate’ representation  $\bar{\pi}$  on the dual vector space  $V^*$  by  $\langle \bar{\pi}(g)\phi, x \rangle = \langle \phi, \pi(g)x \rangle$ . Denoting by  $\pi_0$  the trivial representation, one finds  $\text{Hom}_{\text{Rep } G}(\pi \otimes \bar{\pi}, \pi_0) \cong \text{Hom}_{\text{Rep } G}(\pi, \pi)$ , implying  $\pi \otimes \bar{\pi} \succ \pi_0$ . If  $\pi$  is irreducible, then so is  $\bar{\pi}$  and the multiplicity of  $\pi_0$  in  $\pi \otimes \bar{\pi}$  is one by Schur’s lemma.

Since the above discussion is quite specific to the group situation, it clearly needs to be generalized.

- Let  $(\mathcal{C}, \otimes, \mathbf{1})$  be a strict tensor category and  $X, Y \in \mathcal{C}$ . We say that  $Y$  is a **left dual** of  $X$  if there are morphisms  $e : Y \otimes X \rightarrow \mathbf{1}$  and  $d : \mathbf{1} \rightarrow X \otimes Y$  satisfying

$$\text{id}_X \otimes e \circ d \otimes \text{id}_X = \text{id}_X, \quad e \otimes \text{id}_Y \circ \text{id}_Y \otimes d = \text{id}_Y,$$

or, representing  $e : Y \otimes X \rightarrow \mathbf{1}$  and  $d : \mathbf{1} \rightarrow X \otimes Y$  by  and , respectively,

$$\begin{array}{c} X \\ | \\ d \text{---} \cup \text{---} Y \\ | \\ X \end{array} \quad e = \begin{array}{c} X \\ | \\ X \end{array} \qquad e \text{---} \cap \text{---} Y \\ | \\ Y \quad | \\ X \quad d \end{array} = \begin{array}{c} Y \\ | \\ Y \end{array}$$

( $e$  stands for ‘evaluation’ and  $d$  for ‘dual’). In this situation,  $X$  is called a **right dual** of  $Y$ .

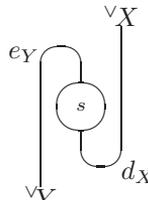
Example:  $\mathcal{C} = \text{Vect}_k^{\text{fin}}$ ,  $X \in \mathcal{C}$ . Let  $Y = X^*$ , the dual vector space. Then  $e : Y \otimes X \rightarrow \mathbf{1}$  is the usual pairing. With the canonical isomorphism  $f : X^* \otimes X \xrightarrow{\cong} \text{End } X$ , we have  $d = f^{-1}(\text{id}_X)$ .

We state some facts:

- (1) Whether an object  $X$  admits a left or right dual is not for us to choose. It is a property of the tensor category.
- (2) If  $Y, Y'$  are left (or right) duals of  $X$  then  $Y \cong Y'$ .
- (3) If  ${}^\vee A, {}^\vee B$  are left duals of  $A, B$ , respectively, then  ${}^\vee B \otimes {}^\vee A$  is a left dual for  $A \otimes B$ , and similarly for right duals.
- (4) If  $X$  has a left dual  $Y$  and a right dual  $Z$ , we may or may not have  $Y \cong Z$ ! (Again, that is a property of  $X$ .)

While duals, if they exist, are unique up to isomorphisms, it is often convenient to make choices. One therefore defines a **left duality** of a strict tensor category  $(\mathcal{C}, \otimes, \mathbf{1})$  to be a map that assigns to each object  $X$  a left dual  ${}^\vee X$  and morphisms  $e_X : {}^\vee X \otimes X \rightarrow \mathbf{1}$  and  $d_X : \mathbf{1} \rightarrow X \otimes {}^\vee X$  satisfying the above identities.

Given a left duality and a morphism,  $s : X \rightarrow Y$  we define

$${}^\vee s = e_Y \otimes \text{id}_{{}^\vee X} \circ \text{id}_{{}^\vee Y} \otimes s \otimes \text{id}_{{}^\vee X} \circ \text{id}_{{}^\vee Y} \otimes d_X =$$


Then  $(X \mapsto {}^\vee X, s \mapsto {}^\vee s)$  is a contravariant functor. (We cannot recover the  $e$ ’s and  $d$ ’s from the functor!) It can be equipped with a natural (anti-)monoidal isomorphism  ${}^\vee(A \otimes B) \rightarrow {}^\vee B \otimes {}^\vee A, {}^\vee \mathbf{1} \rightarrow \mathbf{1}$ . Often, the duality functor comes with a given anti-monoidal structure, e.g. in the case of pivotal categories, cf. Section 3.

- A chosen **right duality**  $X \mapsto (X^\vee, e'_X : X \otimes X^\vee \rightarrow \mathbf{1}, d'_X : \mathbf{1} \rightarrow X^\vee \otimes X)$  also give rise to a contravariant anti-monoidal functor  $X \mapsto X^\vee$ .
- Categories equipped with a left (right) duality are called left (right) **rigid** (or **autonomous**). Categories with left and right duality are called rigid (or autonomous).
- Examples:  $\text{Vect}_k^{\text{fin}}, \text{Rep } G$  are rigid.

- Notice that  ${}^{\vee\vee}X \cong X$  holds if and only if  ${}^{\vee}X \cong X^{\vee}$ , for which there is no general reason.
- If every object  $X \in \mathcal{C}$  admits a left dual  ${}^{\vee}X$  and a right dual  $X^{\vee}$ , and both are isomorphic, we say that  $\mathcal{C}$  has **two-sided duals** and write  $\overline{X}$ . We will only consider such categories, but we will need stronger axioms.
- Let  $\mathcal{C}$  be a  $*$ -category (cf. below) with left duality. If  $({}^{\vee}X, e_X, d_X)$  is a left dual of  $X \in \mathcal{C}$  then  $(X^{\vee} = {}^{\vee}X, d_X^*, e_X^*)$  is a right dual. Thus duals in  $*$ -categories are automatically two-sided. For this reason, duals in  $*$ -category are often axiomatized in a symmetric fashion by saying that a **conjugate**, cf. [70, 172], of an object  $X$  is a triple  $(\overline{X}, r, \overline{r})$ , where  $r : \mathbf{1} \rightarrow \overline{X} \otimes X$ ,  $\overline{r} : \mathbf{1} \rightarrow X \otimes \overline{X}$  satisfy

$$\text{id}_X \otimes r^* \circ \overline{r} \otimes \text{id}_X = \text{id}_X, \quad \text{id}_{\overline{X}} \otimes \overline{r}^* \circ r \otimes \text{id}_{\overline{X}} = \text{id}_{\overline{X}}.$$

It is clear that then  $(\overline{X}, r^*, \overline{r})$  is a left dual and  $(\overline{X}, \overline{r}^*, r)$  a right dual.

- Unfortunately, there is an almost Babylonian inflation of slightly different notions concerning duals, in particular when braidings are involved: A category can be rigid, autonomous, sovereign, pivotal, spherical, ribbon, tortile, balanced, closed, category with conjugates, etc. To make things worse, these terms are not always used in the same way!
- Before we continue the discussion of duality in tensor categories, we will discuss symmetries. For symmetric tensor categories, the discussion of duality is somewhat simpler than in the general case. Proceeding like this seems justified since symmetric (tensor) categories already appeared in the second paper ([178] 1963) on tensor categories.

### 1.6. Additive, linear and $*$ -structure.

- The discussion so far is quite general, but often one encounters categories with more structure.
- We begin with ‘Ab-categories’ (=categories ‘enriched over abelian groups’): For such a category, each  $\text{Hom}(X, Y)$  is an abelian group, and  $\circ$  is bi-additive, cf. [180, Section I.8]. Example: The category **Ab** of abelian groups. In  $\otimes$ -categories, also  $\otimes$  must be bi-additive on the morphisms. Functors of Ab-tensor categories required to be additive on hom-sets.
- If  $X, Y, Z$  are objects in an Ab-category,  $Z$  is called a direct sum of  $X$  and  $y$  if there are morphisms  $X \xrightarrow{u} Z \xrightarrow{u'} X, Y \xrightarrow{v} Z \xrightarrow{v'} Y$  satisfying  $u \circ u' + v \circ v' = \text{id}_Z, u' \circ u = \text{id}_X, v' \circ v = \text{id}_Y$ . An **additive** category is an Ab-category having direct sums for all pairs of objects and a zero object.
- An **abelian category** is an additive category where every morphism has a kernel and a cokernel and every monic (epic) is a kernel (cokernel). We do not have the space to go further into this and must refer to the literature, e.g. [180].
- A category is said to have splitting idempotents (or is ‘Karoubian’) if  $p = p \circ p \in \text{End } X$  implies the existence of an object  $Y$  and of morphisms  $u : Y \rightarrow X, u' : X \rightarrow Y$  such that  $u' \circ u = \text{id}_Y$  and  $u \circ u' = p$ . An

additive category with splitting idempotents is called **pseudo-abelian**. Every abelian category is pseudo-abelian.

- In an abelian category with duals, the functors  $- \otimes X$  and  $X \otimes -$  are automatically exact, cf. [64, Proposition 1.16]. But without rigidity this is far from true.
- A **semisimple** category is an abelian category where every short exact sequence splits.

An alternative, and more pedestrian, way to define semisimple categories is as pseudo-abelian categories admitting a family of simple objects  $X_i, i \in I$  such that every  $X \in \mathcal{C}$  is a finite direct sum of  $X_i$ 's.

Standard examples: The category  $\text{Rep } G$  of finite dimensional representations of a compact group  $G$ , the category  $H - \text{Mod}$  of finite dimensional left modules for a finite dimensional semisimple Hopf algebra  $H$ .

- In  **$k$ -linear** categories, each  $\text{Hom}(X, Y)$  is  $k$ -vector space (often required finite dimensional), and  $\circ$  (and  $\otimes$  in the monoidal case) is bilinear. Functors must be  $k$ -linear. Example:  $\text{Vect}_k$ .
- Pseudo-abelian categories that are  $k$ -linear with finite-dimensional hom-sets are called **Krull-Schmidt** categories. (This is slightly weaker than semisimplicity.)
- A **fusion category** is a semisimple  $k$ -linear category with finite dimensional hom-sets, finitely many isomorphism classes of simple objects and  $\text{End } \mathbf{1} = k$ . We also require that  $\mathcal{C}$  has 2-sided duals.
- A **finite tensor category** (Etingof, Ostrik [85]) is a  $k$ -linear tensor category with  $\text{End } \mathbf{1} = k$  that is equivalent (as a category) to the category of modules over a finite dimensional  $k$ -algebra. (There is a more intrinsic characterization.) Notice that semisimplicity is not assumed.
- Dropping the condition  $\text{End } \mathbf{1} = \text{kid}_k$ , one arrives at a **multi-fusion category** (Etingof, Nikshych, Ostrik [84]).
- Despite the recent work on generalizations, most of these lectures will be concerned with semisimple  $k$ -linear categories satisfying  $\text{End } \mathbf{1} = k \text{ id}_{\mathbf{1}}$ , including infinite ones! (But see the remarks at the end of this section.)
- If  $\mathcal{C}$  is a semisimple tensor category, one can choose representers  $\{X_i, i \in I\}$  of the simple isomorphism classes and define  $N_{i,j}^k \in \mathbb{Z}_+$  by

$$X_i \otimes X_j \cong \bigoplus_{k \in I} N_{i,j}^k X_k.$$

There is a distinguished element  $0 \in I$  such that  $X_0 \cong I$ , thus  $N_{i,0}^k = N_{0,i}^k = \delta_{i,k}$ . By associativity of  $\otimes$  (up to isomorphism)

$$\sum_n N_{i,j}^n N_{n,k}^l = \sum_m N_{i,m}^l N_{j,k}^m \quad \forall i, j, k, l \in I.$$

If  $\mathcal{C}$  has two-sided duals, there is an involution  $i \mapsto \bar{i}$  such that  $\overline{X_i} \cong X_{\bar{i}}$ . One has  $N_{i,j}^0 = \delta_{i,\bar{j}}$ . The quadruple  $(I, \{N_{i,j}^k\}, 0, i \mapsto \bar{i})$  is called the **fusion ring** or **fusion hypergroup** of  $\mathcal{C}$ .

- The above does not work when  $\mathcal{C}$  is not semisimple. But: In any abelian tensor category, one can consider the **Grothendieck ring**  $R(\mathcal{C})$ , the free abelian group generated by the isomorphism classes  $[X]$  of objects in  $\mathcal{C}$ , with a relation  $[X] + [Z] = [Y]$  for every short exact sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  and  $[X] \cdot [Y] = [X \otimes Y]$ .

In the semisimple case, the Grothendieck ring has  $\{[X_i], i \in I\}$  as  $\mathbb{Z}$ -basis and  $[X_i] \cdot [X_j] = \sum_k N_{i,j}^k [X_k]$ . Obviously, an isomorphism of hypergroups gives rise to a ring isomorphism of Grothendieck rings, but the converse is not obvious. While the author is not aware of counterexamples, in order to rule out this annoying eventuality, some authors work with isomorphisms of the Grothendieck **semiring** or the **ordered** Grothendieck ring, cf e.g. [112].

Back to hypergroups:

- The hypergroup contains important information about a tensor category, but it misses that encoded in the associativity constraint. In fact, the hypergroup of  $\text{Rep } G$  for a finite group  $G$  contains exactly the same information as the character table of  $G$ , and it is well known that there are non-isomorphic finite groups with isomorphic character tables. (The simplest example is given by the dihedral group  $D_8 = \mathbb{Z}_4 \rtimes \mathbb{Z}_2$  and the quaternion group  $Q$ , cf. any elementary textbook, e.g. [123].) Since  $D_8$  and  $Q$  have the same number of irreducible representations, the categories  $\text{Rep } D_8$  and  $\text{Rep } Q$  are equivalent (as categories). They are not equivalent as symmetric tensor categories, since this would imply  $D_8 \cong Q$  by the duality theorems of Doplicher and Roberts [70] or Deligne [58] (which we will discuss in Section 3). In fact,  $D_8$  and  $Q$  are already inequivalent as tensor categories (i.e. they are not isocategorical in the sense discussed below). Cf. [254], where fusion categories with the fusion hypergroup of  $D_8$  are classified (among other things).
- On the positive side: (1) If a finite group  $G$  has the same fusion hypergroup (or character table) as a finite simple group  $G'$ , then  $G \cong G'$ , cf. [51]. (The proof uses the classification of finite simple groups.) (2) Compact groups that are abelian or connected are determined by their fusion rings (by Pontrjagin duality, respectively by a result of McMullen [188] and Handelmann [112]. The latter is first proven for simple compact Lie groups and then one deduces the general result via the structure theorem for connected compact groups.)
- If all objects in a semisimple category  $\mathcal{C}$  are invertible, the fusion hypergroup becomes a group. Such fusion categories are called **pointed** and are just the linear versions of the categorical groups encountered earlier. This situation is very special, but:
- To each hypergroup  $\{I, N, 0, i \mapsto \bar{i}\}$  one can associate a group  $G(I)$  as follows: Let  $\sim$  be the smallest equivalence relation on  $I$  such that

$$i \sim j \quad \text{whenever} \quad \exists m, n \in I : i \prec mn \succ j \quad (\text{i.e. } N_{n,m}^i \neq 0 \neq N_{n,m}^j).$$

Now let  $G(I) = I/\sim$  and define

$$[i] \cdot [j] = [k] \text{ for any } k \prec ij, \quad [i]^{-1} = [\bar{i}], \quad e = [0].$$

Then  $G(I)$  is a group, and it has the universal property that every map  $p : I \rightarrow K$ ,  $K$  a group, satisfying  $p(k) = p(i)p(j)$  when  $k \prec ij$  factors through the map  $I \rightarrow G(I)$ ,  $i \mapsto [i]$ .

In analogy to the abelianization of a non-abelian group,  $G(I)$  should perhaps be called the **groupification** of the hypergroup  $I$ . But it was called the **universal grading group** by Gelaki/Nikshych [102], to which this is due in the above generality, since every group-grading on the objects of a fusion category having fusion hypergroup  $I$  factors through the map  $I \rightarrow G(I)$ .

- In the symmetric case (where  $I$  and  $G(I)$  are abelian, but everything else as above) this groupification is due to Baumgärtel/Lledó [21], who spoke of the ‘chain group’. They stated the conjecture that if  $K$  is a compact group, then the (discrete) universal grading group  $G(\text{Rep } K)$  of  $\text{Rep } K$  is the Pontrjagin dual of the (compact) center  $Z(K)$ . Thus: The *center of a compact group  $K$  can be recovered from the fusion ring of  $K$* , even if  $K$  itself in general cannot! This conjecture was proven in [195], but the whole circle of ideas is already implicit in [188].

Example: The representations of  $K = SU(2)$  are labelled by  $\mathbb{Z}_+$  with

$$i \otimes j = |i - j| \oplus \cdots \oplus i + j - 2 \oplus i + j.$$

From this one easily sees that there are two  $\sim$ -equivalence classes, consisting of the even and odd integers. This is compatible with  $Z(SU(2)) = \mathbb{Z}/2\mathbb{Z}$ . Cf. [21].

- There is another application of  $G(\mathcal{C})$ : If  $\mathcal{C}$  is  $k$ -linear semisimple then group of natural monoidal isomorphisms of  $\text{id}_{\mathcal{C}}$  is given by  $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}}) \cong \text{Hom}(G(\mathcal{C}), k^*)$ .
- Given a fusion category  $\mathcal{C}$  (where we have two-sided duals  $\bar{X}$ ), Gelaki/Nikshych [102] define the full subcategory  $\mathcal{C}_{ad} \subset \mathcal{C}$  to be the generated by the objects  $X \otimes \bar{X}$  where  $X$  runs through the simple objects. Notice that  $\mathcal{C}_{ad}$  is just the full subcategory of objects of universal grading zero.

Example: If  $G$  is a compact group then  $(\text{Rep } G)_{ad} = \text{Rep}(G/Z(G))$ .

A fusion category  $\mathcal{C}$  is called **nilpotent** [102] when its upper central series

$$\mathcal{C} \supset \mathcal{C}_{ad} \supset (\mathcal{C}_{ad})_{ad} \supset \cdots$$

leads to the trivial category after finitely many steps.

Example: If  $G$  is a finite group then  $\text{Rep } G$  is nilpotent if and only if  $G$  is nilpotent.

- We call a square  $n \times n$ -matrix  $A$  indecomposable if there is no proper subset  $S \subset \{1, \dots, n\}$  such that  $A$  maps the coordinate subspace  $\text{span}\{e_s \mid s \in$

$S$  into itself. Let  $A$  be an indecomposable square matrix  $A$  with non-negative entries and eigenvalues  $\lambda_i$ . Then the theorem of Perron and Frobenius states that there is a unique non-negative eigenvalue  $\lambda$ , the Perron-Frobenius eigenvalue, such that  $\lambda = \max_i |\lambda_i|$ . Furthermore, the associated eigenspace is one-dimensional and contains a vector with all components non-negative. Now, given a finite hypergroup  $(I, \{N_{i,j}^k\}, 0, i \mapsto \bar{i})$  and  $i \in I$ , define  $N_i \in \text{Mat}(|I| \times |I|)$  by  $(N_i)_{jk} = N_{i,j}^k$ . Due to the existence of duals, this matrix is indecomposable. Now the **Perron-Frobenius dimension**  $d_{FP}(i)$  of  $i \in I$  is defined as the Perron-Frobenius eigenvalue of  $N_i$ . Cf. e.g. [96, Section 3.2]. Then:

$$d_{FP}(i)d_{FP}(j) = \sum_k N_{i,j}^k d_{FP}(k).$$

Also the hypergroup  $I$  has a Perron-Frobenius dimension:  $FP - \dim(I) = \sum_i d_{FP}(i)^2$ . This also defines the PF-dimension of a fusion category, cf. [84]

- **Ocneanu rigidity:** Up to equivalence there are only finitely many fusion categories with given fusion hypergroup. The general statement was announced by Blanchard/A. Wassermann, and a proof is given in [84], using the deformation cohomology theory of Davydov [53] and Yetter [290].
- Ocneanu rigidity was preceded and motivated by several related results on Hopf algebras: Stefan [249] proved that the number of isomorphism classes of semisimple and co-semisimple Hopf algebras of given finite dimension is finite. For Hopf  $*$ -algebras, Blanchard [30] even proved a bound on the number of iso-classes in terms of the dimension. There also is an upper bound on the number of iso-classes of semisimple Hopf algebras with given number of irreducible representations, cf. Etingof's appendix to [224].
- There is an enormous literature on hypergroups. Much of this concerns harmonic analysis on the latter and is not too relevant to tensor categories. But the notion of amenability of hypergroups does have such applications, cf. e.g. [119]. For a review of some aspects of hypergroups, in particular the discrete ones relevant here, cf. [278].
- A considerable fraction of the literature on tensor categories is devoted to categories that are  $k$ -linear over a field  $k$  with finite dimensional Hom-spaces. Clearly this is a rather restrictive condition. It is therefore very remarkable that  $k$ -linearity can actually be deduced under suitable assumptions, cf. [161].
- **$*$ -categories:** A ' $*$ -operation' on a  $\mathbb{C}$ -linear category  $\mathcal{C}$  is a contravariant functor  $*$  :  $\mathcal{C} \rightarrow \mathcal{C}$  which acts trivially on the objects, is antilinear, involutive ( $s^{**} = s$ ) and monoidal ( $(s \otimes t)^* = s^* \otimes t^*$  (when  $\mathcal{C}$  is monoidal)). A  $*$ -operation is called positive if  $s^* \circ s = 0$  implies  $s = 0$ . Categories with (positive)  $*$ -operation are also called hermitian (unitary). We will use ' $*$ -category' as a synonym for 'unitary category'. Example: The category of Hilbert spaces  $\mathcal{HILB}$  with bounded linear maps and  $*$  given by the adjoint.

- It is easy to prove that a finite dimensional  $\mathbb{C}$ -algebra with positive  $*$ -operation is semisimple. Therefore, a unitary category with finite dimensional hom-sets has semisimple endomorphism algebras. If it has direct sums and splitting idempotents then it is semisimple.
- Banach-,  $C^*$ - and von Neumann categories: A Banach category [135] is a  $\mathbb{C}$ -linear additive category, where each  $\text{Hom}(X, Y)$  is a Banach space, and the norms satisfy

$$\|s \circ t\| \leq \|s\| \|t\|, \quad \|s^* \circ s\| = \|s\|^2.$$

(They were introduced by Karoubi with a view to applications in K-theory, cf. [135].) A Banach  $*$ -category is a Banach category with a positive  $*$ -operation. A  $C^*$ -category is a Banach  $*$ -category satisfying  $\|s^* \circ s\| = \|s\|^2$  for any morphism  $s$  (not only endomorphisms). In a  $C^*$ -category, each  $\text{End}(X)$  is a  $C^*$ -algebra. Just as an additive category is a ‘ring with several objects’, a  $C^*$ -category is a “ $C^*$ -algebra with several objects”. Von Neumann categories are defined similarly, cf. [105]. They turned out to have applications to  $L^2$ -cohomology (cf. Farber [88]), representation theory of quantum groups (Woronowicz [280]), subfactors [172], etc.

Remark: A  $*$ -category with finite dimensional hom-spaces and  $\text{End } \mathbf{1} = \mathbb{C}$  automatically is a  $C^*$ -category in a unique way. (Cf. [190].)

- If  $\mathcal{C}$  is a  $C^*$ -tensor category,  $\text{End } \mathbf{1}$  is a commutative  $C^*$ -algebra, thus  $\cong C(S)$  for some compact Hausdorff space  $S$ . Under certain technical conditions, the spaces  $\text{Hom}(X, Y)$  can be considered as vector bundles over  $S$ , or at least as (semi)continuous fields of vector spaces. (Work by Zito [291] and Vasselli [271].) In the case where  $\text{End } \mathbf{1}$  is finite dimensional, this boils down to a direct sum decomposition of  $\mathcal{C} = \bigoplus_i \mathcal{C}_i$ , where each  $\mathcal{C}_i$  is a tensor category with  $\text{End}_{\mathcal{C}_i}(\mathbf{1}_{\mathcal{C}_i}) = \mathbb{C}$ . (In this connection, cf. Baez’ comments a Doplicher-Roberts type theorem for finite groupoids [9].)

## 2. SYMMETRIC TENSOR CATEGORIES

- Many of the obvious examples of tensor categories encountered in Section 1, like the categories  $\mathcal{SET}$ ,  $\text{Vect}_k$ , representation categories of groups and Cartesian categories (tensor product  $\otimes$  given by the categorical product), have an additional piece of structure, to which this section is dedicated.
- A **symmetry** on a tensor category  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \rho, \lambda)$  is a natural isomorphism  $c : \otimes \rightarrow \otimes \circ \sigma$ , where  $\sigma : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  is the flip automorphism of  $\mathcal{C} \times \mathcal{C}$ , such that  $c^2 = \text{id}$ . (I.e., for any two objects  $X, Y$  there is an isomorphism  $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ , natural w.r.t.  $X, Y$  such that  $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ ), where “all properly built diagrams commute”, i.e. the category is coherent. A **symmetric tensor category** (STC) is a tensor category equipped with a symmetry.

We represent the symmetry graphically by

$$c_{X,Y} = \begin{array}{ccc} & Y & X \\ & \diagdown & \diagup \\ & X & Y \end{array}$$

- As for tensor categories, there are two versions of the Coherence Theorem. Version I (Mac Lane [178]): Let  $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, \rho, \lambda)$  be a tensor category. Then a natural isomorphism  $c : \otimes \rightarrow \otimes \circ \sigma$  satisfying  $c^2 = \text{id}$  is a symmetry if and only if

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{c_{X,Y} \otimes \text{id}_Z} & (Y \otimes X) \otimes Z & \xrightarrow{\alpha_{Y,X,Z}} & Y \otimes (X \otimes Z) & \xrightarrow{\text{id}_Y \otimes c_{X,Z}} & Y \otimes (Z \otimes X) \\ \alpha_{X,Y,Z} \downarrow & & & & & & \uparrow \alpha_{Y,Z,X} \\ X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y \otimes Z}} & & & & & (Y \otimes Z) \otimes X \end{array}$$

commutes. (In the strict case, this reduces to  $c_{X,Y \otimes Z} = \text{id}_Y \otimes c_{X,Z} \circ c_{X,Y} \otimes \text{id}_Z$ .)

A **symmetric tensor functor** is a tensor functor  $F$  such that  $F(c_{X,Y}) = c'_{F(X),F(Y)}$ . Notice that a natural transformation between symmetric tensor functors is just a monoidal natural transformation, i.e. there is no new condition.

- Now we can state version II of the Coherence theorem: Every symmetric tensor category is equivalent (by a symmetric tensor functor) to a strict one.
- Examples of symmetric tensor categories:
  - The category  $\mathbb{S}$  defined earlier, when  $c_{n,m} : n + m \rightarrow n + m$  is taken to be the element of  $S_{n+m}$  defined by  $(1, \dots, n+m) \mapsto (n+1, \dots, n+m, 1, \dots, n)$ . It is the free symmetric monoidal category generated by one object.
  - Non-strict symmetric categorical groups were classified by Sinh [245]. We postpone our discussion to Section 4, where we will also consider the braided case.
  - $\text{Vect}_k$ , representation categories of groups: We have the canonical symmetry  $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ ,  $x \otimes y \mapsto y \otimes x$ .
  - The tensor categories obtained using products or coproducts are symmetric.
- Let  $\mathcal{C}$  be a strict STC,  $X \in \mathcal{C}$  and  $n \in \mathbb{N}$ . Then there is a unique homomorphism

$$\Pi_n^X : S_n \rightarrow \text{Aut } X^{\otimes n} \quad \text{such that} \quad \sigma_i \mapsto \text{id}_{X^{\otimes(i-1)}} \otimes c_{X,X} \otimes \text{id}_{X^{\otimes(n-i-1)}}.$$

Proof: This is immediate by the definition of STCs and the presentation

$$S_n = \{ \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i-j| > 1, \sigma_i^2 = 1 \}$$





fin.dim. vector spaces over  $k$  and  $\text{Hom}_{\mathcal{C}}(X, Y) \subset \text{Hom}_{\text{Vect}_k}(X, Y)$ . However, a better way of thinking of a concrete category is as a (abstract) category  $\mathcal{C}$  equipped with a **fiber functor**, i.e. a faithful functor  $E : \mathcal{C} \rightarrow \text{Sets}$ , respectively  $E : \mathcal{C} \rightarrow \text{Vect}_k$ . The latter is required to be monoidal when  $\mathcal{C}$  is monoidal.

- Example:  $G$  a group. Then  $\mathcal{C} := \text{Rep}_k G$  should be considered as an abstract  $k$ -linear  $\otimes$ -category together with a faithful  $\otimes$ -functor  $E : \mathcal{C} \rightarrow \text{Vect}_k$ .
- The point of this that a category  $\mathcal{C}$  may have inequivalent fiber functors!!
- But: If  $k$  is algebraically closed of characteristic zero,  $\mathcal{C}$  is rigid *symmetric*  $k$ -linear with  $\text{End } \mathbf{1} = k$  and  $F, F'$  are *symmetric* fiber functors then  $F \cong F'$  (as  $\otimes$ -functors). (Saavedra Rivano [238, 64]).
- The first non-trivial application of (symmetric) tensor categories probably were the reconstruction theorems of Tannaka [255] (1939!) and Saavedra Rivano [238, 64].

Let  $k$  be algebraically closed. Let  $\mathcal{C}$  be rigid symmetric  $k$ -linear with  $\text{End } \mathbf{1} = k$  and  $E : \mathcal{C} \rightarrow \text{Vect}_k$  faithful tensor functor. (Tannaka did this for  $k = \mathbb{C}$ ,  $\mathcal{C}$  a  $*$ -category and  $E$   $*$ -preserving.) Let  $G = \text{Aut}_{\otimes} E$  be the group of natural monoidal [unitary] automorphisms of  $E$ . Define a functor  $F : \mathcal{C} \rightarrow \text{Rep } G$  [unitary representations] by

$$F(X) = (E(X), \pi_X), \quad \pi_X(g) = g_X \quad (g \in G).$$

Then  $G$  is pro-algebraic [compact] and  $F$  is an equivalence of symmetric tensor  $[*]$ -categories.

Proof: The idea is the following (Grothendieck, Saavedra Rivano [238], cf. Bichon [27]): Let  $E_1, E_2 : \mathcal{C} \rightarrow \text{Vect}_k$  be fiber functors. Define a unital  $k$ -algebra  $A_0(E_1, E_2)$  by

$$A_0(E_1, E_2) = \bigoplus_{X \in \mathcal{C}} \text{Hom}_{\text{Vect}}(E_2(X), E_1(X)),$$

spanned by elements  $[X, s]$ ,  $X \in \mathcal{C}, s \in \text{Hom}(E_2(X), E_1(X))$ , with  $[X, s] \cdot [Y, t] = [X \otimes Y, u]$ , where  $u$  is the composite

$$E_2(X \otimes Y) \xrightarrow{(d_{X,Y}^2)^{-1}} E_2(X) \otimes E_2(Y) \xrightarrow{s \otimes t} E_1(X) \otimes E_1(Y) \xrightarrow{d_{X,Y}^1} E_1(X \otimes Y).$$

This is a unital associative algebra, and  $A(E_1, E_2)$  is defined as the quotient by the ideal generated by the elements  $[X, a \circ E_2(s)] - [Y, E_1(s) \circ a]$ , where  $s \in \text{Hom}_{\mathcal{C}}(X, Y)$ ,  $a \in \text{Hom}_{\text{Vect}}(E_2(Y), E_1(X))$ .

- Remark: Let  $E_1, E_2 : \mathcal{C} \rightarrow \text{Vect}_k$  be fiber functors as above. Then the map

$$X \times Y \mapsto \text{Hom}_{\text{Vect}_k}(E_2(X), E_1(Y))$$

extends to a functor  $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Vect}_k$ . Now the algebra  $A(E_1, E_2)$  is just the **coend**  $\int^X F(X, X)$  of  $F$ , a universal object. Coends are a categorical, non-linear version of traces, but we refrain from going into them since it takes some time to appreciate the concept. (Cf. [180].)

- Now one proves [27, 197]:

- If  $E_1, E_2$  are *symmetric* tensor functors then  $A(E_1, E_2)$  is commutative.
- If  $\mathcal{C}$  is  $*$ -category and  $E_1, E_2$  are  $*$ -preserving then  $A(E_1, E_2)$  is a  $*$ -algebra and has a  $C^*$ -completion.
- If  $\mathcal{C}$  is finitely generated (i.e. there exists a monoidal generator  $Z \in \mathcal{C}$  such that every  $X \in \mathcal{C}$  is direct summand of some  $Z^{\otimes N}$ ) then  $A(E_1, E_2)$  is finitely generated.
- There is a bijection between natural monoidal (unitary) isomorphisms  $\alpha : E_1 \rightarrow E_2$  and  $(*)$ -characters on  $A(E_1, E_2)$ .

Thus: If  $E_1, E_2$  are symmetric and either  $\mathcal{C}$  is finitely generated or a  $*$ -category, the algebra  $A(E_1, E_2)$  admits characters (by the Nullstellensatz or by Gelfand's theory), thus  $E_1 \cong E_2$ . One also finds that  $G = \text{Aut}_{\otimes} E \cong (*)\text{Char}(A(E, E))$  and  $A(E) = \text{Fun}(G)$  (representative respectively continuous functions). This is used to prove that  $F : \mathcal{C} \rightarrow \text{Rep } G$  is an equivalence.

- Remarks: 1. While it has become customary to speak of Tannakian categories, the work of Kreĭn, cf. [158], [118, Section 30], should also be mentioned since it can be considered as a precursor of the later generalizations to non-symmetric categories, in particular in Woronowicz's approach.

2. The uniqueness of the symmetric fiber functor  $E$  implies that  $G$  is unique up to isomorphism.

3. For the above construction, we need to have a fiber functor. Around 1989, Doplicher and Roberts [70], and independently Deligne [58] construct such a functor under weak assumptions on  $\mathcal{C}$ . See below.

4. The uniqueness proof fails if either of  $E_1, E_2$  is not symmetric (or  $\mathcal{C}$  is not symmetric). Given a group  $G$ , there is a tautological fiber functor  $E$ . The fact that there may be (non-symmetric) fiber functors that are not naturally isomorphic to  $E$  reflects the fact that there can be groups  $G'$  such that  $\text{Rep } G \simeq \text{Rep } G'$  as tensor categories, but not as symmetric tensor categories! This phenomenon was independently discovered by Etingof/Gelaki [80], who called such  $G, G'$  **isocategorical** and produced examples of isocategorical but non-isomorphic finite groups, by Davydov [55] and by Izumi and Kosaki [122]. The treatment in [80] relies on the fact that if  $G, G'$  are isocategorical then  $\mathbb{C}G' \cong \mathbb{C}^J$  for some Drinfeld twist  $J$ . A more categorical approach, allowing also an extension to compact groups, will be given in [202]. A group  $G$  is called categorically rigid if every  $G'$  isocategorical to  $G$  is actually isomorphic to  $G$ . (Compact groups that are abelian or connected are categorically rigid in a strong sense since they are determined already by their fusion hypergroups.)

- Consider the free rigid symmetric tensor  $*$ -category  $\mathcal{C}$  with  $\text{End } \mathbf{1} = \mathbb{C}$  generated by one object  $X$  of dimension  $d$ . If  $d \in \mathbb{N}$  then  $\mathcal{C}$  is equivalent to  $\text{Rep } U(d)$  or  $\text{Rep } O(d)$  or  $\text{Rep } Sp(d)$ , depending on whether  $X$  is non-selfdual or orthogonal or symplectic. The proof [9] is straightforward once one has the Doplicher-Roberts theorem.

- The free rigid symmetric categories just mentioned can be constructed in a topological way, in a fashion very similar to the construction of the Temperley-Lieb categories  $TL(\tau)$ . The main difference is that one allows the lines in the pictures defining the morphisms to cross. (But they still live in a plane.) Now one quotients out the negligible morphisms and completes w.r.t. direct sums and splitting idempotents. (In the non-self dual case, the objects are words over the alphabet  $\{+, -\}$  and the lines in the morphisms are directed.) All this is noted in passing by Deligne in a paper [59] dedicated to the exceptional groups! Notice that when  $d \notin \mathbb{N}$ , these categories are examples of rigid symmetric categories that are not Tannakian.
- The above results already establish strong connections between tensor categories and representation theory, but there is much more to say.

### 3. BACK TO GENERAL TENSOR CATEGORIES

- In a general tensor category, left and right duals need not coincide. This can already be seen for the left module category  $H\text{-Mod}$  of a Hopf algebra  $H$ . This category has left and right duals, related to  $S$  and  $S^{-1}$ . ( $S$  must be invertible, but can be aperiodic!) They coincide when  $S^2(x) = uxu^{-1}$  with  $u \in H$ .
- We only consider tensor categories that have isomorphic left and right duals, i.e. two-sided duals, which we denote  $\overline{X}$ .
- If  $\mathcal{C}$  is  $k$ -linear with  $\text{End } \mathbf{1} = k \text{ id}$  and  $\text{End } X = k \text{ id}$  ( $X$  is simple/irreducible), one can canonically define the squared dimensions  $d^2(X) \in k$  by

$$d^2(X) = (e_X \circ d'_X) \cdot (e'_X \circ d_X) \in \text{End } \mathbf{1}.$$

(Since  $X$  is simple, the morphisms  $d, d', e, e'$  are unique up to scalars, and well-definedness of  $d^2$  follows from the equations involving  $(d, e), (d', e')$  bilinearly.) Cf. [191].

- If  $\mathcal{C}$  is a fusion category, we define its **dimension** by  $\dim \mathcal{C} = \sum_i d^2(X_i)$ .
- If  $H$  is a finite dimensional semisimple and co-semisimple Hopf algebra then  $\dim H\text{-Mod} = \dim_k H$ . (A finite dimensional Hopf algebra is co-semisimple if and only if the dual Hopf algebra  $\widehat{H}$  is semisimple.)
- Even if  $\mathcal{C}$  is semisimple, it is not clear whether one can choose roots  $d(X)$  of the above numbers  $d^2(X)$  in such a way that  $d$  is additive and multiplicative!
- In pivotal categories this can be done. A strict **pivotal** category [93, 94] is a strict left rigid category with a monoidal structure on the functor  $X \mapsto {}^\vee X$  and a monoidal equivalence of the functors  $\text{id}_{\mathcal{C}}$  and  $X \mapsto {}^{\vee\vee} X$ . As a consequence, one can define a right duality satisfying  $X^\vee = {}^\vee X$ .

- In a strict pivotal categories we can define left and right traces for every endomorphism:

$$\text{Tr}_X^L(s) = \overline{X} \left( \begin{array}{c} \text{---} e_X \\ \text{---} \text{---} \\ \text{---} s \\ \text{---} \text{---} \\ \text{---} d'_X \end{array} \right) \qquad \text{Tr}_X^R(s) = \left( \begin{array}{c} \text{---} e'_X \\ \text{---} \text{---} \\ \text{---} s \\ \text{---} \text{---} \\ \text{---} d_X \end{array} \right) \overline{X} \qquad (3.1)$$

Notice: In general  $\text{Tr}_X^L(s) \neq \text{Tr}_X^R(s)$ .

- We now define dimensions by  $d(X) = \text{Tr}_X^L(\text{id}_X) \in \text{End } \mathbf{1}$ . One then automatically has  $d(\overline{X}) = \text{Tr}_X^R(\text{id}_X)$ , which can differ from  $d(X)$ . But for simple  $X$  we have  $d(X)d(\overline{X}) = d^2(X)$  with  $d^2(X)$  as above.
- In a pivotal category, we can use the trace to define pairings  $\text{Hom}(X, Y) \times \text{Hom}(Y, X) \rightarrow \text{End } \mathbf{1}$  by  $(s, t) \mapsto \text{Tr}_X^L(t \circ s)$ . In the semisimple  $k$ -linear case with  $\text{End } \mathbf{1}$ , these pairings are non-degenerate for all  $X, Y$ . Cf. e.g. [104]. In general, a morphism  $s : X \rightarrow Y$  is called **negligible** if  $\text{Tr}(t \circ s) = 0$  for all  $t : Y \rightarrow X$ . We call an Ab-category *non-degenerate* if only the zero morphisms are negligible. The negligible morphisms form a monoidal ideal, i.e. composing or tensoring a negligible morphism with any morphism yields a negligible morphism. It follows that one can quotient out the negligible morphisms in a straightforward way, obtaining a non-degenerate category. A non-degenerate abelian category is semisimple [61], but a counterexample given there shows that non-degeneracy plus pseudo-abelianness do not imply semisimplicity!
- A **spherical** category [20] is a pivotal category where the left and right traces coincide. Equivalently, it is a strict autonomous category (i.e. a tensor category equipped with a left and a right duality) for which the resulting functors  $X \mapsto X^\vee$  and  $X \mapsto {}^\vee X$  coincide.

Sphericity implies  $d(X) = d(\overline{X})$ , and if  $\mathcal{C}$  is semisimple, the converse implication holds.

- The Temperley-Lieb categories  $\mathcal{TL}(\tau)$  are spherical.
- A finite dimensional Hopf algebra that is involutive, i.e. satisfies  $S^2 = \text{id}$ , gives rise to a spherical category. (It is known that every semisimple and co-semisimple Hopf algebra is involutive.) More generally, ‘spherical Hopf algebras’, defined as satisfying  $S^2(x) = wxw^{-1}$ , where  $w \in H$  is invertible with  $\Delta(w) = w \otimes w$  and  $\text{Tr}(\theta w) = \text{Tr}(\theta w^{-1})$  for any finitely generated projective left  $H$ -module  $V$ , give rise to spherical categories [20].
- In a  $*$ -category with conjugates, traces of endomorphisms, in particular dimensions of objects, can be defined uniquely without choosing a spherical structure, cf. [70, 172]. The dimension satisfies  $d(X) \geq 1$  for every non-zero  $X$ , and  $d(X) = 1$  holds if and only if  $X$  is invertible. Furthermore, one has

[172] a  $*$ -categorical version of the quantization of the Jones index [126]:

$$d(X) \in \left\{ 2 \cos \frac{\pi}{n}, n = 3, 4, \dots \right\} \cup [2, \infty).$$

On the other hand, every tensor  $*$ -category can be equipped [286] with an (essentially) unique spherical structure such the traces and dimension defined using the latter coincide with those of [172].

- In a  $\mathbb{C}$ -linear fusion category (no  $*$ -operation required!) one has  $d^2(X) > 0$  for all  $X$ , cf. [84].

The following is a very useful application: If  $\mathcal{A} \subset \mathcal{B}$  is a full inclusion of  $\mathbb{C}$ -linear fusion category then  $\dim \mathcal{A} \leq \dim \mathcal{B}$ , and equality holds if and only if  $\mathcal{A} \simeq \mathcal{B}$ .

- In a unitary category,  $\dim \mathcal{C} = FP - \dim \mathcal{C}$ . Categories with the latter property are called **pseudo-unitary** in [84], where it is shown that every pseudo-unitary category admits a unique spherical structure such that  $FP - d(X) = d(X)$  for all  $X$ .
- There are Tannaka-style theorem for not necessarily symmetric categories (Ulbrich [268], Yetter [288], Schauenburg [239]): Let  $\mathcal{C}$  be a  $k$ -linear pivotal category with  $\text{End } \mathbf{1} = \text{kid}_1$  and let  $E : \mathcal{C} \rightarrow \text{Vect}_k$  a fiber functor. Then the algebra  $A(E)$  defined as above admits a coproduct and an antipode, thus the structure of a Hopf algebra  $H$ , and an equivalence  $F : \mathcal{C} \rightarrow \text{Comod } H$  such that  $E = K \circ F$ , where  $K : \text{Comod } H \rightarrow \text{Vect}_k$  is the forgetful functor. (If  $\mathcal{C}$  and  $E$  are symmetric, this  $H$  is a commutative Hopf algebra of functions on the group obtained earlier  $G$ .) Woronowicz proved a similar result [280] for  $*$ -categories, obtaining a compact quantum group (as defined by him [279, 281]). Commutative compact quantum groups are just algebras  $C(G)$  for a compact group, thus one recovers Tannaka’s theorem. Cf. [131] for an excellent introduction to the area of Tannaka-Krein reconstruction.
- Given a fiber functor, can one find an algebraic structure whose *representations* (rather than corepresentations) are equivalent to  $\mathcal{C}$ ? The answer is positive, provided one uses a slight generalization of Hopf algebras, to wit A. van Daele’s ‘Algebraic Quantum Groups’ [269, 270] (or ‘Multiplier Hopf algebras with Haar functional’). They are not necessarily unital algebras equipped with a coproduct  $\Delta$  that takes values in the multiplier algebra  $M(A \otimes A)$  and with a left-invariant Haar-functional  $\mu \in A^*$ . A nice feature of algebraic quantum groups is that they admit a nice version of Pontryagin duality (which is not the case for infinite dimensional ordinary Hopf algebras).

In [200] the following was shown: If  $\mathcal{C}$  is a semisimple spherical ( $*$ -)category and  $E$  a ( $*$ -)fiber functor then there is a discrete multiplier Hopf ( $*$ -)algebra  $(A, \Delta)$  and an equivalence  $F : \mathcal{C} \rightarrow \text{Rep}(A, \Delta)$  such that  $K \circ F = E$ , where  $K : \text{Rep}(A, \Delta) \rightarrow \text{Vect}$  is the forgetful functor. (This  $(A, \Delta)$  is the Pontryagin dual of the  $A(E)$  above.) This theory exploits the semisimplicity

from the very beginning, which makes it quite transparent: One defines

$$A = \bigoplus_{i \in I} \text{End } E(X_i) \quad \text{and} \quad M(A) = \prod_{i \in I} \text{End } E(X_i) \cong \text{Nat } E,$$

where the summation is over the equivalence classes of simple objects in  $\mathcal{C}$ . Now the tensor structures of  $\mathcal{C}$  and  $E$  give rise to a coproduct  $\Delta : A \rightarrow M(A \otimes A)$  in a very direct way.

Notice: This reconstruction is related to the preceding one as follows. Since  $H\text{-comod} \simeq \mathcal{C}$  is semisimple, the Hopf algebra  $H$  has a left-invariant integral  $\mu$ , thus  $(H, \mu)$  is a compact algebraic quantum group, and the discrete algebraic quantum group  $(A, \Delta)$  is just the Pontrjagin dual of the latter.

- In this situation, there is a bijection between braidings on  $\mathcal{C}$  and R-matrices (in  $M(A \otimes A)$ ), cf. [200]. But: The braiding on  $\mathcal{C}$  plays no essential rôle in the reconstruction. (Since [200] works with the category of finite dimensional representations, which in general does not contain the left regular representation, this is more work than e.g. in [137] and requires the use of semisimplicity.)
- Summing up: Linear [braided] tensor categories admitting a fiber functor are (co)representation categories of [(co)quasi-triangular] discrete (compact) quantum groups.

Notice that here ‘Quantum groups’ refers to Hopf algebras and suitable generalizations thereof, but not necessarily to q-deformations of some structure arising from groups!

- WARNING: The non-uniqueness of fiber functors means that there can be non-isomorphic quantum groups whose (co)representation categories are equivalent to the given  $\mathcal{C}$ !

The study of this phenomenon leads to Hopf-Galois theory and is connected (in the  $*$ -case) to the study of ergodic actions of quantum groups on  $C^*$ -algebras. (Cf. e.g. Bichon, de Rijdt, Vaes [28]).

- Despite this non-uniqueness, one may ask whether one can *intrinsically* characterize the tensor categories admitting a fiber functor, thus being related to quantum groups. (Existence of a fiber functor is an extrinsic criterion.) The few known results to this questions are of two types. On the one hand there are some recognition theorems for certain classes of representation categories of quantized enveloping algebras, which will be discussed somewhat later. On the other hand, there are results based on the regular representation, to which we turn now. However, it is only in the symmetric case that this leads to really satisfactory results.
- The left regular representation  $\pi_l$  of a compact group  $G$  (living on  $L^2(G)$ ) has the following well known properties:

$$\begin{aligned} \pi_l &\cong \bigoplus_{\pi \in \widehat{G}} d(\pi) \cdot \pi, && \text{(Peter-Weyl theorem)} \\ \pi_l \otimes \pi &\cong d(\pi) \cdot \pi_l \quad \forall \pi \in \text{Rep } G. && \text{(absorbing property).} \end{aligned}$$

- The second property generalizes to *any* algebraic quantum group’  $(A, \Delta)$ , cf. [201]:

1. Let  $\Gamma = \pi_l$  be the left regular representation. If  $(A, \Delta)$  is discrete, then  $\Gamma$  carries a monoid structure  $(\Gamma, m, \eta)$  with  $\dim \text{Hom}(\mathbf{1}, \Gamma) = \mathbf{1}$ , which we call the **regular monoid**. (Algebras in  $k$ -linear tensor categories satisfying  $\dim \text{Hom}(\mathbf{1}, \Gamma) = \mathbf{1}$  have been called ‘simple’ or ‘haploid’.) If  $(A, \Delta)$  is compact,  $\Gamma$  has a comonoid structure. (And in the finite (=compact + discrete) case, the algebra and coalgebra structures combine to a Frobenius algebra, cf. [191], discussed below.)

2. If  $(A, \Delta)$  is a discrete algebraic quantum group, one has a monoid version of the absorbing property: For every  $X \in \text{Rep}(A, \Delta)$  one has an isomorphism

$$(\Gamma \otimes X, m \otimes \text{id}_X) \cong n(X) \cdot (\Gamma, m) \tag{3.2}$$

of  $(\Gamma, m, \eta)$ -modules in  $\text{Rep}(A, \Delta)$ . (Here  $n(X) \in \mathbb{N}$  is the dimension of the vector space of the representation  $X$ , which in general differs from the categorical dimension.)

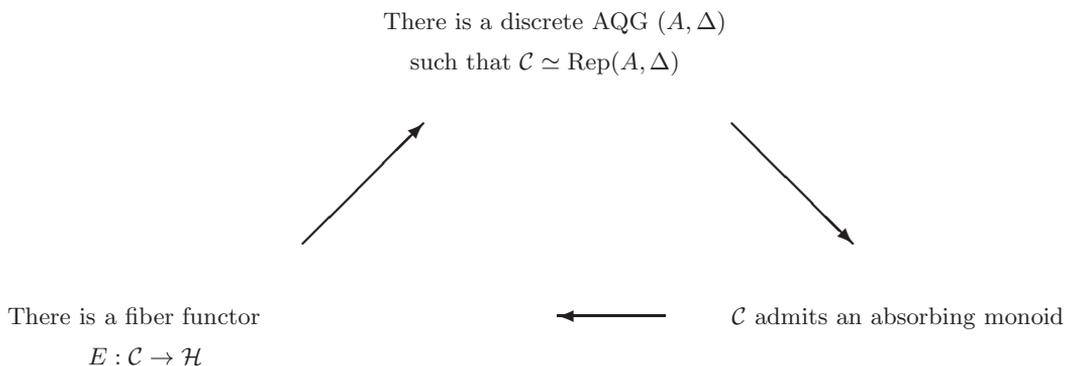
- The following theorem from [201] is motivated by Deligne’s [58]: Let  $\mathcal{C}$  be a  $k$ -linear category and  $(\Gamma, m, \eta)$  a monoid in  $\mathcal{C}$  (more generally, in the associated category  $\text{Ind } \mathcal{C}$  of inductive limits) satisfying  $\dim \text{Hom}(\mathbf{1}, \Gamma) = 1$  and (3.2) for some function  $n : \text{Obj } \mathcal{C} \rightarrow \mathbb{N}$ . Then

$$E(X) = \text{Hom}_{\text{Vect}_k}(\mathbf{1}, \Gamma \otimes X)$$

defines a faithful  $\otimes$ -functor  $E : \mathcal{C} \rightarrow \text{Vect}_k$ , i.e. a fiber functor. (One has  $\dim E(X) = n(X) \forall X$  and  $\Gamma \cong \bigoplus_i n(X_i) X_i$ .) If  $\mathcal{C}$  is symmetric and  $(\Gamma, m, \eta)$  commutative (i.e.  $m \circ c_{\Gamma, \Gamma} = m$ ), then  $E$  is symmetric.

Remark: Deligne considered this only in the symmetric case, but did not make the requirement  $\dim \text{Hom}(\mathbf{1}, \Gamma) = 1$ . This leads to a tensor functor  $E : \mathcal{C} \rightarrow A - \text{Mod}$ , where  $A = \text{Hom}(\mathbf{1}, \Gamma)$  is the commutative  $k$ -algebra of ‘elements of  $\Gamma$ ’ encountered earlier.

- This gives rise to the following implications:



Remarks: 1. This can be considered as an intrinsic characterization of quantum group categories. (Or rather semi-intrinsic, since the regular monoid lives in the Ind-category of  $\mathcal{C}$  rather than  $\mathcal{C}$  itself.)

2. The case of finite  $*$ -categories had been treated in [170], using subfactor theory and a functional analysis.

3. This result is quite unsatisfactory, but I doubt that a better result can be obtained without restriction to *special classes of categories* or adopting a wide *generalization of the notion of quantum groups*. Examples for both will be given below.

4. For a different approach, also in terms of the regular representation, cf. [69].

- Notice that having an absorbing monoid in  $\mathcal{C}$  (or rather  $\text{Ind}(\mathcal{C})$ ) means having an  $\mathbb{N}$ -valued dimension function  $n$  on the hypergroup  $I(\mathcal{C})$  and an associative product on the object  $\Gamma = \bigoplus_{i \in I} n_i X_i$ . The latter is a cohomological condition.

If  $\mathcal{C}$  is finite, one can show using Perron-Frobenius theory that there is only one dimension function, namely the intrinsic one  $i \mapsto d(X_i)$ . Thus a finite category with non-integer intrinsic dimensions cannot be Tannakian (in the above sense).

- We now turn to a very beautiful result of Deligne [58] (simplified considerably by Bichon [27]):

Let  $\mathcal{C}$  be a semisimple  $k$ -linear rigid **even symmetric** category satisfying  $\text{End } \mathbf{1} = k$ , where  $k$  is algebraically closed of characteristic zero. Then there is an absorbing commutative monoid as above. (Thus we have a symmetric fiber functor, implying  $\mathcal{C} \simeq \text{Rep } G$ .)

Sketch: The homomorphisms  $\Pi_n^X : S_n \rightarrow \text{Aut } X^{\otimes n}$  allow to define the idempotents

$$P_{\pm}(X, n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \Pi_n^X(\sigma) \in \text{End}(X^{\otimes n})$$

and their images  $S^n(X), A^n(X)$ , which are direct summands of  $X^{\otimes n}$ . Making crucial use of the evenness assumption on  $\mathcal{C}$ , one proves

$$d(A^n(X)) = \frac{d(X)(d(X) - 1) \cdots (d(X) - n + 1)}{n!} \quad \forall n \in \mathbb{N}.$$

In a  $*$ -category, this must be non-negative  $\forall n$ , implying  $d(X) \in \mathbb{N}$ , cf. [70]. Using this – or assuming it as in [58] – one has  $d(A^{d(X)}(X)) = 1$ , and  $A^{d(X)}(X)$  is called the **determinant** of  $X$ . On the other hand, one can define a commutative monoid structure on

$$S(X) = \bigoplus_{n=0}^{\infty} S^n(X),$$

obtaining the **symmetric algebra**  $(S(X), m, \eta)$  of  $X$ . Let  $Z$  be a  $\otimes$ -generator  $Z$  of  $\mathcal{C}$  satisfying  $\det Z = \mathbf{1}$ . Then the ‘interaction’ between symmetrization (symmetric algebra) and antisymmetrization (determinants)

allows to construct a maximal ideal  $I$  in the commutative algebra  $S(Z)$  such that the quotient algebra  $A = S(Z)/I$  has all desired properties: it is commutative, absorbing and satisfies  $\dim \text{Hom}(\mathbf{1}, A) = 1$ . QED.

Remarks: 1. The absorbing monoid  $A$  constructed in [58, 27] did *not* satisfy  $\dim \text{Hom}(\mathbf{1}, A) = 1$ . Therefore the construction considered above does not give a fiber functor to  $\text{Vect}_{\mathbb{C}}$ , but to  $\Gamma_A - \text{Mod}$ , and one needs to quotient by a maximal ideal in  $\Gamma_A$ . Showing that one can achieve  $\dim \text{Hom}(\mathbf{1}, A) = 1$  was perhaps the main innovation of [197]. This has the advantage that  $(A, m, \eta)$  actually is (isomorphic to) the regular monoid of the group  $G = \text{Nat}_{\otimes} E$ . As a consequence, the latter group can be obtained simply as the automorphism group

$$\text{Aut}(\Gamma, m, \eta) \equiv \{g \in \text{Aut } \Gamma \mid g \circ m = m \circ g \otimes g, g \circ \eta = \eta\}$$

of the monoid – without even mentioning fiber functors!

2. Combining Tannaka’s theorem with those on fiber functors from monoids and with the above, one has the following beautiful

Theorem [70, 58]: Let  $k$  be algebraically closed of characteristic zero and  $\mathcal{C}$  a semisimple  $k$ -linear rigid even symmetric category with  $\text{End } \mathbf{1} = k$ . Assume that all objects have dimension in  $\mathbb{N}$ . Then there is a pro-algebraic group  $G_a$ , unique up to isomorphism, such that  $\mathcal{C} \simeq \text{Rep } G_a$  (finite dimensional rational representations). If  $\mathcal{C}$  is a  $*$ -category then semisimplicity and the dimension condition are redundant, and there is a unique compact group  $G_c$  such that  $\mathcal{C} \simeq \text{Rep } G_c$  (continuous unitary finite-dimensional representations). In this case,  $G_a$  is the complexification of  $G_c$ .

3. If  $\mathcal{C}$  is symmetric but not even, its symmetry can be ‘bosonized’ into an even one, cf. [70]. Then one applies the above result and obtains a group  $G$ . The  $\mathbb{Z}_2$ -grading on  $\mathcal{C}$  given by the twist gives rise to an element  $k \in Z(G)$  satisfying  $k^2 = e$ . Thus  $\mathcal{C} \simeq \text{Rep}(G, k)$  as symmetric category. Cf. also [60].

- The above result has several applications in pure mathematics: It plays a big rôle in the theory of motives [5, 166] and in differential Galois theory and the related Riemann Hilbert problem, cf. [230]. It is used for the classification of triangular Hopf algebras in terms of Drinfeld twists of group algebras (Etingof/Gelaki, cf. [100] and references therein) and for the modularization of braided tensor categories [39, 190], cf. below.

The work of Doplicher and Roberts [70] was motivated by applications to quantum field theory in  $\geq 2 + 1$  dimensions [68, 71], where it leads to a Galois theory of quantum fields, cf. also [111].

- Thus, at least in characteristic zero (in the absence of a  $*$ -operation one needs to impose integrality of all dimensions) rigid symmetric categories with  $\text{End } \mathbf{1} = k \text{id}_{\mathbf{1}}$  are reasonably well understood in terms of compact or pro-affine groups. What about relaxing the last condition? The category of a representations (on continuous fields of Hilbert spaces) of a compact **groupoid**  $\mathcal{G}$  is a symmetric  $C^*$ -tensor category. Since a lot of information is lost in passing from  $\mathcal{G}$  to  $\text{Rep } \mathcal{G}$ , there is no hope of reconstructing  $\mathcal{G}$  up

to isomorphism, but one may hope to find a compact group bundle giving rise to the given category and proving that it is Morita equivalent to  $\mathcal{G}$ . However, there seem to be topological obstructions to this being always the case, cf. [272].

- In this context, we mention related work by Bruguières/Maltsiniotis [184, 40, 37] on Tannaka theory for **quasi quantum groupoids** in a purely algebraic setting.
- We now turn to the characterization of certain *special classes* of tensor categories:
- Combining Doplicher-Roberts reconstruction with the mentioned result of McMullen and Handelmann one obtains a simple prototype: If  $\mathcal{C}$  is an even symmetric tensor  $*$ -category with conjugates and  $\text{End } \mathbf{1} = \mathbb{C}$  whose fusion hypergroup is isomorphic to that of a connected compact Lie group  $G$ , then  $\mathcal{C} \simeq \text{Rep } G$ .
- Kazhdan/Wenzl [145]: Let  $\mathcal{C}$  be a semisimple  $\mathbb{C}$ -linear spherical  $\otimes$ -category with  $\text{End } \mathbf{1} = \mathbb{C}$ , whose fusion hypergroup is isomorphic to that of  $\mathfrak{sl}(N)$ . Then there is a  $q \in \mathbb{C}^*$  such that  $\mathcal{C}$  is equivalent (as a tensor category) to the representation category of the Drinfeld/Jimbo quantum group  $SL_q(N)$  (or one of finitely many twisted versions of it). Here  $q$  is either 1 or not a root of unity and unique up to  $q \rightarrow q^{-1}$ . (For another approach to a characterization of the  $SL_q(N)$ -categories, excluding the root of unity case, cf. [228].)

Furthermore: If  $\mathcal{C}$  is a semisimple  $\mathbb{C}$ -linear rigid  $\otimes$ -category with  $\text{End } \mathbf{1} = \mathbb{C}$ , whose fusion hypergroup is isomorphic to that of the (finite!) representation category of  $SL_q(N)$ , where  $q$  is a primitive root of unity of order  $\ell > N$ , then  $\mathcal{C}$  is equivalent to  $\text{Rep } SL_q(N)$  (or one of finitely many twisted versions).

We will say (a bit) more on quantum groups later. The reason that we mention the Kazhdan/Wenzl result already here is that it does not require  $\mathcal{C}$  to come with a braiding. Unfortunately, the proof is not independent of quantum group theory, nor does it provide a *construction* of the categories.

Beginning of proof: The assumption on the fusion rules implies that  $\mathcal{C}$  has a multiplicative generator  $Z$ . Consider the full monoidal subcategory  $\mathcal{C}_0$  with objects  $\{Z^{\otimes n}, n \in \mathbb{Z}_+\}$ . Now  $\mathcal{C}$  is equivalent to the idempotent completion (‘Karoubification’) of  $\mathcal{C}_0$ . (Aside: Tensor categories with objects  $\mathbb{N}_+$  and  $\otimes = +$  for objects appear quite often: The symmetric category  $\mathbb{S}$ , the braid category  $\mathbb{B}$ , PROPs [179].) A semisimple  $k$ -linear category with objects  $\mathbb{Z}_+$  is called a **monoidal algebra**, and is equivalent to having a family  $\mathcal{A} = \{A_{n,m}\}$  of vector spaces together with semisimple algebra structures on  $A_n = A_{n,n}$  and bilinear operations  $\circ : A_{n,m} \times A_{m,p} \rightarrow A_{n,p}$  and  $\otimes : A_{n,m} \times A_{p,q} \rightarrow A_{n+p,m+q}$  satisfying obvious axioms. A monoidal algebra is **diagonal** if  $A_{n,m} = 0$  for  $n \neq m$  and of **type  $\mathbf{N}$**  if  $\dim A(0, n) = \dim A(n, 0) = 1$  and  $A_{n,m} = 0$  unless

$n \equiv m \pmod{N}$ . If  $\mathcal{A}$  is of type  $N$ , there are exactly  $N$  monoidal algebras with the same diagonal. The possible diagonals arising from type  $N$  monoidal algebras can be classified, using Hecke algebras  $H_n(q)$  (defined later).

- There is an analogous result (Tuba/Wenzl [259]) for categories with the other classical (BCD) fusion rings, but that *does* require the categories to come with a braiding.
- For fusion categories, there are a number of classification results in the case of low rank (number of simple objects) (Ostrik: fusion categories of rank 2 [224], braided fusion categories of rank 3 [225]) or special dimensions, like  $p$  or  $pq$  (Etingof/Gelaki/Ostrik [82]). Furthermore, one can classify **near group categories**, i.e. fusion categories with all simple objects but one invertible (Tambara/Yamagami [254], Siehler [244]).
- In another direction one may try to represent more tensor categories as module categories by *generalizing the notion of Hopf algebras*. We have already encountered a very modest (but useful) generalization, to wit Van Daele’s multiplier Hopf algebras. (But the main rationale for the latter was to repair the breakdown of Pontrjagin duality for infinite dimensional Hopf algebras, which works so nicely for finite dimensional Hopf algebras.)
- Drinfeld’s **quasi-Hopf algebras** [73] go in a different direction: One considers an associative unital algebra  $H$  with a unital algebra homomorphism  $\Delta : H \rightarrow H \otimes H$ , where coassociativity holds only up to conjugation with an invertible element  $\phi \in H \otimes H \otimes H$ :

$$\text{id} \otimes \Delta \circ \Delta(x) = \phi(\Delta \otimes \text{id} \circ \Delta(x))\phi^{-1},$$

where  $(\Delta, \phi)$  must satisfy some identity in order for  $\text{Rep } H$  with the tensor product defined in terms of  $\Delta$  to be (non-strict) monoidal. Unfortunately, duals of quasi-Hopf algebras are not quasi-Hopf algebras. They are useful nevertheless, even for the proof of results concerning ordinary Hopf algebras, like the Kohno-Drinfeld theorem for  $U_q(\mathfrak{g})$ , cf. [73, 74] and [137].

Examples: Given a finite group  $G$  and  $\omega \in Z^3(G, k^*)$ , there is a finite dimensional quasi Hopf algebra  $D^\omega(G)$ , the twisted quantum double of Dijkgraaf/Pasquier/Roche [66]. (We will later define its representation category in a purely categorical way.) Recently, Naidu/Nikshych [205] have given necessary and sufficient conditions on pairs  $(G, [\omega]), (G', [\omega]')$  for  $D^\omega(G) - \text{Mod}, D^{\omega'}(G') - \text{Mod}$  to be equivalent as braided tensor categories. But the question for which pairs  $(G, [\omega])$   $D^\omega(G) - \text{Mod}$  is Tannakian (i.e. admits a fiber functor and therefore is equivalent to the representation category of an ordinary Hopf algebra) seems to be still open.

- There have been various attempts at proving generalized Tannaka reconstruction theorems in terms of quasi-Hopf algebras [182] and “weak quasi-Hopf algebras”. (Cf. e.g. [176, 113].) As it turned out, it is sufficient to consider ‘weak’, but ‘non-quasi’ Hopf algebras:
- Preceded by Hayashi’s ‘face algebras’ [115], which largely went unnoticed, Böhm and Szlachányi [35] and then Nikshych, Vainerman, L. Kadison

introduced **weak Hopf algebras**, which may be considered as finite-dimensional quantum groupoids: They are associative unital algebras  $A$  with *coassociative* algebra homomorphism  $\Delta : A \rightarrow A \otimes A$ , but the axioms  $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$  and  $\varepsilon(\mathbf{1}) = 1$  are weakened.

Weak Hopf algebras are closely related to Hopf algebroids and have various desirable properties: Their duals are weak Hopf algebras, and Pontrjagin duality holds. The categorical dimensions of their representations can be non-integer. And they are general enough to ‘explain’ finite-index depth-two inclusions of von Neumann factors, cf. [215].

- Furthermore, Ostrik [222] proved that every fusion category is the module category of a semisimple weak Hopf algebra. (Again, there was related earlier work by Hayashi [116] in the context of his face algebras [115].)

Proof idea: An  **$R$ -fiber functor** on a fusion category  $\mathcal{C}$  is a faithful tensor functor  $\mathcal{C} \rightarrow \text{Bimod } R$ , where  $R$  is a finite direct sum of matrix algebras. Szlachányi [252]: An  $R$ -fiber functor on  $\mathcal{C}$  gives rise to an equivalence  $\mathcal{C} \simeq A - \text{Mod}$  for a weak Hopf algebra (with base  $R$ ). (Cf. also [110].) How to construct an  $R$ -fiber functor?

Since  $\mathcal{C}$  is semisimple, we can choose an algebra  $R$  such that  $\mathcal{C} \simeq R - \text{Mod}$  (as abelian categories). Since  $\mathcal{C}$  is a module category over itself, we have a  $\mathcal{C}$ -module structure on  $R - \text{Mod}$ . Now use that, for  $\mathcal{C}$  and  $R$  as above, there is a bijection between  $R$ -fiber functors and  $\mathcal{C}$ -module category structures on  $R - \text{Mod}$  (i.e. tensor functors  $\mathcal{C} \rightarrow \text{End}(R - \text{Mod})$ ).

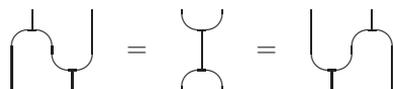
Remarks: 1.  $R$  is highly non-unique: The only requirement was that the number of simple direct summands equals the number of simple objects of  $\mathcal{C}$ . (Thus there is a unique commutative such  $R$ , but even for that, there is no uniqueness of  $R$ -fiber functors.)

2. The above proof uses semisimplicity. (A non-semisimple generalization was announced by Bruguières and Virelizier in 2008.)

- Let  $\mathcal{C}$  be fusion category and  $A$  a weak Hopf algebra such that  $\mathcal{C} \simeq A - \text{Mod}$ . Since there is a dual weak Hopf algebra  $\widehat{A}$ , it is natural to ask how  $\widehat{\mathcal{C}} = \widehat{A} - \text{Mod}$  is related to  $\mathcal{C}$ . (One may call such a category dual to  $\mathcal{C}$ , but must keep in mind that there is one for every weak Hopf algebra  $A$  such that  $\mathcal{C} \simeq A - \text{Mod}$ .)
- Answer:  $\widehat{A} - \text{Mod}$  is (weakly monoidally) Morita equivalent to  $\mathcal{C}$ . This notion (Müger [191]) was inspired by subfactor theory, in particular ideas of Ocneanu, cf. [216, 217]. For this we need the following:
- A **Frobenius algebra** in a strict tensor category is a quintuple  $(A, m, \eta, \Delta, \varepsilon)$ , where  $(A, m, \eta)$  is an algebra,  $(A, \Delta, \varepsilon)$  is a coalgebra and the Frobenius identity

$$m \otimes \text{id}_A \circ \text{id}_A \otimes \Delta = \Delta \circ m = \text{id}_A \otimes m \circ \Delta \otimes \text{id}_A$$

holds. Diagrammatically:

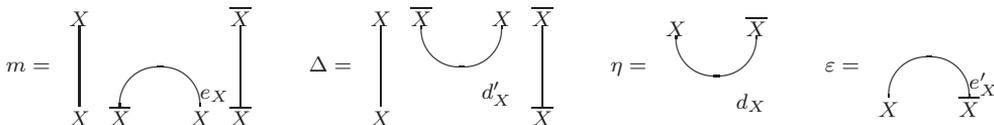


A Frobenius algebra in a  $k$ -linear category is called **strongly separable** if

$$\varepsilon \circ \eta = \alpha \text{id}_{\mathbf{1}}, \quad m \circ \Delta = \beta \text{id}_{\Gamma}, \quad \alpha\beta \in k^*.$$

The roots of this definition go quite far back. F. Quinn [231] discussed them under the name ‘ambialgebras’, and L. Abrams [1] proved that Frobenius algebras in  $\text{Vect}_k^{\text{fin}}$  are the usual Frobenius algebras, i.e.  $k$ -algebras  $V$  equipped with a  $\phi \in V^*$  such that  $(x, y) \mapsto \phi(xy)$  is non-degenerate. Frobenius algebras play a central rôle for topological quantum field theories in 1 + 1 dimensions, cf. e.e. [156].

- Frobenius algebras arise from two-sided duals in tensor categories: Let  $X \in \mathcal{C}$  with two-sided dual  $\overline{X}$ , and define  $\Gamma = X \otimes \overline{X}$ . Then  $\Gamma$  carries a Frobenius algebra structure, cf. [191]:



Verifying the Frobenius identities and strong separability is a trivial exercise. In view of  $\text{End}(V) \cong V \otimes V^*$  in the category of finite dimensional vector spaces, the above Frobenius algebra is called an ‘endomorphism (Frobenius) algebra’.

- This leads to the question whether every (strongly separable) Frobenius algebra in a  $\otimes$ -category arise in this way. The answer is, not quite, but: If  $\Gamma$  is a strongly separable Frobenius algebra in a  $k$ -linear spherical tensor category  $\mathcal{A}$  then there exist
  - a spherical  $k$ -linear 2-category  $\mathcal{E}$  with two objects  $\{\mathfrak{A}, \mathfrak{B}\}$ ,
  - a 1-morphism  $X \in \text{Hom}_{\mathcal{E}}(\mathfrak{B}, \mathfrak{A})$  with 2-sided dual  $\overline{X} \in \text{Hom}_{\mathcal{E}}(\mathfrak{A}, \mathfrak{B})$ , and therefore a Frobenius algebra  $X \circ \overline{X}$  in the  $\otimes$ -category  $\text{End}_{\mathcal{E}}(\mathfrak{A})$ ,
  - a monoidal equivalence  $\text{End}_{\mathcal{E}}(\mathfrak{A}) \xrightarrow{\cong} \mathcal{A}$  mapping the the Frobenius algebra  $X \circ \overline{X}$  to  $\Gamma$ .

Thus every Frobenius algebra in  $\mathcal{A}$  arises from a 1-morphism in a bicategory  $\mathcal{E}$  containing  $\mathcal{A}$  as a corner. In this situation, the tensor category  $\mathcal{B} = \text{End}_{\mathcal{E}}(\mathfrak{B})$  is called weakly monoidally Morita equivalent to  $\mathcal{A}$  and the bicategory  $\mathcal{E}$  is called a Morita context.

- The original proof in [191] was tedious. Assuming mild technical conditions on  $\mathcal{A}$  and strong separability of  $\Gamma$ , the bicategory  $\mathcal{E}$  can simply be obtained as follows:

$$\begin{aligned} \text{Hom}_{\mathcal{E}}(\mathfrak{A}, \mathfrak{A}) &= \mathcal{A}, \\ \text{Hom}_{\mathcal{E}}(\mathfrak{A}, \mathfrak{B}) &= \Gamma - \text{Mod}_{\mathcal{A}}, \\ \text{Hom}_{\mathcal{E}}(\mathfrak{B}, \mathfrak{A}) &= \text{Mod}_{\mathcal{A}} - \Gamma, \\ \text{Hom}_{\mathcal{E}}(\mathfrak{B}, \mathfrak{B}) &= \Gamma - \text{Mod}_{\mathcal{A}} - \Gamma, \end{aligned}$$

with the composition of 1-morphisms given by the usual tensor products of (left and right)  $\Gamma$ -modules. Cf. [285]. (A discussion free of any technical assumptions on  $\mathcal{A}$  was recently given in [163].)

- Weak monoidal Morita equivalence of tensor categories also admits an interpretation in terms of module categories: If  $\mathfrak{A}, \mathfrak{B}$  are objects in a bicategory  $\mathcal{E}$  as above, the category  $\text{Hom}_{\mathcal{E}}(\mathfrak{A}, \mathfrak{B})$  is a left module category over the tensor category  $\text{End}_{\mathcal{E}}(\mathfrak{B})$  and a right module category over  $\mathcal{A} = \text{End}_{\mathcal{E}}(\mathfrak{A})$ . In fact, the whole structure can be formulated in terms of module categories, thereby getting rid of the Frobenius algebras, cf. [85, 84]: Writing  $\mathcal{M} = \text{Hom}_{\mathcal{E}}(\mathfrak{A}, \mathfrak{B})$ , the dual category  $\mathcal{B} = \text{End}_{\mathcal{E}}(\mathfrak{B})$  can be obtained as the tensor category  $\text{HOM}_{\mathcal{A}}(\mathcal{M}, \mathcal{M})$ , denoted  $\mathcal{A}_{\mathcal{M}}^*$  in [85], of right  $\mathcal{A}$ -module functors from  $\mathcal{M}$  to itself.

Since the two pictures are essentially equivalent, the choice is a matter of taste. The picture with Frobenius algebras and the bicategory  $\mathcal{E}$  is closer to subfactor theory. What speaks in favor of the module category picture is the fact that non-isomorphic algebras in  $\mathcal{A}$  can have equivalent module categories, thus give rise to the same  $\mathcal{A}$ -module category. (But not in the case of commutative algebras!)

- Morita equivalence of tensor categories indeed is an equivalence relation, denoted  $\approx$ . (In particular,  $\mathcal{B}$  contains a strongly separable Frobenius algebra  $\widehat{\Gamma}$  such that  $\widehat{\Gamma} - \text{Mod}_{\mathcal{B}} - \widehat{\Gamma} \simeq \mathcal{A}$ .)
- As mentioned earlier, the left regular representation of a finite dimensional Hopf algebra  $H$  gives rise to a Frobenius algebra  $\Gamma$  in  $H - \text{Mod}$ .  $\Gamma$  is strongly separable if and only if  $H$  is semisimple and cosemisimple. In this case, one finds for the ensuing Morita equivalent category:

$$\mathcal{B} = \Gamma - \text{Mod}_{H - \text{Mod}} - \Gamma \simeq \widehat{H} - \text{Mod}.$$

(This is a situation encountered earlier in subfactor theory.) Actually, in this case the Morita context  $\mathcal{E}$  had been defined independently by Tambara [253].

The same works for weak Hopf algebras, thus for any semisimple and co-semisimple weak Hopf algebra we have  $A - \text{Mod} \approx \widehat{A} - \text{Mod}$ , provided the weak Hopf algebra is Frobenius, i.e. has a non-degenerate integral. (It is unknown whether every weak Hopf algebra is Frobenius.)

- The above concept of Morita equivalence has important applications: If  $\mathcal{C}_1, \mathcal{C}_2$  are Morita equivalent (spherical) fusion categories then
  - (1)  $\dim \mathcal{C}_1 = \dim \mathcal{C}_2$ .
  - (2)  $\mathcal{C}_1$  and  $\mathcal{C}_2$  give rise to the same triangulation TQFT in 2+1 dimensions (as defined by Barrett/Westbury [19] and S. Gelfand/Kazhdan [104], generalizing the Turaev/Viro TQFT [265, 262] to non-braided categories. Cf. also Ocneanu [218].)

This fits nicely with the known fact (Kuperberg [159], Barrett/Westbury [18]) that, the spherical categories  $H - \text{Mod}$  and  $\widehat{H} - \text{Mod}$  (for a

semisimple and co-semisimple Hopf algebra  $H$ ) give rise to the same triangulation TQFT.

- (3) The braided centers  $Z_1(\mathcal{C}_1), Z_1(\mathcal{C}_2)$  (to be discussed in the next section) are equivalent as braided tensor categories. This is quite immediate by a result of Schauenburg [240].
- We emphasize that (just like  $\text{Vect}_k$ ) a fusion category can contain many (strongly separable) Frobenius algebras, thus it can be Morita equivalent to many other tensor categories. In view of this, studying (Frobenius) algebras in fusion categories is an important and interesting subject. (Even more so in the braided case.)
- Example: Commutative algebras in a representation category  $\text{Rep } G$  (for  $G$  finite) are the same as commutative algebras carrying a  $G$ -action by algebra automorphisms. The condition  $\dim \text{Hom}(\mathbf{1}, \Gamma) = 1$  means that the  $G$ -action is ergodic. Such algebras correspond to closed subgroups  $H \subset G$  via  $\Gamma_H = C(G/H)$ . Cf. [155].
- Algebras in and module categories over the category  $\mathcal{C}_k(G, \omega)$  defined in Section 1 were studied in [223].
- A **group theoretical category** is a fusion category that is weakly Morita equivalent (or ‘dual’) to a pointed fusion category, i.e. one of the form  $\mathcal{C}_k(G, \omega)$  (with  $G$  finite and  $[\omega] \in H^3(G, \mathbb{T})$ ). (The original definition [222] was in terms of quadruples  $(G, H, \omega, \psi)$  with  $H \subset G$  finite groups,  $\omega \in Z^3(G, \mathbb{C}^*)$  and  $\psi \in C^2(H, \mathbb{C}^*)$  such that  $d\psi = \omega|_H$ , but the two notions are equivalent by Ostrik’s analysis of module categories of  $\mathcal{C}_k(G, \omega)$  [222].) For more on group theoretical categories cf. [203, 101].
- The above considerations are closely related to subfactor theory (at finite Jones index): A **factor** is a von Neumann algebra with center  $\mathbb{C}\mathbf{1}$ . For an inclusion  $N \subset M$  of factors, there is a notion of index  $[M : N] \in [1, +\infty]$  (not necessarily integer!!), cf. [126, 169]. One has  $[M : N] < \infty$  if and only if the canonical  $N$ - $M$ -bimodule  $X$  has a dual 1-morphism  $\overline{X}$  in the bicategory of von Neumann algebras, bimodules and their intertwiners. Motivated by Ocneanu’s bimodule picture of subfactors [216, 217] one observes that the bicategory with the objects  $\{N, M\}$  and bimodules generated by  $X, \overline{X}$  is a Morita context. On the other hand, a single factor  $M$  gives rise to a certain tensor  $*$ -category  $\mathcal{C}$  (consisting of  $M - M$ -bimodules or the endomorphisms  $\text{End } M$ ) such that, by Longo’s work [170], the Frobenius algebras (“Q-systems” [170]) in  $\mathcal{C}$  are (roughly) in bijection with the subfactors  $N \subset M$  with  $[M : N] < \infty$ . (Cf. also the introduction of [191].)

#### 4. BRAIDED TENSOR CATEGORIES

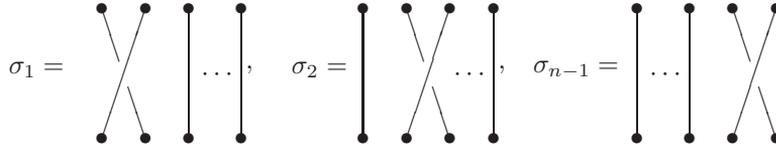
- The symmetric groups have the well known presentation

$$S_n = \{\sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i-j| > 1, \sigma_i^2 = 1\}.$$

Dropping the last relation, one obtains the **Braid groups**:

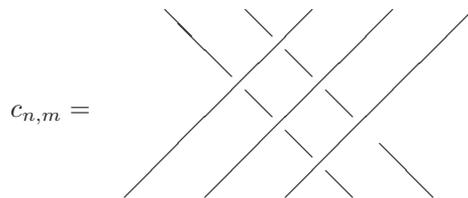
$$B_n = \{\sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i-j| > 1\}.$$

They were introduced by Artin in 1928, but had appeared implicitly in much earlier work by Hurwitz, cf. [141]. They have a natural geometric interpretation:



Note:  $B_n$  is infinite for all  $n \geq 2$ ,  $B_2 \cong \mathbb{Z}$ . The representation theory of  $B_n, n \geq 3$  is difficult. It is known that all  $B_n$  are linear, i.e. they have faithful finite dimensional representations  $B_n \hookrightarrow GL(m, \mathbb{C})$  for suitable  $m = m(n)$ . Cf. Kassel/Turaev [141].

- Analogously, one can drop the condition  $c_{Y,X} \circ c_{X,Y} = \text{id}$  on a symmetric tensor category. This leads to the concept of a **braiding**, due to Joyal and Street [128, 132], i.e. a family of natural isomorphisms  $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$  satisfying two hexagon identities but not necessarily the condition  $c^2 = \text{id}$ . Notice that without the latter condition, one needs to require two hexagon identities, the second being obtained from the first one by the replacement  $c_{X,Y} \rightsquigarrow c_{Y,X}^{-1}$  (which does nothing when  $c^2 = \text{id}$ ). (The latter is the non-strict generalization of  $c_{X \otimes Y, Z} = c_{X, Z} \otimes \text{id}_Y \circ \text{id}_X \otimes c_{Y, Z}$ .) A **braided tensor category** (BTC) now is a tensor category equipped with a braiding.
- In analogy to the symmetric case, given a BTC  $\mathcal{C}$  and  $X \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$ , one has a homomorphism  $\Pi_n^X : B_n \rightarrow \text{Aut}(X^{\otimes n})$ .
- The most obvious example of a BTC that is not symmetric is provided by the braid category  $\mathbb{B}$ . In analogy to the symmetric category  $\mathbb{S}$ , it is defined by  $\text{Obj } \mathbb{B} = \mathbb{Z}_+$ ,  $\text{End}(n) = B_n$ ,  $n \otimes m = n + m$ , while on the morphisms  $\otimes$  is defined by juxtaposition of braid diagrams. The definition of the braiding  $c_{n,m} \in \text{End}(n + m) = B_{n+m}$  is illustrated by the example  $(n, m) = (3, 2)$ :



- If  $\mathcal{C}$  is a strict BTC and  $X \in \mathcal{C}$ , there is a unique braided tensor functor  $F : \mathbb{B} \rightarrow \mathcal{C}$  such that  $F(1) = X$  and  $F(c_{2,2}) = c_{X,X}$ . Thus  $\mathbb{B}$  is the **free braided tensor category** generated by one object.
- Centralizer and center  $Z_2$ :  
 If  $\mathcal{C}$  is a BTC, we say that two objects  $X, Y$  **commute** if  $c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}$ . If  $\mathcal{D} \subset \mathcal{C}$  is subcategory (or just subset of  $\text{Obj } \mathcal{C}$ ), we define the

**centralizer**  $\mathcal{C} \cap \mathcal{D}' \subset \mathcal{C}$  as the full subcategory defined by

$$\text{Obj}(\mathcal{C} \cap \mathcal{D}') = \{X \in \mathcal{C} \mid c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y} \quad \forall Y \in \mathcal{D}'\}.$$

Now, the **center**  $Z_2(\mathcal{C})$  is

$$Z_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'.$$

Notice that  $\mathcal{C} \cap \mathcal{D}'$  is monoidal and  $Z_2(\mathcal{C})$  is symmetric! In fact, a BTC  $\mathcal{C}$  is symmetric if and only if  $\mathcal{C} = Z_2(\mathcal{C})$ . Apart from ‘central’, the objects of  $Z_2(\mathcal{C})$  have been called ‘degenerate’ [232] or ‘transparent’ [39].

- We thus see that STC are maximally commutative BTCs. Does it make sense to speak of maximally non-commutative BTCs?  $\mathbb{B}$  is an example since  $\text{Obj } Z_2(\mathbb{B}) = \{0\}$ . Braided fusion categories with ‘trivial’ center will turn out to be just Turaev’s modular categories, cf. Section 5.
- Since the definition of BTCs is quite natural if one knows the braid groups, one may wonder why they appeared more than 20 years after symmetric categories. Most likely, this was a consequence of a lack of really interesting examples. When they finally appeared in [128], this was mainly motivated by developments internal to category theory (and homotopy theory). It is a remarkable historical accident that this happened at the same time as (and independently from) the development of quantum groups, which dramatically gained in popularity in the wake of Drinfeld’s talk [72].
- In 1971 it was shown [68] that certain representation theoretic considerations for quantum field theories in spacetimes of dimension  $\geq 2 + 1$  lead to symmetric categories. Adapting this theory to  $1 + 1$  dimensions inevitably leads to braided categories, as was finally shown in 1989, cf. [90]. That this was not done right after the appearance of [68] must be considered as a missed opportunity.
- As promised, we will briefly look at braided categorical groups. Consider  $\mathcal{C}(G)$  for  $G$  abelian. As shown in [132] – and in much more detail in the preprints [128] – the braided categorical groups  $\mathcal{C}$  with  $\pi_0(\mathcal{C}) \cong G$  (isomorphism classes of objects) and  $\pi_1(\mathcal{C}) \cong A$  ( $\text{End } \mathbf{1}$ ) are classified by the group  $H_{ab}^3(G, A)$ , where  $H_{ab}^n(G, A)$  refers to the Eilenberg-Mac Lane cohomology theory for *abelian* groups, cf. [177]. (Whereas  $H^3(G, A)$  can be defined in terms of topological cohomology theory as  $H^3(K(G, 1), A)$  of the Eilenberg-Mac Lane space  $K(G, 1)$ , one has  $H_{ab}^3(G, A) := H^4(K(G, 2), A)$ . This group also has a description in terms of quadratic functions  $q : G \rightarrow A$ . The subgroup of  $H_{ab}^3(G, A)$  corresponding to symmetric braidings is isomorphic to  $H^5(K(G, 3), A)$ , cf. [46].)
- Duality: Contrary to the symmetric case, in the presence of a (non-symmetric) braiding, having a left duality is not sufficient for a nice theory: If we define a right duality in terms of a left duality and the braiding, the left and right traces will fail to have all the properties they do have in the symmetric case. Therefore, some additional concepts are needed:
- A **twist** for a braided category with left duality is a natural family  $\{\Theta_X \in \text{End } X, X \in \mathcal{C}\}$  of isomorphisms (i.e. a natural isomorphism of the functor

$\text{id}_{\mathcal{C}}$ ) satisfying

$$\Theta_{X \otimes Y} = \Theta_X \otimes \Theta_Y \circ c_{Y,X} \circ c_{X,Y}, \quad \Theta_{\mathbf{1}} = \text{id}_{\mathbf{1}}, \quad \vee(\Theta_X) = \Theta_{\vee X}.$$

Notice: If  $c_{Y,X} \circ c_{X,Y} \neq \text{id}$  then the natural isomorphism  $\Theta$  is *not monoidal* and  $\Theta = \text{id}$  is not a legal twist!

- A **ribbon category** is a strict braided tensor category equipped with a left duality and a twist.
- Let  $\mathcal{C}$  be a ribbon category with left duality  $X \mapsto (\vee X, e_X, d_X)$  and twist  $\Theta$ . We define a right duality  $X \mapsto (X^\vee, e'_X, d'_X)$  by  $X^\vee = \vee X$  and (2.2). Now one can show, cf. e.g. [137], that the maps  $\text{End } X \rightarrow \text{End } \mathbf{1}$  defined as in (3.1) coincide and that  $\text{Tr}(s) := \text{Tr}_L(s) = \text{Tr}_R(s)$  has the trace property and behaves well under tensor products, as previously in the symmetric case. Writing  $\overline{X} = \vee X = X^\vee$ , one finds that  $\mathcal{C}$  is a spherical category in the sense of [20]. Conversely, if  $\mathcal{C}$  is spherical and braided, then *defining*

$$\Theta_X = (\text{Tr}_X \otimes \text{id}_X)(c_{X,X}),$$

$\{\Theta_X, X \in \mathcal{C}\}$  satisfies the axioms of a twist and thus forms a ribbon structure together with the left duality. (Cf. Yetter [289], based on ideas of Deligne, and Barrett/Westbury [20].)

(Personally, I prefer to consider the twist as a derived structure, thus talking about spherical categories with a braiding, rather than about ribbon categories. In some situations, e.g. when the center  $Z_1(\mathcal{C})$  is involved, this is advantageous. This also is the approach of the Rome school [71, 172].)

- So far, our only example of a non-symmetric braided category is the free braided category  $\mathbb{B}$ , which is not rigid. In the remainder of this section, we will consider three main ‘routes’ to braided categories: (A) the topological route, (B) the “non-perturbative approach” via quantum doubles and categorical centers, and (C) the “perturbative approach” via deformation (or ‘quantization’) of symmetric categories.
- We briefly mention one construction of an interesting braided category that doesn’t seem to fit nicely into one of our routes: While the usual representation category of a group is symmetric, the category of representations of the general linear group  $GL_n(\mathbb{F}_q)$  over a finite field with the *external* tensor product of representations turns out to be braided and non-symmetric, cf. [133].

#### 4.1. Route A: Free braided categories (tangles) and their quotients.

- Combining the ideas behind the Temperley-Lieb categories  $\text{TL}(\tau)$  (which have duals) and the braid category  $\mathbb{B}$  (which is braided but has no duals), one arrives at the categories of **tangles** (Turaev [260], Yetter [287]. See also [262, 137].) One must distinguish between categories of **unoriented tangles** having  $\text{Obj } U\text{-}\mathcal{TAN} = \mathbb{Z}_+$  with tensor product (of objects) given by addition and **oriented tangles**, based on  $\text{Obj } O\text{-}\mathcal{TAN} = \{+, -\}^*$  (i.e. finite words in  $\pm$ ,  $\mathbf{1} = \emptyset$ ) with concatenation as tensor product. In either case, the morphisms are given as sets of pictures as in Figure 1, or else by

linear combinations of such pictures with coefficients in a commutative ring or field. All this is just as in the discussion of the free symmetric categories at the end of Section 2. The only difference is that one must distinguish between over- and undercrossings of the lines; for technical reasons it is more convenient to do this in terms of pictures embedded in 3-space.

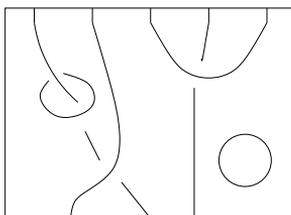


FIGURE 1. An unoriented 3-5 tangle

There also is a category  $O\text{-}\mathcal{TA}\mathcal{N}$  of oriented tangles, where the objects are finite words in  $\pm, \mathbf{1} = \emptyset$  and the lines in the morphisms are directed, in a way that is compatible with the signs of the objects. It is clear that the morphisms in  $\text{End}(\mathbf{1})$  in  $U\text{-}\mathcal{TA}\mathcal{N}$  ( $O\text{-}\mathcal{TA}\mathcal{N}$ ) are just the unoriented (oriented) links.

While the definition is intuitively natural, the details are tedious and we refer to the textbooks [262, 137, 290]. In particular, we omit discussing *ribbon* tangles.

- The tangle categories are pivotal, in fact spherical, thus ribbon categories.  $O\text{-}\mathcal{TA}\mathcal{N}$  is the free ribbon category generated by one element, cf. [243].
- Let  $\mathcal{C}$  be a ribbon category. Then one can define a category  $\mathcal{C}\text{-}\mathcal{TA}\mathcal{N}$  of  $\mathcal{C}$ -labeled oriented tangles and a ribbon tensor functor  $F_{\mathcal{C}} : \mathcal{C}\text{-}\mathcal{TA}\mathcal{N} \rightarrow \mathcal{C}$ . (This is the rigorous rationale behind the diagrammatic calculus for braided tensor categories!)

Let  $\mathcal{C}$  be a ribbon category and  $X$  a self-dual object. Given an unoriented tangle, we can label every edge by  $X$ . This gives a composite map

$$\{\text{links}\} \xrightarrow{\cong} \text{Hom}_{U\text{-}\mathcal{TA}\mathcal{N}}(0, 0) \longrightarrow \text{Hom}_{\mathcal{C}\text{-}\mathcal{TA}\mathcal{N}}(0, 0) \xrightarrow{F_{\mathcal{C}}} \text{End}_{\mathcal{C}}\mathbf{1}.$$

In particular, if  $\mathcal{C}$  is  $k$ -linear with  $\text{End}\mathbf{1} = k\text{id}$ , we obtain a map from  $\{\text{links}\}$  to  $k$ , which is easily seen to be a knot invariant. If  $\mathcal{C} = U_q(\mathfrak{sl}(2))\text{-Mod}$  and  $X$  is the fundamental object, one essentially obtains the Jones polynomial. Cf. [260, 234]. (The other objects of  $\mathcal{C}$  give rise to the colored Jones polynomials, which are much studied in the context of the volume conjecture for hyperbolic knots.)

- So far, all our examples of braided categories have come from topology. In a sense, they are quite trivial, since they are just the universal braided (ribbon) categories freely generated by one object. Furthermore, we are primarily interested in linear categories. Of course, we can apply the  $k$ -linearization functor  $\mathcal{CAT} \rightarrow k\text{-lin.}\text{-}\mathcal{CAT}$ . But the categories we obtain

have infinite dimensional hom-sets and are not more interesting than the original ones. (This should be contrasted to the symmetric case, where this construction produces the representation categories of the classical groups, cf. Section 2.)

- Thus in order to obtain interesting  $k$ -linear ribbon categories from the tangle categories, we must reduce the infinite dimensional hom-spaces to finite dimensional ones.

We consider the following analogous situation in the context of associative algebras: The braid group  $B_n$  ( $n > 1$ ) is infinite, thus the group algebra  $\mathbb{C}B_n$  is infinite dimensional. But this algebra has finite dimensional quotients, e.g. the **Hecke algebra**  $H_n(q)$ , the unital  $\mathbb{C}$ -algebra generated by  $\sigma_1, \dots, \sigma_{n-1}$ , modulo the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ when } |i - j| > 1, \quad \sigma_i^2 = (q - 1)\sigma_i + q\mathbf{1}.$$

This algebra is finite dimensional for any  $q$ , and for  $q = 1$  we have  $H_n(q) \cong \mathbb{C}S_n$ . In fact,  $H_n(q)$  is isomorphic to  $\mathbb{C}S_n$ , thus semisimple, whenever  $q$  is not a root of unity, but this isomorphism is highly non-trivial. Cf. e.g. [164].

The idea now is to do a similar thing on the level of categories, or to ‘categorify’ the Hecke algebras or other quotients of  $\mathbb{C}B_n$  like the Birman/Murakami/Wenzl- (BMW-)-algebras [29].

- We have seen that ribbon categories give rise to knot invariants. One can go the other way and construct  $k$ -linear ribbon categories from link invariants. This approach was initiated in [262, Chapter XII], where a topological construction of the representation category of  $U_q(\mathfrak{sl}(2))$  was given. A more general approach was studied in [267]. A  $k$ -valued link invariant  $G$  is said to **admit functorial extension to tangles** if there exists a tensor functor  $F : U - \mathcal{TAN} \rightarrow k - \text{Mod}$  whose restriction to  $\text{End}_{U - \mathcal{TAN}}(0) \cong \{\text{links}\}$  equals  $G$ .

For any  $X \in U - \mathcal{TAN}$ ,  $f \in \text{End}(X)$ , let  $L_f$  be the link obtained by closing  $f$  on the right, and define  $\text{Tr}_G(f) = G(L_f)$ . If  $\mathcal{C}$  is the  $k$ -linearization of  $U - \mathcal{TAN}$ , it is shown in [267], under weak assumptions on  $G$ , that the idempotent and direct sum completion of the quotient of  $\mathcal{C}$  by the ideal of negligible morphisms is a semisimple ribbon category with finite dimensional hom-sets. Cf. [267].

Example: Applying the above procedure  $G = V_t$ , the Jones polynomial, one obtains a Temperley-Lieb category  $\mathcal{TL}_\tau$ , which in turn is equivalent to a category  $U_q(\mathfrak{sl}(2)) - \text{Mod}$ . Cf. [262, Chapter XII]. Applying it to the Kauffman polynomial [142], one obtains the quantized BTCs of types BCD, cf. [267]. The general theory in [267] is quite nice, but it should be noted that the assumption of functorial extendability to tangles is rather strong: It implies that the resulting semisimple category admits a fiber functor and therefore is the representation category of a discrete quantum group. Furthermore, the application of the general formalism of [267] to the Kauffman polynomial used input from (q-deformed) quantum group

theory for the proof of functorial extension to tangles and of modularity. This drawback was repaired by Beliakova/Blanchet, cf. [22, 23].

Blanchet [31] gave a similar construction with HOMFLY polynomial [92], obtaining the type A categories. (The HOMFLY polynomial is an invariant for oriented links, thus one must work with oriented tangles.)

Remark: The ribbon categories of BCD type arising from the Kauffman polynomial give rise to topological quantum field theories. The latter can even be constructed directly from the Kauffman bracket, bypassing the categories, cf. [32]. This construction actually preceded those mentioned above.

- The preceding constructions reinforce the close connection between braided categories and knot invariants. It is important to realize that this reasoning is not circular, since the polynomials of Jones, HOMFLY, Kauffman can (nowadays) be constructed in rather elementary ways, independently of categories and quantum groups, cf. e.g. [167]. Since the knot polynomials are defined in terms of skein relations, we speak of the skein construction of the quantum categories, which arguably is the simplest known so far.
- In the case  $q = 1$ , the skein constructions of the ABCD categories reduce to the construction of the categories arising from classical groups mentioned in Section 2. (This happens since  $q = 1$  corresponds to parameters in the knot polynomials for which they fail to distinguish over- from under-crossings. Then one can replace the tangle categories by symmetric categories of non-embedded cobordisms (oriented or not) as in [59].)
- Concerning the exceptional Lie algebras and their quantum categories, inspired by work of Cvitanovic, cf. [52] for a book-length treatment, and by Vogel [273], Deligne conjectured [59] that there is a one parameter family of symmetric tensor categories  $\mathcal{C}_t$  specializing to  $\text{Rep } G$  for the exceptional Lie groups at certain values of  $t$ . This is still unproven, but see [48, 63, 62] for work resulting from this conjecture. (For the  $E_n$ -categories, including the  $q$ -deformed ones, cf. [277].)
- In a similar vein, Deligne defined [61] a one parameter family of rigid symmetric tensor categories  $\mathcal{C}_t$  such that  $\mathcal{C}_t \simeq \text{Rep } S_t$  for  $t \in \mathbb{N}$ . These categories were studied further in [49]. (Recall that  $S_n$  is considered as the  $GL_n(\mathbb{F}_1)$  where  $\mathbb{F}_1$  is the ‘field with one element’, cf. [248].)
- More generally, one can define linear categories by generators and relations, cf. e.g. [160].

**4.2. Route B: Doubles and centers.** We begin with a brief look at Hopf algebras.

- **Quasi-triangular** Hopf algebras (Drinfeld, 1986 [72]): If  $H$  is a Hopf algebra and  $R$  an invertible element of (possibly a completion of)  $H \otimes H$ , satisfying

$$R\Delta(\cdot)R^{-1} = \sigma \circ \Delta(\cdot), \quad \sigma(x \otimes y) = y \otimes x,$$

$$(\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{12}.$$

$$(\varepsilon \otimes \text{id})(R) = (\text{id} \otimes \varepsilon)(R) = \mathbf{1}.$$

If  $(V, \pi), (V', \pi') \in H - \text{Mod}$ , the definition  $c_{(V, \pi), (V', \pi')} = \Sigma_{V, V'}(\pi \otimes \pi')(R)$  produces a braiding for  $H - \text{Mod}$ .

- But this has only shifted the problem: How to get quasi-triangular Hopf algebras? To this purpose, Drinfeld [72] gave the quantum double construction  $H \rightsquigarrow D(H)$ , which associates a quasi-triangular Hopf algebra  $D(H)$  to a Hopf algebra  $H$ . Cf. also [137].
- Soon after, an analogous categorical construction was given by Drinfeld (unpublished), Joyal/Street [130] and Majid [181]: The (braided) center  $Z_1(\mathcal{C})$ , defined as follows.

Let  $\mathcal{C}$  be a strict tensor category and let  $X \in \mathcal{C}$ . A half braiding  $e_X$  for  $X$  is a family  $\{e_X(Y) \in \text{Hom}_{\mathcal{C}}(X \otimes Y, Y \otimes X), Y \in \mathcal{C}\}$  of isomorphisms, natural w.r.t.  $Y$ , satisfying  $e_X(\mathbf{1}) = \text{id}_X$  and

$$e_X(Y \otimes Z) = \text{id}_Y \otimes e_X(Z) \circ e_X(Y) \otimes \text{id}_Z \quad \forall Y, Z \in \mathcal{C}.$$

Now, the **center**  $Z_1(\mathcal{C})$  of  $\mathcal{C}$  has as objects pairs  $(X, e_X)$ , where  $X \in \mathcal{C}$  and  $e_X$  is a half braiding for  $X$ . The morphisms are given by

$$\text{Hom}_{Z_1(\mathcal{C})}((X, e_X), (Y, e_Y)) = \{t \in \text{Hom}_{\mathcal{C}}(X, Y) \mid \text{id}_X \otimes t \circ e_X(Z) = e_Y(Z) \circ t \otimes \text{id}_X \quad \forall Z \in \mathcal{C}\}.$$

The tensor product of objects is given by  $(X, e_X) \otimes (Y, e_Y) = (X \otimes Y, e_{X \otimes Y})$ , where

$$e_{X \otimes Y}(Z) = e_X(Z) \otimes \text{id}_Y \circ \text{id}_X \otimes e_Y(Z).$$

The tensor unit is  $(\mathbf{1}, e_{\mathbf{1}})$  where  $e_{\mathbf{1}}(X) = \text{id}_X$ . The composition and tensor product of morphisms are inherited from  $\mathcal{C}$ . Finally, the braiding is given by

$$c((X, e_X), (Y, e_Y)) = e_X(Y).$$

(The author finds this definition is much more transparent than that of  $D(H)$  even though *a priori* little is known about  $Z_1(\mathcal{C})$ .)

- Just as the centralizer  $\mathcal{C} \cap \mathcal{D}'$  generalizes  $Z_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$ , there is a version of  $Z_1$  relative to a subcategory  $\mathcal{D} \subset \mathcal{C}$ , cf. [181].
- $Z_1(\mathcal{C})$  is categorical version (generalization) of Hopf algebra quantum double in the following sense: If  $H$  is a finite dimensional Hopf algebra, there is an equivalence

$$Z_1(H - \text{Mod}) \simeq D(H) - \text{Mod} \tag{4.1}$$

of braided tensor categories, cf. e.g. [137]. (If  $H$  is infinite dimensional, one still has an equivalence between  $Z_1(H - \text{Mod})$  and the category of Yetter-Drinfeld modules over  $H$ .)

- If  $\mathcal{C}$  is a category and  $\mathcal{D} := Z_0(\mathcal{C}) = \text{End}(\mathcal{C})$  is its tensor category of endofunctors, then  $Z_1(\mathcal{D})$  is trivial. (This may be considered as the categorification of the fact that the center (in the usual sense) of the endomorphism monoid  $\text{End}(S)$  of a set  $S$  is trivial, i.e. equal to  $\{\text{id}_S\}$ .) But in general, the braided center of a tensor category is a non-trivial braided category that is not symmetric. Unfortunately, this doesn't seem to have been studied

thoroughly. Presently, strong results on  $Z_1(\mathcal{C})$  exist only in the case where  $\mathcal{C}$  is a fusion category.

- There are abstract categorical considerations, quite unrelated to topology and quantum groups, that provide rationales for studying BTCs:

(A): A second, compatible, multiplication functor on a tensor category gives rise to a braiding, and conversely, cf. [132]. (This is a higher dimensional version of the Eckmann-Hilton argument mentioned earlier.)

(B): Recall that tensor categories are bicategories with one object. Now, braided tensor categories turn out to be monoidal bicategories with one object, which in turn are weak 3-categories with one object and one 1-morphism. Thus braided (and symmetric) categories really are a manifestation of the existence of  $n$ -categories for  $n > 1$ !

- Baez-Dolan [10] conjectured the following ‘periodic table’ of ‘ $k$ -tuply monoidal  $n$ -categories’:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$k = 0$	sets	categories	2-categories	3-categories	...
$k = 1$	monoids	monoidal categories	monoidal 2-categories	monoidal 3-categories	...
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories	braided monoidal 3-categories	...
$k = 3$	”	symmetric monoidal categories	‘sylleptic’ monoidal 2-categories	?	...
$k = 4$	”	”	symmetric monoidal 2-categories	?	...
$k = 5$	”	”	”	symmetric monoidal 3-categories	...
$k = 6$	”	”	”	”	...

In particular, one expects to find ‘center constructions’ from each structure in the table to the one underneath it. For the column  $n = 1$  these are the centers  $Z_0, Z_1, Z_2$  discussed above. For  $n = 0$  they are given by the endomorphism monoid of a set and the ordinary center of a monoid. The column  $n = 2$  is also relatively well understood, cf. Crans [50]. There is an accepted notion of a non-strict 3-category (i.e.  $n = 3, k = 0$ ) (Gordon/Power/Street [108]), but there are many competing definitions of weak higher categories. We refrain from moving any further into this subject. See e.g. [13].

- With this heuristic preparation, one can give a high-brow interpretation of  $Z_1(\mathcal{C})$ , cf. [132, 250]: Let  $\mathcal{C}$  be tensor category and  $\Sigma\mathcal{C}$  the corresponding bicategory with one object. Then the category  $\text{End}(\Sigma\mathcal{C})$  of endofunctors of  $\Sigma\mathcal{C}$  is a monoidal bicategory (with natural transformations as 1-morphisms and ‘modifications’ as 2-morphisms). Now,  $\mathcal{D} = \text{End}_{\text{End}(\Sigma\mathcal{C})}(\mathbf{1})$  is a tensor

category with two compatible  $\otimes$ -structures (categorifying  $\mathbf{End}\ \mathbf{1}$  in a tensor category), thus braided, and it is equivalent to  $Z_1(\mathcal{C})$ .

- For further abstract considerations on the center  $Z_1$ , consider the work of Street [250, 251] and of Bruguières and Virelizier [41, 42].
- If  $\mathcal{C}$  is braided there is a braided embedding  $\iota_1 : \mathcal{C} \hookrightarrow Z_1(\mathcal{C})$ , given by  $X \mapsto (X, e_X)$ , where  $e_X(Y) = c(X, Y)$ . Defining  $\tilde{\mathcal{C}}$  to be the tensor category  $\mathcal{C}$  with ‘opposite’ braiding  $\tilde{c}_{X,Y} = c_{Y,X}^{-1}$ , there is an analogous embedding  $\tilde{\iota} : \mathcal{C} \hookrightarrow Z_1(\mathcal{C})$ . In fact, one finds that the images of  $\iota, \iota'$  are each others’ centralizers:

$$Z_1(\mathcal{C}) \cap \iota(\mathcal{C})' = \tilde{\iota}(\tilde{\mathcal{C}}), \quad Z_1(\mathcal{C}) \cap \tilde{\iota}(\tilde{\mathcal{C}})' = \iota(\mathcal{C}).$$

Cf. [192]. On the one hand, this is an instance of the double commutant principle, and on the other hand, this establishes one connection

$$\iota(\mathcal{C}) \cap \tilde{\iota}(\tilde{\mathcal{C}}) = \iota(Z_2(\mathcal{C})) = \tilde{\iota}(Z_2(\tilde{\mathcal{C}})),$$

between  $Z_1$  and  $Z_2$  which suggests that “ $Z_1(\mathcal{C}) \simeq \mathcal{C} \times \tilde{\mathcal{C}}$ ” when  $Z_2(\mathcal{C})$  is “trivial”. We will return to both points in the next section.

#### 4.3. Route C: Deformation of groups or symmetric categories.

- As for Route B, there is a more traditional approach via deformation of Hopf algebras and a somewhat more recent one focusing directly on deformation of tensor categories.
- ( $C_1$ ): The earlier approach to braided categories relies on deformation of Hopf algebras related to groups. For lack of space we will limit ourselves to providing just enough information as needed for the discussion of the more categorical approach. For more, we refer to the textbooks, in particular [137, 47, 124, 173]. In any case, one chooses a simple (usually compact) Lie group  $G$  and considers either the enveloping algebra  $U(\mathfrak{g})$  of its Lie algebra  $\mathfrak{g}$  in terms of Serre’s generators and relations [242], or one departs from the algebra  $\text{Fun}(G)$  of regular functions on  $G$ , which can also be described in terms of finitely many relations, cf. e.g. [279]. In a nutshell, one inserts factors of a ‘deformation parameter’  $q$  into the presentation of  $U(\mathfrak{g})$  or  $\text{Fun}(G)$  in such a way that for  $q \neq 1$  one still obtains a (non-trivial) Hopf algebra. Quantum group theory began with the discovery that this is possible at all.
- Obviously, this ‘definition’ is a farcical caricature. But there is some truth in it: In the mathematical literature on quantum groups, cf. e.g. [137, 47, 124, 173], it is all but impossible to find a comment on the origin of the presentation of the quantum group under study and of the underlying motivation. While the initiators of quantum group theory from the Leningrad school (Faddeev, Kulish, Semenov-Tian-Shansky, Sklyanin, Reshetikhin, Drinfeld and others) were very well aware of these origins, this knowledge has now almost faded into obscurity. (This certainly has to do with the fact that the applications to theoretical physics for which

quantum groups were invented in the first place are still exclusively pursued by physicists, cf. e.g. [106].) One point of this section will be that – quite independently of the original physical motivation – the categorical approach to quantum deformation is mathematically better motivated.

- In what follows, we will concentrate on the enveloping algebra approach. The usual Drinfeld-Jimbo presentation of the quantized enveloping algebra is as follows, Consider the algebra  $U_q(\mathfrak{g})$  generated by elements  $E_i, F_i, K_i, K_i^{-1}, 1 \leq i \leq r$ , satisfying the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \\ E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0,$$

where  $\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[k]_{q_i}! [m-k]_{q_i}!}, [m]_{q_i}! = [m]_{q_i} [m-1]_{q_i} \dots [1]_{q_i}, [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$  and  $q_i = q^{a_i}$ . This is a Hopf algebra with coproduct  $\Delta$  and counit  $\varepsilon$  defined by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \\ \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

One should distinguish between Drinfeld’s [72] formal approach, where one constructs a Hopf algebra  $H$  over the ring  $\mathbb{C}[[\hbar]]$  of formal power series in such a way that  $H/\hbar H$  is isomorphic to the enveloping algebra  $U(\mathfrak{g})$ , and the non-formal deformation of Jimbo [125], who obtains an honest quasi-triangular Hopf algebra  $U_q(\mathfrak{g})$  (over  $\mathbb{C}$ ) for any value  $q \in \mathbb{C}$  of a deformation parameter. (In this approach, the properties of the resulting Hopf algebra depend heavily on whether  $q$  is a root of unity or not. In the formal approach, this distinction obviously does not arise.) The relation between both approaches becomes clear by inserting  $q = e^\hbar$  in Jimbo’s definition and considering the result as a Hopf algebra over  $\mathbb{C}[[\hbar]]$ .

- (C<sub>2</sub>): As mentioned, one can obtain non-symmetric braided categories directly by ‘deforming’ symmetric categories. This approach was initiated by Cartier [45] and worked out in more detail in [137, Appendix] and [140]. (These works were all motivated by applications to Vassiliev link invariants, which we cannot discuss here.)

Let  $\mathcal{S}$  be a strict symmetric Ab-category. Now an **infinitesimal braiding** on  $\mathcal{S}$  is a natural family of endomorphisms  $t_{X,Y} : X \otimes Y \rightarrow X \otimes Y$  satisfying

$$c_{X,Y} \circ t_{X,Y} = t_{Y,X} \circ c_{X,Y} \quad \forall X, Y,$$

$$t_{X,Y \otimes Z} = t_{X,Y} \otimes \text{id}_Z + c_{X,Y}^{-1} \otimes \text{id}_Z \circ \text{id}_Y \otimes t_{X,Z} \circ c_{X,Y} \otimes \text{id}_Z \quad \forall X, Y, Z.$$

Strict symmetric Ab-categories equipped with an infinitesimal braiding were called **infinitesimal symmetric**. (We would prefer to call them symmetric categories equipped with an infinitesimal braiding.)

- Example: If  $H$  is a Hopf algebra, there is a bijection between infinitesimal braidings  $t$  on  $\mathcal{S} = H - \text{Mod}$  and elements  $t \in \text{Prim}(H) \otimes \text{Prim}(H)$  (where  $\text{Prim}(H) = \{x \in H \mid \Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x\}$ ) satisfying  $t_{21} = t$  and  $[t, \Delta(H)] = 0$ , given by  $t_{X,Y} = (\pi_X \otimes \pi_Y)(t)$ .
- Now we can define the formal deformation of a symmetric category associated to an infinitesimal braiding: Let  $\mathcal{S}$  be a strict  $\mathbb{C}$ -linear symmetric category with finite dimensional hom-sets and let  $t$  be an infinitesimal braiding for  $\mathcal{S}$ . We write  $\mathcal{S}[[\hbar]]$  for the  $\mathbb{C}[[\hbar]]$ -linear category obtained by extension of scalars. (I.e.  $\text{Obj } \mathcal{S}[[\hbar]] = \text{Obj } \mathcal{S}$  and  $\text{Hom}_{\mathcal{S}[[\hbar]]}(X, Y) = \text{Hom}_{\mathcal{S}}(X, Y) \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$ .) Also the functor  $\otimes : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  lifts to  $\mathcal{S}[[\hbar]]$ . For objects  $X, Y, Z$ , define

$$\alpha_{X,Y,Z} = \Theta_{KZ}(h t_{X,Y} \otimes \text{id}_Z, h \text{id}_X \otimes t_{Y,Z}), \quad \tilde{c}_{X,Y} = c_{X,Y} \circ e^{ht_{X,Y}/2}.$$

Here  $\Theta_{KZ}$  is a Drinfeld associator [73], i.e. a formal power series

$$\Theta_{KZ}(A, B) = \sum_{w \in \{A, B\}^*} c_w w$$

in two non-commuting variables  $A, B$ , where  $c_w \in \mathbb{C}$ , satisfying certain identities. (Cf. [137, Chapter XIX, (8.27)-(8.29)].) Then  $(\mathcal{S}[[\hbar]], \otimes, \mathbf{1}, \alpha)$  is a (non-strict) tensor category with associativity constraint  $\alpha$ , trivial unit constraints and  $\tilde{c}$  a braiding. If  $\mathcal{S}$  is rigid, then  $(\mathcal{S}[[\hbar]], \otimes, \mathbf{1}, \alpha, \tilde{c})$  admits a ribbon structure.

- Application: Let  $\mathfrak{g}$  be a simple Lie algebra/ $\mathbb{C}$ . Let  $\mathcal{S} = \mathfrak{g} - \text{Mod}$  and define  $\{t_{X,Y}\}$  be as in the example, corresponding to  $t = (\sum_i x_i \otimes x^i + x^i \otimes x_i)/2$ , where  $x_i, x^i$  are dual bases of  $\mathfrak{g}$  w.r.t. the Killing form. Then  $[t, \Delta(\cdot)] = 0$  and one can prove

$$(\mathcal{S}[[\hbar]], \otimes, \mathbf{1}, \alpha, \tilde{c}) \simeq U_{\hbar}(\mathfrak{g}) - \text{Mod} \tag{4.2}$$

as  $\mathbb{C}[[\hbar]]$ -linear ribbon categories. (The proof is a corollary of the proof of the Kohno-Drinfeld theorem [73, 74], cf. also [137].)

Remark: 1. Obviously, we have cheated: The main difficulty resides in the definition of  $\Theta_{KZ}$ ! Giving the latter and proving its properties requires ca. 10-15 pages of rather technical material (but no Lie theory). Le and Murakami explicitly wrote down an associator; cf. e.g. [137, Remark XIX.8.3]. Drinfeld also gave a non-constructive proof of existence of an associator defined over  $\mathbb{Q}$ , cf. [74].

2. The above is relevant for a more conceptual approach to the theory of finite-type knot invariants (Vassiliev invariants), cf. [45, 140].

3. A disadvantage of the above is that we obtain only a formal deformation of  $\mathcal{S}$ . If  $\mathfrak{g}$  is a simple Lie algebra and  $\mathcal{S} = \mathfrak{g} - \text{Mod}$ , we know by (4.2), that we obtain the  $\mathbb{C}[[\hbar]]$ -category  $U_{\hbar}(\mathfrak{g}) - \text{Mod}$ . On the other hand, thanks to the work of Jimbo [125] and others [173, 124] we know that there is a non-formal version  $U_q(\mathfrak{g})$  of the quantum group with  $\mathbb{C}$ -linear representation category. One would therefore hope that the  $\mathbb{C}$ -linear categories  $U_q(\mathfrak{g}) - \text{Mod}$  can be obtained directly as deformations of the module

categories  $U(\mathfrak{g}) - \text{Mod}$ . Indeed, for numerical  $q \in \mathbb{C} \setminus \mathbb{Q}$ , with some more analytical effort one can make sense of  $\alpha_q = \Theta_{KZ}(h t_{X,Y} \otimes \text{id}_Z, h \text{id}_X \otimes t_{Y,Z})$  as an element of  $\text{End}(X \otimes Y \otimes Z)$  and define a non-formal,  $\mathbb{C}$ -linear category  $\mathcal{C}(\mathfrak{g}, q)$  and prove an equivalence

$$\mathcal{C}(\mathfrak{g}, q) = (\mathcal{S}, \otimes, \mathbf{1}, \alpha_q, \tilde{c}_q) \simeq U_q(\mathfrak{g}) - \text{Mod}$$

of  $\mathbb{C}$ -linear ribbon categories. This was done by Kazhdan and Lusztig [144], but see also the nice recent exposition by Neshveyev/Tuset [209].

- Fact: If  $q \in \mathbb{C}^*$  is generic, i.e. *not* a root of unity, then  $\mathcal{C}(\mathfrak{g}, q) := U_q(\mathfrak{g}) - \text{Mod}$  is a semisimple braided ribbon category whose fusion hypergroup is isomorphic to that of  $U(\mathfrak{g})$ , thus of the category of  $\mathfrak{g}$ -modules, cf. [124, 173]. But it is not symmetric for  $q \neq 1$ , thus certainly not equivalent to the latter. In fact,  $U_q(\mathfrak{g}) - \text{Mod}$  and  $U(\mathfrak{g}) - \text{Mod}$  are already inequivalent as  $\otimes$ -categories. (Recall that associativity constraints  $\alpha$  can be considered as generalized 3-cocycles, and the  $\alpha_q$  for different  $q$  are not cohomologous.)
- We have briefly discussed the Cartier/Kassel/Turaev formal deformation quantization of symmetric categories equipped with an infinitesimal braiding. There is a cohomology theory for Ab- tensor categories and tensor functors that classifies deformations due to Davydov [53] and Yetter [290].

Definition: Let  $F : \mathcal{C} \rightarrow \mathcal{C}'$  a tensor functor. Define  $T_n : \mathcal{C}^n \rightarrow \mathcal{C}$  by  $X_1 \times \cdots \times X_n \mapsto X_1 \otimes \cdots \otimes X_n$ . ( $T_0(\emptyset) = \mathbf{1}$ ,  $T_1 = \text{id}$ .) Let  $C_F^n(\mathcal{C}) = \text{End}(T_n \circ F^{\otimes n})$ . ( $C_F^0(\mathcal{C}) = \text{End } \mathbf{1}'$ .) For a fusion category, this is finite dimensional. Define  $d : C_F^n(\mathcal{C}) \rightarrow C_F^{n+1}(\mathcal{C})$  by

$$df = \text{id} \otimes f_{2, \dots, n+1} - f_{12, \dots, n+1} + f_{1, 23, \dots, n+1} - \cdots + (-1)^n f_{1, \dots, n(n+1)} + (-1)^{n+1} f_{1, \dots, n} \otimes \text{id},$$

where, e.g.,  $f_{12,3, \dots, n+1}$  is defined in terms of  $f$  using the isomorphism  $d_{X_1, X_2}^F : F(X_1) \otimes F(X_2) \rightarrow F(X_1 \otimes X_2)$  coming with the tensor functor  $F$ .

One has  $d^2 = 0$ , thus  $(C^i, d)$  is a complex. Now  $H_F^i(\mathcal{C})$  is the cohomology of this complex, and  $H^i(\mathcal{C}) = H_F^i(\mathcal{C})$  for  $F = \text{id}_{\mathcal{C}}$ .

In low dimensions one finds that  $H_F^1$  classifies derivations of the tensor functor  $F$ ,  $H_F^2$  classifies deformations of the tensor structure  $\{d_{X,Y}^F\}$  of  $F$ .  $H^3(\mathcal{C})$  classifies deformations of the associativity constraint  $\alpha$  of  $\mathcal{C}$ .

Examples: 1. If  $\mathcal{C}$  is fusion then  $H^i(\mathcal{C}) = 0 \forall i > 0$ . This implies Ocneanu rigidity, cf. [84].

2. If  $\mathfrak{g}$  is a reductive algebraic group with Lie algebra  $\mathfrak{g}$  and  $\mathcal{C} = \text{Rep } G$  (algebraic representations). Then  $H^i(\mathcal{C}) \cong (\Lambda^i \mathfrak{g})^G \forall i$ . If  $\mathfrak{g}$  is simple then  $H^1(\mathcal{C}) = H^2(\mathcal{C}) = 0$ , but  $H^3(\mathcal{C})$  is one-dimensional, corresponding to a one-parameter family of deformations  $\mathcal{C}$ . According to [84] “it is easy to guess that this deformation comes from an actual deformation, namely the deformation of  $O(G)$  to the quantum group  $O_q(G)$ ”. It is not clear to this author whether this suggestion should be considered as proven. If so, together with the one-dimensionality of  $H^3(\mathfrak{g} - \text{Mod})$  it provides a very satisfactory ‘explanation’ for the existence of the quantized categories  $\mathcal{C}(\mathfrak{g}, q) \simeq U_q(\mathfrak{g}) - \text{Mod}$ .

- In analogy to the result of Kazhdan and Wenzl mentioned in Section 3, Tuba and Wenzl [259] proved that a semisimple ribbon category with the fusion hypergroup isomorphic to that of a simple classical Lie algebra  $\mathfrak{g}$  of BCD type (i.e. orthogonal or symplectic) is equivalent to the category  $\mathcal{C}(\mathfrak{g}, q)$ , with  $q = 1$  or not a root of unity, or one of finitely many twisted versions thereof. Notice that in contrast to the Kazhdan/Wenzl result [145], this result needs the category to be braided! (Again, this is a characterization, not a construction of the categories.)
- Finkelberg [89] proved a braided equivalence between  $\mathcal{C}(\mathfrak{g}, q)$ ,  $q = e^{i\pi/m\kappa}$ , where  $m = 1$  for ADE,  $m = 2$  for BCD and  $m = 3$  for  $G_2$ , and the ribbon category  $\tilde{\mathcal{O}}_\kappa$  of integrable representations of the affine Lie algebra  $\hat{\mathfrak{g}}$  of central charge  $c = \kappa - \check{h}$ , where  $\check{h}$  is the dual Coxeter number of  $\mathfrak{g}$ .

The category  $\tilde{\mathcal{O}}_\kappa$  plays an important rôle in conformal field theory, either in terms of vertex operator algebras or via the representation theory of loop groups (Wassermann [275], Toledo-Laredo [256]). This is the main reason for the relevance of quantum groups to CFT.

- Finally, we briefly discuss the connection between routes (B) and (C) to BTCs: In order to find an R-matrix for the Hopf algebra  $U_q(\mathfrak{g})$  one traditionally uses the quantum double, appealing to an isomorphism  $U_q(\mathfrak{g}) \cong D(B_q(\mathfrak{g}))/I$ , where  $B_q(\mathfrak{g})$  is the  $q$ -deformation of a Borel subalgebra of  $\mathfrak{g}$  and  $I$  an ideal in  $D(B_q(\mathfrak{g}))$ . Now  $R_{U_q(\mathfrak{g})} = (\phi \otimes \phi)(R_{D(B_q(\mathfrak{g}))})$ , where  $\phi$  is the quotient map. Since a surjective Hopf algebra homomorphism  $H_1 \rightarrow H_2$  corresponds to a full monoidal inclusion  $H_2 - \text{Mod} \hookrightarrow H_1 - \text{Mod}$ , and recalling the connection (4.1) between Drinfeld’s double construction and the braided center  $Z_1$ , we conclude that the BTC  $U_q(\mathfrak{g}) - \text{Mod}$  is a full monoidal subcategory of  $Z_1(B_q(\mathfrak{g}) - \text{Mod})$  (with the inherited braiding). Therefore, also in the deformation approach, the braiding can be understood as ultimately arising from the  $Z_1$  center construction.
- Question: It is natural to ask whether a similar observation also holds for  $q$  a root of unity, i.e., whether the modular categories  $\mathcal{C}(\mathfrak{g}, q)$ , for  $q$  a root of unity, can be understood as full  $\otimes$ -subcategories of  $Z_1(\mathcal{D})$ , where  $\mathcal{D}$  is a fusion category corresponding to the deformed Borel subalgebra  $B_q(\mathfrak{g})$ . Very recently, Etingof and Gelaki [81] gave an affirmative answer in some cases.

Remark: In the next section, we will discuss a criterion that allows to recognize the quantum doubles  $Z_1(\mathcal{C})$  of fusion categories.

### 5. MODULAR CATEGORIES

- Turaev [261, 262]: A **modular** category is a fusion category that is ribbon (alternatively, spherical and braided) such that the matrix  $S = (S_{i,j})$

$$S_{i,j} = \text{Tr}_{X \otimes Y}(c_{Y,X} \circ c_{X,Y}), \quad i, j \in I(\mathcal{C}),$$

where  $I(\mathcal{C})$  is the set of simple objects modulo isomorphism, is invertible.

- A fusion category that is ribbon is modular if and only if  $\dim \mathcal{C} \neq 0$  and the center  $Z_2(\mathcal{C})$  is trivial. (In the sense of consisting only of the objects

$\mathbf{1} \oplus \cdots \oplus \mathbf{1}$ .) (This was proven by Rehren [232] for  $*$ -categories and by Beliakova/Blanchet [23] in general. Cf. also [39] and [2].)

Thus: Modular categories are braided fusion categories with trivial center, i.e. the maximally non-symmetric ones. (This definition seems more conceptual than the original one in terms of invertibility of  $S$ .)

- Why are these categories called ‘modular’? Let  $S$  as above and  $T = \text{diag}(\omega_i)$ , where  $\Theta_{X_i} = \omega_i \text{id}_{X_i}$ ,  $i \in I$ . Then

$$S^2 = \alpha C, \quad (ST)^3 = \beta C, \quad (\alpha\beta \neq 0)$$

where  $C_{i,j} = \delta_{i,\bar{j}}$ , thus  $S, T$  give rise to a projective representation of the modular group  $SL(2, \mathbb{Z})$  (which has a presentation  $\{s, t \mid (st)^3 = s^2 = c, c^2 = e\}$ ). Cf. [232, 262].

- At first sight, this is somewhat mysterious. Notice:  $SL(2, \mathbb{Z})$  is the mapping class group of the 2-torus  $S^1 \times S^1$ . Now, by work of Reshetikhin/Turaev [235, 262], providing a rigorous version of ideas of Witten, every modular category gives rise to a **topological quantum field theory** in  $2 + 1$  dimensions. Every such TQFT in turn gives rise to projective representation of the mapping class groups of all closed surfaces, and for the torus one obtains just the above representation of  $SL(2, \mathbb{Z})$ . Cf. [262, 15]. We don’t have the time to say more about TQFTs.
- Turaev’s motivation came from conformal field theory (CFT). (Cf. e.g. Moore-Seiberg [189]). In fact, there is a (rigorous) definition of **rational chiral CFTs** (using von Neumann algebras) and their representations, for which one can prove that the latter are unitary modular (Kawahigashi, Longo, Müger [143]). Most of the examples considered in the (heuristic) physics literature fit into this scheme. (E.g. the loop group models: [275, 282] and the minimal Virasoro models with  $c < 1$  [168].)

In the context of vertex operator algebras, similar results were proven by Huang [121].

- It is natural to ask whether there are less complicated ways to produce modular categories? The answer is positive; we will reconsider our three routes to braided categories.
- Route A: Recall that the classical categories can be obtained from the linearized tangle categories (type A: oriented tangles, types BCD: unoriented tangles), dividing by ideals defined in terms of the knot polynomials of HOMFLY and Kauffman. At roots of unity, this leads to modular categories, cf. [267, 31, 23].
- Route  $C_1$ : H. Andersen et al. [4], Turaev/Wenzl [266] (and others): Let  $\mathfrak{g}$  be a simple Lie algebra and  $q$  a primitive root of unity. Then  $U_q(\mathfrak{g}) - \text{Mod}$  gives rise to a modular category  $\mathcal{C}(\mathfrak{g}, q)$ . (Using tilting modules, dividing by negligible morphisms, etc.)
- Let  $q$  be primitive root of unity of order  $\ell$ . Then  $\mathcal{C}(\mathfrak{g}, q)$  has a positive  $*$ -operation (i.e. is unitary) if  $\ell$  is even (Kirillov Jr. [152], Wenzl [276]) and is not unitarizable for odd  $\ell$  (Rowell [236]).

- Characterization theorem: A braided fusion category with the fusion hypergroup of  $\mathcal{C}(\mathfrak{g}, q)$ , where  $\mathfrak{g}$  is a simple Lie algebra of BCD type and  $q$  a root of unity, is equivalent to  $\mathcal{C}(\mathfrak{g}, q)$  or one of finitely many twisted versions. (Tuba/Wenzl [259])
- Before we reconsider Route B, we assume that we already have a braided fusion category, or **pre-modular category**.

As we have seen, failure of modularity is due to non-trivial center  $Z_2(\mathcal{C})$ . Idea: Given a braided (but not symmetric) category with *even* center  $Z_2(\mathcal{C})$ , kill the latter, using the Deligne / Doplicher-Roberts theorem:  $Z_2(\mathcal{C}) \simeq \text{Rep } G$ . The latter contains a commutative (Frobenius) algebra  $\Gamma$  corresponding to the regular representation of  $G$ . Now  $\Gamma - \text{Mod}_{\mathcal{C}}$  is modular. (Bruguères [39], Müger [190]). This construction can be interpreted as Galois closure in a Galois theory for BTCs, cf. [190].

- Route B to braided categories: Quantum doubles: If  $G$  is a finite group then  $D(G) - \text{Mod}$  and  $D^\omega(G) - \text{Mod}$  are modular (Bantay [16], Altschuler/Coste [3]). If  $H$  is a finite-dimensional semisimple and cosemisimple Hopf algebra then  $D(H) - \text{Mod}$  is modular (Etingof/Gelaki [79]). If  $A$  is a finite-dimensional weak Hopf algebra then  $D(A) - \text{Mod}$  modular (Nikshych/Turaev/Vainerman [214]).
- The center  $Z_1$  of a left/right rigid, pivotal, spherical category has the same properties. In particular, the center of a spherical category is spherical and braided, thus a ribbon category. (Under weaker assumptions, this is not true, and existence of a twist for the center, if desired, must be enforced by a categorical version of the ribbonization of a Hopf algebra, cf. [139].)
- The braided center  $Z_1$ : If  $\mathcal{C}$  is spherical fusion category and  $\dim \mathcal{C} \neq 0$  then  $Z_1(\mathcal{C})$  is modular and  $\dim Z_1(\mathcal{C}) = (\dim \mathcal{C})^2$ . (Müger [192].)

Comments on the proof: Semisimplicity not difficult. Next, one finds a Frobenius algebra  $\Gamma$  in  $\mathcal{D} = \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$  such that the dual category  $\Gamma - \text{Mod}_{\mathcal{D}} - \Gamma$  is equivalent to  $Z_1(\mathcal{C})$ , implying  $\dim Z_2(\mathcal{C}) = (\dim \mathcal{C})^2$ . Here  $\Gamma = \bigoplus_i X_i \boxtimes X_i^{\text{op}}$ , which is again a coend and can exist also in non-semisimple categories.

- This contains all the earlier modularity results on  $D(G) - \text{Mod}$  and  $D(H) - \text{Mod}$ , but also for  $D^\omega(G) - \text{Mod}$  since:

$$D^\omega(G) - \text{Mod} \simeq Z_1(\mathcal{C}_k(G, \omega)).$$

(Using work by Hausser/Nilg [114] or Panaite [226] on quantum double of quasi Hopf-algebras.)

- Modularity of  $Z_1(\mathcal{C})$  also follows by combination of Ostrik's result that every fusion category arises from a weak Hopf algebra  $A$ , combined with modularity of  $D(A) - \text{Mod}$  [214], provided one proves  $D(A) - \text{Mod} \simeq Z_1(A - \text{Mod})$ , generalizing the known result for Hopf algebras. But the purely categorical proof avoiding weak Hopf algebras seems preferable, not least since it probably extends to finite non-semisimple categories.
- In the Morita context having  $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$  and  $Z_1(\mathcal{C})$  as its corners, the two off-diagonal categories are equivalent to  $\mathcal{C}$  and  $\mathcal{C}^{\text{op}}$ , and their structures as  $\mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ -module categories are the obvious ones. Therefore, the center can

also be understood as (using the notation of EO):

$$Z_1(\mathcal{C}) \simeq (\mathcal{C} \boxtimes \mathcal{C}^{\text{op}})_{\mathcal{C}}^*.$$

A (somewhat sketchy) proof of this equivalence can be found in [223, Prop. 2.5].

- We give another example for a purely categorical result that can be proven using weak Hopf algebras: Radford’s formula for  $S^4$  has a generalization to weak Hopf algebras [213], and this can be used to prove that in every fusion category, there exists an isomorphism of tensor functors  $\text{id} \rightarrow ****$ , cf. [83]. (Notice that in every pivotal category we have  $\text{id} \cong **$ , thus here it is important that we understand ‘fusion’ just to mean existence of two-sided duals. But in [84] it is conjectured that every fusion category admits a pivotal structure.)
- If  $\mathcal{C}$  is already modular then there is a braided equivalence  $Z_1(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{op}}$ , cf. [192]. Thus, every modular category  $\mathcal{M}$  is full subcategory of  $Z_1(\mathcal{C})$  for some fusion category. (This probably is not very useful for the classification of modular categories, since there are ‘more fusion categories than modular categories’: Recall from Section 3 that  $\mathcal{C}_1 \approx \mathcal{C}_2 \Rightarrow Z_1(\mathcal{C}_1) \simeq Z_1(\mathcal{C}_2)$ . (For converse, see below.)
- There is a “Double commutant theorem” for modular categories (Müger [193], inspired by Ocneanu [219]): Let  $\mathcal{M}$  a modular category and a  $\mathcal{C} \subset \mathcal{M}$  a replete full tensor subcategory. Then:
  - (1)  $(\mathcal{M} \cap (\mathcal{M} \cap \mathcal{C}')') = \mathcal{C}$ .
  - (2)  $\dim \mathcal{C} \cdot \dim(\mathcal{M} \cap \mathcal{C}') = \dim \mathcal{M}$ ,
  - (3) If, in addition  $\mathcal{C}$  is modular, then also  $\mathcal{D} = \mathcal{M} \cap \mathcal{C}'$  is modular and  $\mathcal{M} \simeq \mathcal{C} \boxtimes \mathcal{D}$ . (Thus every full inclusion of modular categories arises from a direct product.)

These results indicate that ‘modular categories are better behaved than finite groups’.

- Corollary: If  $\mathcal{M}$  is modular and  $\mathcal{S} \subset \mathcal{M}$  symmetric then  $\mathcal{S} \subset \mathcal{M} \cap \mathcal{S}'$ . Thus

$$(\dim \mathcal{S})^2 \leq \dim \mathcal{S} \cdot \dim(\mathcal{M} \cap \mathcal{S}') = \dim \mathcal{M},$$

implying  $\dim \mathcal{S} \leq \sqrt{\dim \mathcal{M}}$ . Notice that the bound is satisfied by  $\text{Rep } G \subset D^{(\omega)}(G) - \text{Mod}$ . In fact, existence of a symmetric subcategory attaining the bound characterizes the representation categories of twisted doubles, cf. below.

- On the other hand, consider  $\mathcal{C} \subset \mathcal{M}$  with  $\mathcal{M}$  modular. We have  $\mathcal{M} \cap \mathcal{C}' \supset Z_2(\mathcal{C})$ , implying  $\dim \mathcal{M} \geq \dim \mathcal{C} \cdot \dim Z_2(\mathcal{C})$ . This provides a lower bound on the dimension of a modular category containing a given pre-modular subcategory as a full tensor subcategory. In [193] it was conjectured that this bound can always be attained.
- It is natural to ask how primality of  $D(G) - \text{Mod}$  is related to simplicity of  $G$ . It turns out that the two properties are independent. On the one hand, there are non-simple finite groups for which  $D(G) - \text{Mod}$  is prime. (This is a corollary of the classification of the full fusion subcategories of

$D(G) - \text{Mod}$  given in [206].) On the other hand, for  $G = \mathbb{Z}/p\mathbb{Z}$  one finds that  $D(G) - \text{Mod}$  is prime if and only if  $p = 2$ . For  $p$  an odd prime,  $D(G) - \text{Mod}$  has two prime factors, both of which are modular categories with  $p$  invertible objects, cf. [193]. But for every finite simple non-abelian  $G$ , one finds that  $D(G) - \text{Mod}$  is prime. In fact, it has only one replete full tensor subcategory at all, namely  $\text{Rep } G$ . Thus all these categories are mutually inequivalent: The classification of prime modular categories contains that of finite simple groups.

- If  $\mathcal{C}$  is symmetric and  $(\Gamma, m, \eta)$  a commutative algebra in  $\mathcal{C}$ , then  $\Gamma - \text{Mod}_{\mathcal{C}}$  is again symmetric and

$$\dim \Gamma - \text{Mod}_{\mathcal{C}} = \frac{\dim \mathcal{C}}{d(\Gamma)}. \tag{5.1}$$

Now, if  $\mathcal{C}$  is only braided,  $\Gamma - \text{Mod}_{\mathcal{C}}$  is a fusion category satisfying (5.1), but in general it fails to be braided! (Unless  $\Gamma \in Z_2(\mathcal{C})$ , as was the case in the context of modularization.)

- Example: Given a BTC  $\mathcal{C} \supset \mathcal{S} \simeq \text{Rep } G$ , let  $\Gamma$  be the regular monoid in  $\mathcal{S}$  as considered in Section 3. Then  $\mathcal{C} \rtimes \mathcal{S} := \Gamma - \text{Mod}_{\mathcal{C}}$  is fusion category, but it is braided only if  $\mathcal{S} \subset Z_2(\mathcal{C})$ , as in the discussion of modularization. In general, one obtains a **braided crossed G-category** as defined by Turaev [263, 264] (cf. also Carrasco and Moreno [44]), i.e. a tensor category with  $G$ -grading  $\partial$  on the objects, a  $G$ -action  $\gamma$  such that  $\partial(\gamma_g(X)) = g\partial Xg^{-1}$  and a ‘braiding’  $c_{X,Y} : X \otimes Y \xrightarrow{\cong} \gamma_{\partial X}(Y) \otimes X$ . The degree zero part is  $\Gamma - \text{Mod}_{\mathcal{C} \cap \mathcal{S}'} \simeq \Gamma - \text{Mod}_{\mathcal{C}}^0$  (cf. below). (Kirillov Jr. [153, 154], Müger [194]). This construction has an interesting connection to conformal orbifold models ([196, 199]).
- Even if  $\Gamma \notin Z_2(\mathcal{C})$ , there is a full tensor subcategory  $\Gamma - \text{Mod}_{\mathcal{C}}^0 \subset \Gamma - \text{Mod}_{\mathcal{C}}$  that is braided. Calling a module  $(X, \mu) \in \Gamma - \text{Mod}_{\mathcal{C}}$  **dyslectic** if

$$\mu \circ c_{X,\Gamma} = \mu \circ c_{\Gamma,X}^{-1},$$

one finds that the full subcategory  $\Gamma - \text{Mod}_{\mathcal{C}}^0$  of dyslectic modules is not only monoidal, but also inherits the braiding from  $\mathcal{C}$ , cf. Pareigis [227]. This was rediscovered by Kirillov and Ostrik [155] who in addition proved that if  $\mathcal{C}$  is modular then  $\Gamma - \text{Mod}_{\mathcal{C}}^0$  is modular and the following identity, similar to (5.1) but different, holds:

$$\dim \Gamma - \text{Mod}_{\mathcal{C}}^0 = \frac{\dim \mathcal{C}}{d(\Gamma)^2}.$$

Remark: Analogous results were previously obtained by Böckenhauer, Evans and Kawahigashi [34] in an operator algebraic context. While the transposition of their work to tensor  $*$ -categories is immediate, removing the  $*$ -assumption requires some work.

- The above implies (for  $*$ -categories, but also in general over  $\mathbb{C}$  by [84]) that  $d(\Gamma) \leq \sqrt{\dim \mathcal{C}}$  for commutative Frobenius algebras in modular categories. (The above bound on the dimension of full symmetric categories follows

from this, since the regular monoid in  $\mathcal{S}$  is a commutative Frobenius algebra  $\Gamma$  with  $d(\Gamma) = \dim \mathcal{S}$ .)

- All these facts have applications to chiral conformal field theories in the operator algebraic framework, reviewed in more detail in [198]:

Longo/Rehren [171]: Finite local extensions of a CFT  $A$  are classified by the ‘local Q-systems’ ( $\approx$  commutative Frobenius algebras) in  $\text{Rep } A$ , which is a  $*$ -BTC.

Böckenhauer/Evans [33], [198]: If  $B \supset A$  is the finite local extension corresponding to the commutative Frobenius algebra  $\Gamma \in \text{Rep } A$ , then  $\text{Rep } B \simeq \Gamma - \text{Mod}_{\text{Rep } A}^0$ .

Analogous results for vertex operator algebras were formulated by Kirillov and Ostrik [155].

Remark: It is perhaps not completely absurd to compare these results to local class field theory, where finite Galois extensions of a local field  $k$  are shown to be in bijection to finite index subgroups of  $k^*$ .

- Drinfeld, Gelaki, Nikshych and Ostrik [75], and independently Kitaev and the author, observed that every commutative Frobenius algebra  $\Gamma$  in a modular category  $\mathcal{M}$  gives rise to a braided equivalence

$$Z_1(\Gamma - \text{Mod}_{\mathcal{M}}) \simeq \mathcal{M} \boxtimes \Gamma - \widetilde{\text{Mod}}_{\mathcal{M}}^0. \tag{5.2}$$

Taking  $\Gamma = \mathbf{1}$ , one recovers the fact  $Z_1(\mathcal{M}) \simeq \mathcal{M} \boxtimes \widetilde{\mathcal{M}}$ . The latter raises the question whether one can find a smaller fusion category  $\mathcal{C}$  such that  $\mathcal{M} \subset Z_1(\mathcal{C})$ . The answer given by (5.2) is that the bigger a commutative algebra one can find in  $\mathcal{M}$ , the smaller one can take  $\mathcal{C}$  to be. In particular, if  $\Gamma - \text{Mod}_{\mathcal{M}}^0$  is trivial (which is equivalent to  $d(\Gamma)^2 = \dim \mathcal{M}$  over  $\mathbb{C}$ ) then  $\mathcal{M} \simeq Z_1(\Gamma - \text{Mod}_{\mathcal{M}})$  is not just contained in a center of a fusion category but is such a center. In fact, this criterion identifies the modular categories of the form  $Z_1(\mathcal{C})$  since, conversely, cf. [57], one finds that the center  $Z_1(\mathcal{C})$  of a fusion category contains a commutative Frobenius algebra  $\Gamma$  of the maximal dimension  $d(\Gamma) = \sqrt{\dim Z_1(\mathcal{C})} = \dim \mathcal{C}$  such that

$$\Gamma - \text{Mod}_{Z_1(\mathcal{C})}^0 \text{ trivial,} \quad \Gamma - \text{Mod}_{Z_1(\mathcal{C})} \simeq \mathcal{C}.$$

- As an application one obtains that if  $\mathcal{M}$  is modular and  $\mathcal{S} \subset \mathcal{M}$  symmetric and even such that  $\dim \mathcal{S} = \sqrt{\dim \mathcal{M}}$  then  $\mathcal{M} \simeq D^\omega(G) - \text{Mod}$ , where  $\mathcal{S} \simeq \text{Rep } G$  and  $\omega \in Z^3(G, \mathbb{T})$ .

This has an application in CFT: If  $A$  is a chiral CFT with trivial representation category  $\text{Rep } A$  (i.e.  $A$  is ‘holomorphic’) acted upon by finite group  $G$ . Then  $\text{Rep } A^G \simeq D^\omega(G) - \text{Mod}$ . (Together with the results of [143], this proves the folk conjecture, having its roots in [67, 66], that the representation category of a ‘holomorphic chiral orbifold CFT’ is given by a category  $D^\omega(G) - \text{Mod}$ .)

- As shown in [191], a weak monoidal Morita equivalence  $\mathcal{C}_1 \approx \mathcal{C}_2$  of fusion categories implies  $Z_1(\mathcal{C}_1) \simeq Z_1(\mathcal{C}_2)$ . (This is an immediate corollary of the

definition of  $\approx$ , combined with [240].) The converse is true for group theoretical categories (Naidu/Nikshych [205]), and a general proof is announced by Nikshych.

- By definition, a group theoretical category  $\mathcal{C}$  is weakly Morita equivalent (dual) to  $\mathcal{C}_k(G, \omega)$  for a finite group  $G$  and  $[\omega] \in H^3(G, \mathbb{T})$ . Thus  $Z_1(\mathcal{C}) \simeq Z_1(\mathcal{C}_k(G, \omega)) \simeq D^\omega(G) - \text{Mod}$ . The converse is also true.

Therefore, with  $\mathcal{M}$  modular and  $\mathcal{C}$  fusion we have:

contains	$\mathcal{M}$	$Z_1(\mathcal{C})$
maximal comm. FA $\Gamma$	$\mathcal{M} \simeq Z_1(\mathcal{C})$	always true
maximal STC $\mathcal{S}$	$\mathcal{M} \simeq D^\omega(G) - \text{Mod}$	$\mathcal{C}$ is group theoretical

- What can we say about non-commutative (Frobenius) algebras in modular categories? We first look at the symmetric case. Let thus  $\mathcal{C}$  be a rigid symmetric  $k$ -linear tensor category and  $\Gamma$  a strongly separable Frobenius algebra in  $\mathcal{C}$ . Define  $p \in \text{End } \Gamma$  by

$$p = (\text{Tr}_\Gamma \otimes \text{id}_\Gamma)(\Delta \circ m \circ c_{\Gamma, \Gamma}) = \begin{array}{c} \Gamma \\ \text{[Diagram 1]} \\ \Gamma \end{array} = \begin{array}{c} \Gamma \\ \text{[Diagram 2]} \\ \Gamma \end{array} \quad (5.3)$$

(The fourfold vertex in the right diagram represents the morphism  $m^{(2)} = m \circ m \otimes \text{id}$ .) Then  $p$  is idempotent (up to a scalar) and its kernel is an ideal. Thus the image of  $p$  is a commutative Frobenius subalgebra of  $\Gamma$ . The latter is called the **center** of  $\Gamma$  since it is the ordinary center in the case  $\mathcal{C} = \text{Vect}_k^{\text{fin}}$ .

- Application to TQFT: Every finite dimensional semisimple  $k$ -algebra  $A$  gives rise to a TQFT in 1+1 dimensions via triangulation (Fukuma/Hosono/Kawai [99]). By the classification of TQFTs in 1+1 dimensions [65, 1, 156], this TQFT corresponds to a commutative Frobenius algebra  $B$  (in  $\text{Vect}_k^{\text{fin}}$ ), with  $A = V(S^1)$  and the product arising from the pants cobordism. The latter is given by the vector space associated with the circle and the multiplication is given by the pants cobordism. One finds  $B = Z(A)$ , and  $B$  arises exactly as the image of  $A$  under the above projection  $p$ . (This works since every semisimple algebra is a Frobenius algebra.)
- If  $\mathcal{C}$  is braided, but not symmetric, we must choose between  $c_{\Gamma, \Gamma}$  and  $c_{\Gamma, \Gamma}^{-1}$  in the definition (5.3) of the idempotent  $p$ . This implies that a non-commutative Frobenius algebra will typically have two different centers, called the left and right centers  $\Gamma_l, \Gamma_r$ . Remarkably, one then obtains an

equivalence

$$E : \Gamma_l - \text{Mod}_{\mathcal{C}}^0 \xrightarrow{\simeq} \Gamma_r - \text{Mod}_{\mathcal{C}}^0$$

of modular categories, cf. Böckenhauer, Evans, Kawahigashi [34], Ostrik [222] and Fröhlich, Fuchs, Runkel, Schweigert [98, 95]. Conversely, if  $\mathcal{C}$  is modular, every triple  $(\Gamma_l, \Gamma_r, E)$  as above arises from a non-commutative algebra in  $\mathcal{C}$ , [157]. (The latter is unique only up to Morita equivalence.)

- This is relevant for the classification of CFTs in two dimensions: The latter are constructed from a pair  $(A_l, A_r)$  of chiral CFTs and some algebraic datum (‘modular invariant’) specifying how the two chiral CFTs are glued together. In the left-right symmetric case, where the two chiral theories coincide  $A_l = A_r = A$ , the above result indicates that Frobenius algebras in  $\mathcal{C} = \text{Rep } A$  are the structure to use. This is substantiated by a construction, using TQFTs, of a ‘topological from a modular category  $\mathcal{C}$  and a Frobenius algebra  $\Gamma \in \mathcal{C}$ , cf. Fuchs, Runkel, Schweigert, cf. [97] and sequels.
- The Frobenius algebras in / module categories of  $SU_q(2) - \text{Mod}$  can be classified in terms of ADE graphs. (Quantum MacKay correspondence.) Cf. Böckenhauer, Evans [33], Kirillov Jr. and Ostrik [155], Etingof/Ostrik [86].
- These results should be extended to other Lie groups. If  $SU(2)$  already leads to the ADE graphs (“ubiquitous” according to [117]), the other classical groups should give rise to very interesting algebraic-combinatorial structures, cf. e.g. [220, 221].
- More generally, when the two chiral theories  $A_l, A_r$ , and therefore the associated modular categories  $\mathcal{C}_l, \mathcal{C}_r$  differ, it is better to work with triples  $(\Gamma_l, \Gamma_r, E)$ , where  $\Gamma_{l/r} \in \mathcal{C}_{l/r}$  are commutative algebras and  $E : \Gamma_l - \text{Mod}_{\mathcal{C}_l}^0 \rightarrow \Gamma_r - \text{Mod}_{\mathcal{C}_r}^0$  is a braided equivalence. (By the above, in the left-right symmetric case  $\mathcal{C}_l = \mathcal{C}_r = \mathcal{C}$ , this is equivalent to the study of non-commutative Frobenius algebras  $\Gamma \in \mathcal{C}$ .) Now one finds [198] a bijection between such triples and commutative algebras  $\Gamma \in \mathcal{C}_l \boxtimes \widetilde{\mathcal{C}}_r$  of the maximal dimension  $d(\Gamma) = \sqrt{\dim \mathcal{C}_l \cdot \dim \mathcal{C}_r}$ . (This is a categorical version of Rehren’s approach [233] to the classification of modular invariants. It is based on studying local extensions  $\mathcal{A} \supset A_l \boxtimes \widetilde{A}_r$ , corresponding to commutative algebras  $\Gamma \in \mathcal{C}_l \boxtimes \widetilde{\mathcal{C}}_r$ .)
- There also is a concept of a center of an algebra  $A$  in a not-necessarily braided tensor category  $\mathcal{C}$ , to wit the **full center** defined in [56] by a universal property. While the full center is a commutative algebra in the braided center  $Z_1(\mathcal{C})$  of  $\mathcal{C}$ , as apposed to in  $\mathcal{C}$  like the above notions of center, there are connections between these constructions.
- We close this section giving three more reasons why modular categories are interesting:
  1. They have many connections with **number theory**:
    - Rehren [232], Turaev [262]:

$$\sum_i d_i^2 = \left| \sum_i d_i^2 \omega_i \right|^2.$$

In the pointed case (all simple objects have dimension one) this reduces to  $|\sum_i \omega_i| = \pm\sqrt{|I|}$ . For suitable  $\mathcal{C}$ , this reproduces Gauss' evaluation of Gauss sums. (Gauss actually also determined the sign of his sums.)

- The elements of  $T$  matrix are roots of unity, and the elements of  $S$  are cyclotomic integers [36, 78].
- For related integrality properties in  $Y=TQFSs$ , cf. Masbaum, Roberts, Wenzl [186, 187] and Bruguières [38]).
- The congruence subgroup property: Let  $N = \text{ord } T (< \infty)$ . Then

$$\ker(\pi : SL(2, \mathbb{Z}) \rightarrow GL(|I|, \mathbb{C})) \supset \Gamma(N) \equiv \ker(SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/N\mathbb{Z})).$$

For the modular categories arising from rational CFTs, this had been known in many cases and widely believed to be true in general. Considerable progress was made by Bantay [17], whose arguments were made rigorous by Xu [283] using algebraic quantum field theory. Bantay's work inspired a proof [247] by Sommerhäuser and Zhu for modular Hopf algebras, using the higher Frobenius-Schur indicators defined by Kashina and Sommerhäuser [136]. Finally, Ng and Schauenburg proved the congruence property for all modular categories along similar lines, cf. [212], beginning with a categorical version of the higher Frobenius-Schur indicators [211].

2. A modular category  $\mathcal{M}$  gives rise to a surgery TQFT in  $2 + 1$  dimensions (Reshetikhin, Turaev [235, 262]). In particular, this works for  $\mathcal{M} = Z_1(\mathcal{C})$  when  $\mathcal{C}$  is spherical fusion categories  $\mathcal{C}$  with  $\dim \mathcal{C} \neq 0$ . Since such a category  $\mathcal{C}$  also defines a TQFT via triangulation [19, 104], it is natural to expect an isomorphism  $RT_{\mathcal{M}} = BWGK_{\mathcal{C}}$  of TQFTs. (When  $\mathcal{C}$  is itself modular, this is indeed true by  $Z_1(\mathcal{C}) \simeq \mathcal{C} \boxtimes \tilde{\mathcal{C}}$  and Turaev's work in [262].) Recently, a general proof of this result was announced by Turaev and Virelizier, based on the work of Bruguières and Virelizier [41, 42], partially joint with S. Lack. (Notice in any case that the surgery construction provides more TQFTs than the triangulation approach, since not all modular categories are centers.)

3. We close with the hypothetical application of modular categories to topological quantum computing [274]. There are actually two different approaches to topological quantum computing: The one initiated by M. Freedman, using TQFTs in  $2 + 1$  dimension and the one due to A. Kitaev using  $d = 2$  quantum spin systems. However, in both proposals, the modular representation categories are central. Cf. also Z. Wang, E. Rowell et al. [120, 237].

## 6. SOME OPEN PROBLEMS

- (1) Characterize the hypergroups arising from a fusion category. (Probably hopeless.) Or at least those corresponding to (connected) compact groups.
- (2) Find an algebraic structure whose representation categories give all semisimple pivotal categories, generalizing Ostrik's result [222]. Perhaps this will

be something like the quantum groupoids defined by Lesieur and Enock [165]?

- (3) Classify all prime modular categories. (The next challenge after the classification of finite simple groups...)
- (4) Give a direct construction of the fusion categories associated with the two Haagerup subfactors [109, 7, 8].
- (5) Prove that every braided fusion category  $\mathcal{C}/\mathbb{C}$  embeds fully into a modular category  $\mathcal{M}$  with  $\dim \mathcal{M} = \dim \mathcal{C} \cdot \dim Z_2(\mathcal{C})$ . (This is the optimum allowed by the double commutant theorem, cf. [193].)
- (6) Find the most general context in which an analytic (i.e. non-formal) version of the Cartier/ Kassel/ Turaev [45, 140] formal deformation quantization of a symmetric tensor category  $\mathcal{S}$  with infinitesimal braiding can be given. (I.e. give an abstract version of the Kazhdan/Lusztig construction of Drinfeld’s category [144] that does not suppose  $\mathcal{S} = \text{Rep } G$ .)
- (7) Generalize the proof of modularity of  $Z_1(\mathcal{C})$  for semisimple fusion categories to not necessarily semisimple finite categories (in the sense of [85]), using Lyubashenko’s definition [175] of modularity.
- (8) Likewise for the triangulation TQFT [265, 19, 104]. Generalize the relation to surgery TQFT to the non-semisimple case. (For the non-semisimple version of the RT-TQFT in [151].)
- (9) Hard non-commutative analysis: Every countable  $C^*$ -tensor category with conjugates and  $\text{End } \mathbf{1} = \mathbb{C}$  embeds fully into the  $C^*$ -tensor category of bi-modules over  $L(F_\infty)$  and, for any infinite factor  $M$ , into  $\text{End}(L(F_\infty) \overline{\otimes} M)$ . Here  $F_\infty$  is the free group with countably many generators and  $L(F_\infty)$  the type  $II_1$  factor associated to its left regular representation. (This would extend and conceptualize the results of Popa/Shlyakhtenko [229] on the universality of the factor  $L(F_\infty)$  in subfactor theory.)
- (10) Give satisfactory categorical interpretations for various generalizations of quasi-triangular Hopf algebras, e.g. dynamical quantum groups [77] and Toledano-Laredo’s quasi-Coxeter algebras [257]. Soibelman’s ‘meromorphic tensor categories’ [246] and the ‘categories with cylinder braiding’ of tom Dieck and Häring-Oldenburg [258] might be relevant – and in any case they deserve further study.

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*Disclaimer:* While the following bibliography is quite extensive, it should be clear that it has no pretense whatsoever at completeness. Therefore the absence of this or that reference should not be construed as a judgment of its relevance. The choice of references was guided by the principal thrust of these lectures, namely linear categories. This means that the subjects of quantum groups and low dimensional topology, but also general categorical algebra are touched upon only tangentially.

## REFERENCES

- [1] L. Abrams: Two-dimensional topological quantum field theories and Frobenius algebras, *J. Knot Th. Ramif.* **5**, 569-587 (1996). [127](#), [148](#)
- [2] D. Altschuler, A. Bruguières: Handle slide for a sovereign category. Preprint, 2001. [143](#)
- [3] D. Altschuler, A. Coste: Invariants of 3-manifolds from finite groups. *In: Proc. XXth Int. Conf. Diff. Geom. Meth. in Theor. Phys.*, New York, 1991, pp. 219–233, World Scientific, 1992. Quasi-quantum groups, knots, three-manifolds, and topological field theory. *Comm. Math. Phys.* **150**, 83-107 (1992). [144](#)
- [4] H. H. Andersen; P. Polo; K. X. Wen: Representations of quantum algebras. *Invent. Math.* **104**, 1-59 (1991). H. H. Andersen: Tensor products of quantized tilting modules. *Comm. Math. Phys.* **149**, 149-159 (1992). [143](#)
- [5] Y. André: *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*. Soc. Math. France, 2004. [123](#)
- [6] A. Ardizzoni: The category of modules over a monoidal category: Abelian or not? *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.*, Vol. L, 167-185 (2004). [103](#)
- [7] M. Asaeda, U. Haagerup: Exotic subfactors of finite depth with Jones indices  $(5 + \sqrt{13})/2$  and  $(5 + \sqrt{17})/2$ . *Comm. Math. Phys.* **202**, 1-63 (1999). [151](#)
- [8] M. Asaeda, S. Yasuda: On Haagerup's list of potential principal graphs of subfactors. *Comm. Math. Phys.* **286**, 1141-1157 (2009). [151](#)
- [9] J. C. Baez: Higher-dimensional algebra. II. 2-Hilbert spaces. *Adv. Math.* **127**, 125-189 (1997). [111](#), [116](#)
- [10] J. C. Baez, J. Dolan: Categorification. In: E. Getzler, M. Kapranov (eds.): *Higher Category Theory*. *Contemp. Math.* **230**, 1-36 (1998). [137](#)
- [11] J. C. Baez; J. Dolan: Higher-dimensional algebra. III.  $n$ -categories and the algebra of opetopes. *Adv. Math.* **135**, 145-206 (1998). [103](#)
- [12] J. C. Baez; A. D. Lauda: Higher-dimensional algebra V: 2-groups. *Theor. Appl. Categ.* **12**, 423-491 (2004). [102](#)
- [13] J. C. Baez; J. P. May (eds.): *Towards higher categories*. Springer Verlag, 2010. [137](#), [152](#), [158](#)
- [14] J. C. Baez; M. Shulman: Lectures on  $n$ -categories and cohomology. In [\[13\]](#). ([arXiv:math/0608420](#)) [100](#)
- [15] B. Bakalov, A. Kirillov Jr.: *Lectures on tensor categories and modular functors*. American Mathematical Society, 2001. [143](#)
- [16] P. Bantay: Orbifolds and Hopf algebras. *Phys. Lett.* **B245**, 477-479 (1990). Orbifolds, Hopf algebras, and the Moonshine. *Lett. Math. Phys.* **22**, 187-194 (1991). [144](#)
- [17] P. Bantay: The kernel of the modular representation and the Galois action in RCFT. *Comm. Math. Phys.* **233**, 423-438 (2003). [150](#)
- [18] J. W. Barrett, B. W. Westbury: The equality of 3-manifold invariants. *Math. Proc. Camb. Phil. Soc.* **118**, 503-510 (1995). [128](#)
- [19] J. W. Barrett, B. W. Westbury: Invariants of piecewise-linear 3-manifolds. *Trans. Amer. Math. Soc.* **348**, 3997-4022 (1996). [128](#), [150](#), [151](#)
- [20] J. W. Barrett, B. W. Westbury: Spherical categories. *Adv. Math.* **143**, 357-375 (1999). [118](#), [132](#)

- [21] H. Baumgärtel, F. Lledó: Duality of compact groups and Hilbert  $C^*$ -systems for  $C^*$ -algebras with a nontrivial center. *Int. J. Math.* **15**, 759-812 (2004). [109](#)
- [22] A. Beliakova, C. Blanchet: Skein construction of idempotents in Birman-Murakami-Wenzl algebras. *Math. Ann.* **321**, 347-373 (2001). [135](#)
- [23] A. Beliakova, C. Blanchet: Modular categories of types B, C and D. *Comment. Math. Helv.* **76**, 467-500 (2001). [135](#), [143](#)
- [24] J. Benabou: Catégories avec multiplication. *C. R. Acad. Sci. Paris* **256**, 1887-1890 (1963). [95](#), [99](#)
- [25] J. Benabou: Algèbre élémentaire dans les catégories avec multiplication. *C. R. Acad. Sci. Paris* **258**, 771-774 (1964). [103](#)
- [26] J. Benabou: Introduction to bicategories. *In: Reports of the Midwest Category Seminar. Lecture Notes in Mathematics* 47, pp. 1-77. Springer Verlag, 1967. [103](#)
- [27] J. Bichon: Trivialisations dans les catégories tannakiennes. *Cah. Topol. Geom. Diff. Catég.* **39**, 243-270 (1998). [115](#), [122](#), [123](#)
- [28] J. Bichon, A. De Rijdt, S. Vaes: Ergodic coactions with large multiplicity and monoidal equivalences of quantum groups. *Comm. Math. Phys.* **262**, 703-728 (2006). [120](#)
- [29] J. S. Birman, H. Wenzl: Braids, link polynomials and a new algebra. *Trans. Amer. Math. Soc.* **313**, 249-273 (1989). [134](#)
- [30] E. Blanchard: On finiteness of the number of  $N$ -dimensional Hopf  $C^*$ -algebras. *In: A. Gheondea, R. N. Gologan and D. Timotin (eds.): Operator theoretical methods.* (Timișoara, 1998), 39-46, Theta Found., Bucharest, 2000. [110](#)
- [31] C. Blanchet: Hecke algebras, modular categories and 3-manifolds quantum invariants. *Topology* **39**, 193-223 (2000). [135](#), [143](#)
- [32] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel: Three-manifold invariants derived from the Kauffman bracket. *Topology* **31**, 685-699 (1992). *Topological quantum field theories derived from the Kauffman bracket.* *Topology* **34**, 883-927 (1995). [135](#)
- [33] J. Böckenhauer, D. E. Evans: Modular invariants, graphs and  $\alpha$ -induction for nets of subfactors I-III. *Comm. Math. Phys.* **197**, 361-386 (1998), **200**, 57-103 (1999), **205**, 183-228 (1999). [147](#), [149](#)
- [34] J. Böckenhauer, D. E. Evans, Y. Kawahigashi: On  $\alpha$ -induction, chiral generators and modular invariants for subfactors. *Comm. Math. Phys.* **208**, 429-487 (1999). *Chiral structure of modular invariants for subfactors.* *Comm. Math. Phys.* **210**, 733-784 (2000). [146](#), [149](#)
- [35] G. Böhm, K. Szlachányi: A coassociative  $C^*$ -quantum group with nonintegral dimensions. *Lett. Math. Phys.* **38**, 437-456 (1996). [125](#)
- [36] J. de Boer, J. Goeree: Markov traces and  $II_1$  factors in conformal field theory. *Comm. Math. Phys.* **139**, 267-304 (1991). [150](#)
- [37] A. Bruguières: Théorie tannakienne non commutative. *Comm. Alg.* **22**, 5817-5860 (1994). *Dualité tannakienne pour les quasi-groupeïdes quantiques.* *Comm. Alg.* **25**, 737-767 (1997). [124](#)
- [38] A. Bruguières, Alain: Tresses et structure entière sur la catégorie des représentations de  $SL_N$  quantique. *Comm. Algebra* **28**, 1989-2028 (2000). [150](#)
- [39] A. Bruguières: Catégories prémodulaires, modularisations et invariants de variétés de dimension 3. *Math. Ann.* **316**, 215-236 (2000). [123](#), [131](#), [143](#), [144](#)
- [40] A. Bruguières, G. Maltsiniotis: Quasi-groupeïdes quantiques. *C. R. Acad. Sci. Paris Sér. I Math.* **319**, 933-936 (1994) [124](#)
- [41] A. Bruguières, A. Virelizier: Categorical centers and Reshetikhin-Turaev invariants. *Acta Math. Vietnam.* **33**, 255-277 (2008). [138](#), [150](#)
- [42] A. Bruguières, A. Virelizier: Quantum double of Hopf monads and categorical centers. To appear in *Trans. AMS.* (Preprint: The double of a Hopf monad. [arXiv:0812.2443.](#)) [138](#), [150](#)

- [43] D. Calaque, P. Etingof: Lectures on tensor categories. *In*: B. Enriquez (ed.): *Quantum groups*. IRMA Lectures in Mathematics and Theoretical Physics, 12. European Mathematical Society, 2008. [95](#)
- [44] P. Carrasco, J. M. Moreno: Categorical G-crossed modules and 2-fold extensions. *J. Pure Appl. Alg.* **163**, 235-257 (2001). [146](#)
- [45] P. Cartier: Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds. *C. R. Acad. Sci. Paris Sér. I Math.* **316**, 1205-1210 (1993). [139](#), [140](#), [151](#)
- [46] A. M. Cegarra, E. Khmaladze: Homotopy classification of graded Picard categories. *Adv. Math.* **213**, 644-686 (2007). [131](#)
- [47] V. Chari, A. Pressley: *A guide to quantum groups*. Cambridge University Press, 1995. [138](#)
- [48] A. M. Cohen, R. de Man: Computational evidence for Deligne's conjecture regarding exceptional Lie groups. *C. R. Acad. Sci., Paris, Sér. I* **322**, 427-432 (1996). On a tensor category for the exceptional Lie groups. *In*: P. Dräxler et al. (eds.): *Computational methods for representations of groups and algebras*. *Progr. Math.* **173**, Birkhäuser Verlag, 1999. [135](#)
- [49] J. Comes, V. Ostrik: On blocks of Deligne's category  $\text{Rep}(S_t)$ . [arXiv:0910.5695](#). [135](#)
- [50] S. E. Crans: Generalized centers of braided and sylleptic monoidal 2-categories. *Adv. Math.* **136**, 183-223 (1998). [137](#)
- [51] G. Y. Chen: A new characterization of finite simple groups. *Chinese Sci. Bull.* **40**, 446-450 (1995). [108](#)
- [52] P. Cvitanović: *Group theory. Birdtracks, Lie's, and exceptional groups*. Princeton University Press, 2008. [135](#)
- [53] A. A. Davydov: Twisting of monoidal structures. [q-alg/9703001](#). [110](#), [141](#)
- [54] A. A. Davydov: Monoidal categories. *J. Math. Sci. (New York)* **88**, 457-519 (1998). [102](#)
- [55] A. A. Davydov: Galois algebras and monoidal functors between categories of representations of finite groups. *J. Alg.* **244**, 273-301 (2001). [116](#)
- [56] A. A. Davydov: Centre of an algebra. To appear in *Adv. Math.* ([arXiv:0908.1250](#)). [149](#)
- [57] A. A. Davydov, M. Müger, D. Nikshych, V. Ostrik: In preparation. [147](#)
- [58] P. Deligne: Catégories tannakiennes. *In*: P. Cartier et al. (eds.): *Grothendieck Festschrift*, vol. II, pp. 111-195. Birkhäuser Verlag, 1991. [108](#), [116](#), [121](#), [122](#), [123](#)
- [59] P. Deligne: La série exceptionnelle de groupes de Lie. *C. R. Acad. Sci., Paris, Sér. I* **322**, 321-326 (1996). [117](#), [135](#)
- [60] P. Deligne: Catégories tensorielles. *Moscow Math. J.* **2**, 227-248 (2002). [123](#)
- [61] P. Deligne: La catégorie des représentations du groupe symétrique  $S_t$ , lorsque  $t$  n'est pas un entier naturel. *In*: *Algebraic groups and homogeneous spaces*, 209-273, *Tata Inst. Fund. Res. Stud. Math.*, Tata Inst. Fund. Res., Mumbai, 2007. [118](#), [135](#)
- [62] P. Deligne, B. H. Gross: On the exceptional series, and its descendants. *C. R. Math. Acad. Sci. Paris* **335**, 877-881 (2002). [135](#)
- [63] P. Deligne, R. de Man: La série exceptionnelle de groupes de Lie. II. *C. R. Acad. Sci., Paris, Sér. I* **323**, 577-582 (1996). [135](#)
- [64] P. Deligne, J. S. Milne: Tannakian categories. *Lecture notes in mathematics* **900**, 101-228. Springer Verlag, 1982. [107](#), [115](#)
- [65] R. Dijkgraaf: A Geometrical Approach to Two Dimensional Conformal Field Theory, Ph.D. thesis (Utrecht, 1989). [148](#)
- [66] R. Dijkgraaf, V. Pasquier, P. Roche: Quasi Hopf algebras, group cohomology and orbifold models. *Nucl. Phys. B (Proc. Suppl.)* **18B**, 60-72 (1990). [125](#), [147](#)
- [67] R. Dijkgraaf, C. Vafa, E. Verlinde, H. Verlinde: The operator algebra of orbifold models. *Comm. Math. Phys.* **123**, 485-527 (1989). [147](#)
- [68] S. Doplicher, R. Haag, J. E. Roberts: Local observables and particle statistics I & II. *Comm. Math. Phys.* **23**, 199-230 (1971), **35**, 49-85 (1974). [97](#), [123](#), [131](#)
- [69] S. Doplicher, C. Pinzari and J. E. Roberts, An algebraic duality theory for multiplicative unitaries. *Int. J. Math.* **12**, 415-459 (2001). C. Pinzari and J. E. Roberts, Regular objects, multiplicative unitaries and conjugation. *Int. J. Math.* **13** (2002) 625-665. [122](#)

- [70] S. Doplicher, J. E. Roberts: A new duality theory for compact groups. *Invent. Math.* **98**, 157–218 (1989). [106](#), [108](#), [116](#), [118](#), [122](#), [123](#)
- [71] S. Doplicher, J. E. Roberts: Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. *Comm. Math. Phys.* **131**, 51-107 (1990). [123](#), [132](#)
- [72] V. G. Drinfeld: Quantum groups. *In: Proc. Int. Congr. Math, Berkeley 1986.* [131](#), [135](#), [136](#), [139](#)
- [73] V. G. Drinfeld: Quasi-Hopf algebras. *Leningrad Math. J.* **1**, 1419-1457 (1990). [125](#), [140](#)
- [74] V. G. Drinfeld: On quasitriangular quasi-Hopf algebras and on a group that is closely connected with  $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ . *Leningrad Math. J.* **2**, 829-860 (1991). [125](#), [140](#)
- [75] V. Drinfeld, S. Gelaki, D. Nikshych, V. Ostrik: Group-theoretical properties of nilpotent modular categories. [math.QA/0704.0195](#). [147](#)
- [76] S. Eilenberg, J. C. Moore: Adjoint functors and triples. *Ill. J. Math.* **9**, 381-398 (1965). [103](#)
- [77] P. Etingof; A. Varchenko: Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Comm. Math. Phys.* **196** 591-640 (1998). [151](#)
- [78] P. Etingof: On Vafa’s theorem for tensor categories. *Math. Res. Lett.* **9**, 651-657 (2002). [150](#)
- [79] P. Etingof, S. Gelaki: Some properties of finite-dimensional semisimple Hopf algebras. *Math. Res. Lett.* **5**, 191-197 (1998). [144](#)
- [80] P. Etingof, S. Gelaki: Isocategorical groups. *Int. Math. Res. Not.* 2001:2, 59–76. [116](#)
- [81] P. Etingof, S. Gelaki: The small quantum group as a quantum double. *J. Alg.* **322**, 2580-2585 (2009). [142](#)
- [82] P. Etingof, S. Gelaki, V. Ostrik: Classification of fusion categories of dimension  $pq$ . *Int. Math. Res. Not.* 2004:57, 3041–3056. [125](#)
- [83] P. Etingof, D. Nikshych, V. Ostrik: An analogue of Radford’s  $S^4$  formula for finite tensor categories. *Int. Math. Res. Not.* 2004:54, 2915–2933. [145](#)
- [84] P. Etingof, D. Nikshych, V. Ostrik: On fusion categories. *Ann. Math.* **162**, 581-642 (2005). [100](#), [107](#), [110](#), [119](#), [128](#), [141](#), [145](#), [146](#)
- [85] P. Etingof, V. Ostrik: Finite tensor categories. *Mosc. Math. J.* **4**, 627-654, 782–783 (2004). [107](#), [128](#), [151](#)
- [86] P. Etingof, V. Ostrik: Module categories over representations of  $\text{SL}_q(2)$  and graphs. *Math. Res. Lett.* **11**, 103-114 (2004). [149](#)
- [87] D. E. Evans, Y. Kawahigashi: *Quantum symmetries on operator algebras*. Oxford University Press, 1998.
- [88] M. Farber: Von Neumann categories and extended  $L^2$ -cohomology. *K-Theory* **15**, 347-405 (1998). [111](#)
- [89] M. Finkelberg: An equivalence of fusion categories. *Geom. Funct. Anal.* **6**, 249-267 (1996). [142](#)
- [90] K. Fredenhagen, K.-H. Rehren, B. Schroer: Superselection sectors with braid group statistics and exchange algebras I. General theory. *Comm. Math. Phys.* **125**, 201-226 (1989). [97](#), [131](#)
- [91] M. H. Freedman, A. Kitaev, M. J. Larsen, Z. Wang: Topological quantum computation. *Mathematical challenges of the 21st century.* *Bull. Amer. Math. Soc.* **40**, 31-38 (2003).
- [92] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu: A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc.* **12**, 239-246 (1985). [135](#)
- [93] P. Freyd, D. Yetter: Braided compact closed categories with applications to low-dimensional topology. *Adv. Math.* **77**, 156-182 (1989). [117](#)
- [94] P. Freyd, D. Yetter: Coherence theorems via knot theory. *J. Pure Appl. Alg.* **78**, 49-76 (1992). [99](#), [117](#)
- [95] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert: Correspondences of ribbon categories. *Adv. Math.* **199**, 192-329 (2006). [149](#)

- [96] J. Fröhlich, T. Kerler: *Quantum groups, quantum categories and quantum field theory*. (Lecture Notes in Mathematics vol. 1542.) Springer Verlag, 1993. [110](#)
- [97] J. Fuchs, I. Runkel, C. Schweigert: TFT construction of RCFT correlators. I. Partition functions. Nucl. Phys. **B646**, 353-497 (2002). [149](#)
- [98] J. Fuchs, C. Schweigert: Category theory for conformal boundary conditions. Fields Inst. Commun. **39**, 25-71 (2003). [103](#), [149](#)
- [99] M. Fukuma, S. Hosono, H. Kawai: Lattice topological field theory in two dimensions. Comm. Math. Phys. **161**, 157-175 (1994). [148](#)
- [100] S. Gelaki: Semisimple triangular Hopf algebras and Tannakian categories. In: M. D. Fried, Y. Ihara (eds.): *Arithmetic fundamental groups and noncommutative algebra*. Proc. Sympos. Pure Math. **70**, 497-515 (2002). [123](#)
- [101] S. Gelaki, D. Naidu: Some properties of group-theoretical categories. J. Alg. **322**, 2631-2641 (2009). [129](#)
- [102] S. Gelaki, D. Nikshych: Nilpotent fusion categories. Adv. Math. **217**, 1053-1071 (2008). [109](#)
- [103] S. Gelfand, D. Kazhdan: Examples of tensor categories. Invent. Math. **109**, 595-617 (1992). [114](#)
- [104] S. Gelfand, D. Kazhdan: Invariants of three-dimensional manifolds. Geom. Funct. Anal. **6**, 268-300 (1996). [118](#), [128](#), [150](#), [151](#)
- [105] P. Ghez, R. Lima, J. E. Roberts:  $W^*$ -categories. Pac. J. Math. **120**, 79-109 (1985). [111](#)
- [106] C. Gomez, M. Ruiz-Altaba, G. Sierra: *Quantum groups in two-dimensional physics*. Cambridge University Press, 1996. [139](#)
- [107] F. Goodman, H. Wenzl: Ideals in Temperley-Lieb category. Appendix to: M. H. Freedman: A magnetic model with a possible Chern-Simons phase. Comm. Math. Phys. **234**, 129-183 (2003). [97](#)
- [108] R. Gordon, A. J. Power, R. Street: Coherence for tricategories. Mem. Amer. Math. Soc. **117** (1995). [137](#)
- [109] U. Haagerup: Principal graphs of subfactors in the index range  $4 < [M : N] < 3 + \sqrt{2}$ . In: H. Araki, Y. Kawahigashi, H. Kosaki (eds.): *Subfactors*. (Kyuzeso, 1993), 1-38, World Sci. Publ., River Edge, NJ, 1994. [151](#)
- [110] Phung Ho Hai: Tannaka-Krein duality for Hopf algebroids. Israel J. Math. **167**, 193-225 (2008). [126](#)
- [111] H. Halvorson: Algebraic Quantum Field Theory. In: J. Butterfield and J. Earman (eds.): *Handbook of the Philosophy of Physics*. Elsevier, 2006. [123](#), [159](#)
- [112] D. Handelman: Representation rings as invariants for compact groups and limit ratio theorems for them. Int. J. Math. **4**, 59-88 (1993). [108](#)
- [113] R. Häring-Oldenburg: Reconstruction of weak quasi Hopf algebras. J. Alg. **194**, 14-35 (1997). [125](#)
- [114] F. Hausser, Frank; F. Nill: Doubles of quasi-quantum groups. Comm. Math. Phys. **199**, 547-589 (1999). [144](#)
- [115] T. Hayashi: Face algebras. I. A generalization of quantum group theory. J. Math. Soc. Japan **50**, 293-315 (1998). [125](#), [126](#)
- [116] T. Hayashi: A canonical Tannaka duality for finite semisimple tensor categories. [arXiv:math.QA/9904073](#). [126](#)
- [117] M. Hazewinkel, W. Hesselink, D. Siersma, F. D. Veldkamp: The ubiquity of Coxeter-Dynkin diagrams (an introduction to the  $A-D-E$  problem). Nieuw Arch. Wisk. **25**, 257-307 (1977). [149](#)
- [118] E. Hewitt, K. A. Ross: *Abstract harmonic analysis*, Vol. II. Springer Verlag, 1970. [116](#)
- [119] F. Hiai, M. Izumi: Amenability and strong amenability for fusion algebras with applications to subfactor theory. Int. J. Math. **9**, 669-722 (1998). [110](#)
- [120] Seung-moon Hong, E. Rowell, Z. Wang: On exotic modular tensor categories. Commun. Contemp. Math. **10**, Suppl. 1, 1049-1074 (2008). [150](#)

- [121] Y.-Z. Huang: Vertex operator algebras, the Verlinde conjecture, and modular tensor categories. Proc. Natl. Acad. Sci. USA **102**, 5352-5356 (2005). [143](#)
- [122] M. Izumi; H. Kosaki, Hideki: On a subfactor analogue of the second cohomology. Rev. Math. Phys. **14**, 733-757 (2002). [116](#)
- [123] G. James, M. Liebeck: *Representations and characters of groups*. Cambridge University Press, 1993, 2001. [108](#)
- [124] J. C. Jantzen: *Lectures on quantum groups*. American Mathematical Society, 1996. [138](#), [140](#), [141](#)
- [125] M. Jimbo: A  $q$ -difference analogue of  $U(g)$  and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63-69 (1985). A  $q$ -analogue of  $U(gl(N+1))$ , Hecke algebra and the Yang-Baxter equation. Lett. Math. Phys. **11**, 247-245 (1986). [139](#), [140](#)
- [126] V. F. R. Jones: Index for subfactors. Invent. Math. **72**, 1-25 (1983). [119](#), [129](#)
- [127] V. F. R. Jones: A polynomial invariant for knots via von Neumann algebras. Bull. Amer. Math. Soc. **12**, 103-111 (1985). [97](#)
- [128] A. Joyal, R. Street: Braided monoidal categories. Macquarie Mathematics Reports Nos. 850067, Dec. 1985 (<http://www.math.mq.edu.au/~street/BMC850067.pdf>) and 860081, Nov. 1986 (<http://www.maths.mq.edu.au/~street/JS1.pdf>). [102](#), [130](#), [131](#)
- [129] A. Joyal, R. Street: The geometry of tensor calculus. I. Adv. Math. **88**, 55-112 (1991). [99](#)
- [130] A. Joyal, R. Street: Tortile Yang-Baxter operators in tensor categories. J. Pure Appl. Alg. **71**, 43-51 (1991). [136](#)
- [131] A. Joyal, R. Street: An introduction to Tannaka duality and quantum groups. In: Proceedings of the conference on category theory, Como 1990. Lecture notes in mathematics **1488**, 413-492 (1991). [119](#)
- [132] A. Joyal, R. Street: Braided tensor categories. Adv. Math. **102**, 20-78 (1993). [130](#), [131](#), [137](#)
- [133] A. Joyal, R. Street: The category of representations of the general linear groups over a finite field. J. Alg. **176**, 908-946 (1995). [132](#)
- [134] A. Joyal, R. Street, D. Verity: Traced monoidal categories. Math. Proc. Cambridge Philos. Soc. **119**, 447-468 (1996). [113](#)
- [135] M. Karoubi: *K-theory. An introduction*. Springer Verlag, 1978. [111](#)
- [136] Y. Kashina, Y. Sommerhäuser, Y. Zhu: On higher Frobenius-Schur indicators. Mem. Amer. Math. Soc. **181** (2006), no. 855. [150](#)
- [137] C. Kassel: *Quantum groups*. Springer Verlag, 1995. [99](#), [100](#), [102](#), [120](#), [125](#), [132](#), [133](#), [136](#), [138](#), [139](#), [140](#)
- [138] C. Kassel, M. Rosso and V. Turaev: *Quantum groups and knot invariants*. SMF, 1997.
- [139] C. Kassel, V. G. Turaev: Double construction for monoidal categories. Acta Math. **175**, 1-48 (1995). [144](#)
- [140] C. Kassel, V. G. Turaev: Chord diagram invariants of tangles and graphs. Duke Math. J. **92**, 497-552 (1998). [139](#), [140](#), [151](#)
- [141] C. Kassel, V. G. Turaev: *Braid groups*. Springer Verlag, 2008. [130](#)
- [142] L. H. Kauffman: An invariant of regular isotopy. Trans. Amer. Math. Soc. **318**, 417-471 (1990). [134](#)
- [143] Y. Kawahigashi, R. Longo, M. Müger: Multi-interval subfactors and modularity of representations in conformal field theory. Comm. Math. Phys. **219**, 631-669 (2001). [143](#), [147](#)
- [144] D. Kazhdan, G. Lusztig: Tensor structures arising from affine Lie algebras I-IV. J. Amer. Math. Soc. **6**, 905-947, 949-1011 (1993); **7**, 335-381, 383-453. (1994). [141](#), [151](#)
- [145] D. Kazhdan and H. Wenzl, Reconstructing tensor categories. Adv. Sov. Math. **16** (1993) 111-136. [124](#), [142](#)
- [146] G. M. Kelly: On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. J. Algebra **1**, 397-402 (1964). [100](#)
- [147] G. M. Kelly: Tensor products in categories. J. Algebra **2**, 15-37 (1965). [100](#)
- [148] G. M. Kelly, M. L. Laplaza: Coherence for compact closed categories. J. Pure Appl. Algebra **19**, 193-213 (1980). [102](#), [113](#)

- [149] G. M. Kelly, S. Mac Lane: Coherence in closed categories. *J. Pure Appl. Algebra* **1**, 97–140 (1971) and erratum on p. 219. [113](#)
- [150] G. M. Kelly, R. Street: Review of the elements of 2-categories. *Lecture Notes in Mathematics* 420, pp. 75-103. Springer Verlag, 1974. [103](#)
- [151] T. Kerler, V. V. Lyubashenko: *Non-semisimple topological quantum field theories for 3-manifolds with corners*. Springer Verlag, 2001. [151](#)
- [152] A. A. Kirillov Jr.: On an inner product in modular tensor categories. *J. Amer. Math. Soc.* **9**, 1135-1169 (1996). On inner product in modular tensor categories. II. Inner product on conformal blocks and affine inner product identities. *Adv. Theor. Math. Phys.* **2**, 155-180 (1998). [143](#)
- [153] A. Kirillov Jr.: Modular categories and orbifold models I & II. *Comm. Math. Phys.* **229**, 309-335 (2002) & [math.QA/0110221](#). [146](#)
- [154] A. Kirillov Jr.: On  $G$ -equivariant modular categories. [math.QA/0401119](#). [146](#)
- [155] A. Kirillov Jr., V. Ostrik: On  $q$ -analog of McKay correspondence and ADE classification of  $sl(2)$  conformal field theories. *Adv. Math.* **171**, 183-227 (2002). [129](#), [146](#), [147](#), [149](#)
- [156] J. Kock: *Frobenius algebras and 2D topological quantum field theories*. Cambridge University Press, 2004. [127](#), [148](#)
- [157] L. Kong, I. Runkel: Morita classes of algebras in modular tensor categories. *Adv. Math.* **219**, 1548-1576 (2008). Cardy algebras and sewing constraints. I. *Comm. Math. Phys.* **292**, 871-912 (2009). [149](#)
- [158] M. Kreĭn: A principle of duality for bicomact groups and quadratic block algebras. *Doklady Akad. Nauk SSSR (N.S.)* **69**, 725-728 (1949). [116](#)
- [159] G. Kuperberg: Involutory Hopf algebras and 3-manifold invariants. *Intern. J. Math.* **2**, 41-66 (1991). [128](#)
- [160] G. Kuperberg: Spiders for rank 2 Lie algebras. *Comm. Math. Phys.* **180**, 109-151 (1996). [135](#)
- [161] G. Kuperberg: Finite, connected, semisimple, rigid tensor categories are linear. *Math. Res. Lett.* **10**, 411-421 (2003). [110](#)
- [162] S. Lack: A 2-categories companion. In [\[13\]](#). ([math/0702535](#)). [103](#)
- [163] A. D. Lauda: Frobenius algebras and ambidextrous adjunctions. *Theory Appl. Categ.* **16**, 84-122 (2006). [128](#)
- [164] G. I. Lehrer: A survey of Hecke algebras and the Artin braid groups. *In: J. S. Birman, A. Libgober (eds.): Braids*, *Contemp. Math.* **78**, 365-385 (1988). [134](#)
- [165] F. Lesieur: *Measured quantum groupoids*. *Mém. Soc. Math. Fr. (N.S.)* **109** (2007). M. Enock: *Measured quantum groupoids in action*. *Mém. Soc. Math. Fr. (N.S.)* **114** (2008). [151](#)
- [166] M. Levine: *Mixed motives*. American Mathematical Society, 1998. [123](#)
- [167] W. B. R. Lickorish: *An introduction to knot theory*. Springer Verlag, 1997. [135](#)
- [168] T. Loke: *Operator algebras and conformal field theory of the discrete series representations of  $Diff(S^1)$* , PhD thesis Cambridge (UK), 1994. [143](#)
- [169] R. Longo: Index of subfactors and statistics of quantum fields I & II. *Comm. Math. Phys.* **126**, 217-247 (1989) & **130**, 285-309 (1990). [97](#), [129](#)
- [170] R. Longo: A duality for Hopf algebras and for subfactors I. *Comm. Math. Phys.* **159**, 133-150 (1994). [122](#), [129](#)
- [171] R. Longo, K.-H. Rehren: Nets of subfactors. *Rev. Math. Phys.* **7**, 567-597 (1995). [147](#)
- [172] R. Longo, J. E. Roberts: A theory of dimension. *K-Theory* **11**, 103-159 (1997). [106](#), [111](#), [118](#), [119](#), [132](#)
- [173] G. Lusztig: *Introduction to Quantum Groups*. Birkhäuser Verlag, 1993. [138](#), [140](#), [141](#)
- [174] V. V. Lyubashenko: Tangles and Hopf algebras in braided categories. *J. Pure Appl. Alg.* **98**, 245-278 (1995).
- [175] V. V. Lyubashenko: Modular transformations for tensor categories. *J. Pure Appl. Alg.* **98**, 279-327 (1995). [151](#)

- [176] G. Mack, V. Schomerus: Quasi Hopf quantum symmetry in quantum theory. Nucl. Phys. **B370**, 185-230 (1992). [125](#)
- [177] S. Mac Lane: Cohomology theory of abelian groups. Proceedings of the ICM 1950, vol. II, p. 8-14. [131](#)
- [178] S. Mac Lane: Natural associativity and commutativity. Rice Univ. Studies **49**, 28-46 (1963). [95](#), [99](#), [100](#), [106](#), [112](#)
- [179] S. Mac Lane: Categorical algebra. Bull. Amer. Math. Soc. **71**, 40-106 (1965). [124](#)
- [180] S. Mac Lane: *Categories for the working mathematician*. 2nd ed. Springer Verlag, 1998. [95](#), [99](#), [100](#), [101](#), [102](#), [103](#), [106](#), [115](#)
- [181] S. Majid: Representations, duals and quantum doubles of monoidal categories. Rend. Circ. Mat. Palermo Suppl. **26**, 197-206 (1991). [136](#)
- [182] S. Majid: Reconstruction theorems and rational conformal field theories. Internat. J. Modern Phys. A **6**, 4359-4374 (1991). [125](#)
- [183] S. Majid: *Foundations of quantum group theory*. Cambridge University Press, 1995.
- [184] G. Maltsiniotis: Groupoïdes quantiques. C. R. Acad. Sci. Paris Sér. I Math. **314**, 249-252 (1992). [124](#)
- [185] G. Maltsiniotis: Traces dans les catégories monoïdales, dualité et catégories monoïdales fibrées. Cah. Topol. Géom. Diff. Catég. **36**, 195-288 (1995). [113](#)
- [186] G. Masbaum, J. D. Roberts: A simple proof of integrality of quantum invariants at prime roots of unity. Math. Proc. Cambr. Philos. Soc. **121**, 443-454 (1997). [150](#)
- [187] G. Masbaum, H. Wenzl: Integral modular categories and integrality of quantum invariants at roots of unity of prime order. J. Reine Angew. Math. **505**, 209-235 (1998). [150](#)
- [188] J. R. McMullen: On the dual object of a compact connected group. Math. Z. **185**, 539-552 (1984). [108](#), [109](#)
- [189] G. Moore, N. Seiberg: Classical and quantum conformal field theory. Comm. Math. Phys. **123**, 177-254 (1989). [143](#)
- [190] M. Müger: Galois theory for braided tensor categories and the modular closure. Adv. Math. **150**, 151-201 (2000). [111](#), [123](#), [144](#)
- [191] M. Müger: From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories. J. Pure Appl. Alg. **180**, 81-157 (2003). [117](#), [121](#), [126](#), [127](#), [129](#), [147](#)
- [192] M. Müger: From subfactors to categories and topology II. The quantum double of tensor categories and subfactors. J. Pure Appl. Alg. **180**, 159-219 (2003). [138](#), [144](#), [145](#)
- [193] M. Müger, On the structure of modular categories. Proc. Lond. Math. Soc. **87**, 291-308 (2003). [145](#), [146](#), [151](#)
- [194] M. Müger: Galois extensions of braided tensor categories and braided crossed G-categories. J. Alg. **277**, 256-281 (2004). [146](#)
- [195] M. Müger: On the center of a compact group. Intern. Math. Res. Not. 2004:51, 2751-2756 (2004). [109](#)
- [196] M. Müger: Conformal orbifold theories and braided crossed G-categories. Comm. Math. Phys. **260**, 727-762 (2005). [146](#)
- [197] M. Müger: Abstract duality for symmetric tensor \*-categories. Appendix to [\[111\]](#), p. 865-922. [104](#), [114](#), [115](#), [123](#)
- [198] M. Müger: On Superselection theory of quantum fields in low dimensions. Invited lecture at the XVIth International Congress on Mathematical Physics. Prague, August 3-8, 2009. To appear in the proceedings. ([arXiv:0909.2537.](#)) [147](#), [149](#)
- [199] M. Müger: On the structure of braided crossed G-categories. Appendix to V. Turaev: *Homotopy TQFTs*. European Mathematical Society, to appear 2010. [146](#)
- [200] M. Müger, J. E. Roberts, L. Tuset: Representations of algebraic quantum groups and reconstruction theorems for tensor categories. Alg. Repres. Theor. **7**, 517-573 (2004). [119](#), [120](#)

- [201] M. Müger, L. Tuset: Monoids, embedding functors and quantum groups. *Int. J. Math.* **19**, 1-31 (2008). [121](#)
- [202] M. Müger: Isocategorical compact groups. In preparation. [116](#)
- [203] D. Naidu: Categorical Morita equivalence for group-theoretical categories. *Comm. Alg.* **35**, 3544-3565 (2007). [129](#)
- [204] D. Naidu: Crossed pointed categories and their equivariantizations. Preprint, 2009. Available from <http://www.math.tamu.edu/~dnaidu/publications.html>.
- [205] D. Naidu, D. Nikshych: Lagrangian subcategories and braided tensor equivalences of twisted quantum doubles of finite groups. *Comm. Math. Phys.* **279**, 845-872 (2008). [125](#), [148](#)
- [206] D. Naidu, D. Nikshych, S. Witherspoon: Fusion subcategories of representation categories of twisted quantum doubles of finite groups. *Int. Math. Res. Not.* **22**, 4183-4219(2009). [146](#)
- [207] S. Natale: Frobenius-Schur indicators for a class of fusion categories. *Pac. J. Math.* **221**, 353-377 (2005).
- [208] S. Natale: On the exponent of tensor categories coming from finite groups. *Israel J. Math.* **162**, 253-273 (2007).
- [209] S. Neshveyev, L. Tuset: Notes on the Kazhdan-Lusztig theorem on equivalence of the Drinfeld category and the category of  $U_q(g)$ -modules. [arXiv:0711.4302](#). [141](#)
- [210] S.-H. Ng, P. Schauenburg: Frobenius-Schur indicators and exponents of spherical categories. *Adv. Math.* **211**, 34-71 (2007).
- [211] S.-H. Ng, P. Schauenburg: Higher Frobenius-Schur Indicators for Pivotal Categories. *Contemp. Math.* **441**, 63-90 (2007). [150](#)
- [212] S.-H. Ng, P. Schauenburg: Congruence subgroups and generalized Frobenius-Schur indicators. [arXiv:0806.2493](#). [150](#)
- [213] D. Nikshych: On the structure of weak Hopf algebras. *Adv. Math.* **170**, 257-286 (2002). [145](#)
- [214] D. Nikshych, V. Turaev, L. Vainerman: Invariants of Knots and 3-manifolds from Quantum Groupoids. *Topol. Appl.* **127**, 91-123 (2003). [144](#)
- [215] D. Nikshych, L. Vainerman: A characterization of depth 2 subfactors of  $II_1$  factors. *J. Funct. Anal.* **171**, 278-307 (2000). A Galois correspondence for  $II_1$  factors and quantum groupoids. *J. Funct. Anal.* **178**, 113-142 (2000). [126](#)
- [216] A. Ocneanu: Quantized groups, string algebras and Galois theory for algebras. *In: Operator algebras and applications*, Vol. 2, pp. 119-172, Cambridge Univ. Press, 1988. [126](#), [129](#)
- [217] A. Ocneanu (notes by Y. Kawahigashi): Quantum symmetry, differential geometry of finite graphs and classification of subfactors. Lectures given at Tokyo Univ. 1990. [126](#), [129](#)
- [218] A. Ocneanu: An invariant coupling between 3-manifolds and subfactors, with connections to topological and conformal quantum field theory. Unpublished manuscript. Ca. 1991. [128](#)
- [219] A. Ocneanu: Chirality for operator algebras. *In: H. Araki, Y. Kawahigashi and H. Kosaki (eds.): Subfactors*. pp 39-63. World Scientific Publ., 1994. [145](#)
- [220] A. Ocneanu: Operator algebras, topology and subgroups of quantum symmetry—construction of subgroups of quantum groups. *In: Taniguchi Conference on Mathematics Nara 1998*, 235-263, *Adv. Stud. Pure Math.*, 31, Math. Soc. Japan, Tokyo, 2001. [149](#)
- [221] A. Ocneanu: The classification of subgroups of quantum  $SU(N)$ . *In: Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000)*, 133-159, *Contemp. Math.* 294, 2002. [149](#)
- [222] V. Ostrik: Module categories, weak Hopf algebras and modular invariants. *Transf. Groups* **8**, 177-206 (2003). [104](#), [126](#), [129](#), [149](#), [150](#)
- [223] V. Ostrik: Module categories over the Drinfeld double of a finite group. *Int. Math. Res. Not.* 2003:27, 1507-1520. [100](#), [129](#), [145](#)
- [224] V. Ostrik: Fusion categories of rank 2. *Math. Res. Lett.* **10**, 177-183 (2003). [110](#), [125](#)
- [225] V. Ostrik: Pre-modular categories of rank 3. *Moscow Math. J.* **8**, 111-118 (2008). [125](#)
- [226] F. Panaite, F. Van Oystaeyen: Quasi-Hopf algebras and the centre of a tensor category. *In: S. Caenepeel et al. (eds.): Hopf algebras and quantum groups*. *Lect. Notes Pure Appl. Math.* **209**, 221-235 (2000). [144](#)

- [227] B. Pareigis: On braiding and dyslexia. *J. Algebra* **171**, 413-425 (1995). [146](#)
- [228] C. Pinzari: The representation category of the Woronowicz quantum group  $SU(d)$  as a braided tensor  $C^*$ -category. *Int. J. Math.* **18**, 113-136 (2007). [124](#)
- [229] S. Popa, D. Shlyakhtenko: Universal properties of  $L(F_\infty)$  in subfactor theory. *Acta Math.* **191**, 225-257 (2003). [151](#)
- [230] M. van der Put, M. F. Singer: *Galois theory of linear differential equations*. Springer Verlag, 2003. [123](#)
- [231] F. Quinn: Lectures on axiomatic topological quantum field theory. *In: D. S. Freed and K. K. Uhlenbeck (eds.): Geometry and quantum field theory*. IAS/Park City Mathematics Series, 1. American Mathematical Society, 1995. [127](#)
- [232] K.-H. Rehren: Braid group statistics and their superselection rules. *In: D. Kastler (ed.): The algebraic theory of superselection sectors. Introduction and recent results*. World Scientific, 1990. [131](#), [143](#), [149](#)
- [233] K.-H. Rehren: Locality and modular invariance in 2D conformal QFT. *Fields Inst. Commun.* **20**, 297-319 (2001). [149](#)
- [234] N. Yu. Reshetikhin, V. G. Turaev: Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys.* **127**, 1-26 (1990). [133](#)
- [235] N. Yu. Reshetikhin, V. G. Turaev: Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.* **103**, 547-598 (1991). [143](#), [150](#)
- [236] E. Rowell: On a family of non-unitarizable ribbon categories. *Math. Z.* **250**, 745-774 (2005). [143](#)
- [237] E. Rowell, R. Stong, Z. Wang: On classification of modular tensor categories. *Comm. Math. Phys.* **292**, 343-389 (2009). [150](#)
- [238] N. Saaveda Rivano: *Catégories Tannakiennes*. Springer Verlag, 1972. [101](#), [115](#)
- [239] P. Schauenburg: Tannaka duality for arbitrary Hopf algebras. *Algebra Berichte*, 66. Verlag Reinhard Fischer, Munich, 1992. [119](#)
- [240] P. Schauenburg: The monoidal center construction and bimodules. *J. Pure Appl. Alg.* **158**, 325-346 (2001). [129](#), [148](#)
- [241] P. Schauenburg: Turning monoidal categories into strict ones. *New York J. Math.* **7**, 257-265 (2001). [102](#)
- [242] J.-P. Serre: *Complex Semisimple Lie Algebras*. Springer, 1987. [138](#)
- [243] M. C. Shum: Tortile tensor categories. *J. Pure Appl. Alg.* **93**, 57-110 (1994). [133](#)
- [244] J. Siehler: Near-group categories. *Algebr. Geom. Topol.* **3**, 719-775 (2003). [125](#)
- [245] Sinh Hong Xuan: *Gr-Catégories*. PhD thesis Université Paris VII, 1975. [100](#), [102](#), [112](#)
- [246] Y. Soibelman: Meromorphic tensor categories, quantum affine and chiral algebras I. *Contemp. Math.* **248**, 437-451 (1998). [151](#)
- [247] Y. Sommerhäuser, Y. Zhu: Hopf algebras and congruence subgroups. [arXiv:0710.0705](#) [150](#)
- [248] C. Soulé: Les variétés sur le corps à un élément. *Mosc. Math. J.* **4**, 217-244 (2004). [135](#)
- [249] D. Ştefan: The set of types of  $n$ -dimensional semisimple and cosemisimple Hopf algebras is finite. *J. Alg.* **193**, 571-580 (1997). [110](#)
- [250] R. Street: The quantum double and related constructions. *J. Pure Appl. Alg.* **132**, 195-206 (1998). [137](#), [138](#)
- [251] R. Street: The monoidal centre as a limit. *Theory Appl. Categ.* **13**, 184-190 (2004). [138](#)
- [252] K. Szlachányi: Finite quantum groupoids and inclusions of finite type. *In: R. Longo (ed.): Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*. Fields Inst. Commun. **20**, 297-319 (2001). [126](#)
- [253] D. Tambara: A duality for modules over monoidal categories of representations of semisimple Hopf algebras. *J. Algebra* **241**, 515-547 (2001). [128](#)
- [254] D. Tambara, S. Yamagami: Tensor categories with fusion rules of self-duality for finite abelian groups. *J. Alg.* **209**, 692-707 (1998). [108](#), [125](#)
- [255] T. Tannaka: Über den Dualitätssatz der nichtkommutativen topologischen Gruppen. *Tôhoku Math. Journ.* **45**, 1-12 (1939). [95](#), [115](#)

- [256] V. Toledano Laredo: Positive energy representations of the loop groups of non-simply connected Lie groups. *Comm. Math. Phys.* **207**, 307-339 (1999). [142](#)
- [257] V. Toledano Laredo: Quasi-Coxeter algebras, Dynkin diagram cohomology, and quantum Weyl groups. *Int. Math. Res. Pap. IMRP* 2008, Art. ID rpn009, 167 pp. [151](#)
- [258] T. tom Dieck: Categories of rooted cylinder ribbons and their representations. *J. Reine Angew. Math.* **494**, 35-63 (1998). T. tom Dieck, Tammo; R. Häring-Oldenburg: Quantum groups and cylinder braiding. *Forum Math.* **10**, 619-639 (1998). [151](#)
- [259] I. Tuba, H. Wenzl: Hans On braided tensor categories of type  $BCD$ . *J. Reine Angew. Math.* **581**, 31-69 (2005). [125](#), [142](#), [144](#)
- [260] V. G. Turaev: Operator invariants of tangles, and  $R$ -matrices. *Math. USSR-Izv.* **35**, 411-444 (1990). [132](#), [133](#)
- [261] V. G. Turaev: Modular categories and 3-manifold invariants. *Int. J. Mod. Phys.* **B6**, 1807-1824 (1992). [142](#)
- [262] V. G. Turaev: *Quantum invariants of knots and 3-manifolds*. Walter de Gruyter, 1994. [97](#), [128](#), [132](#), [133](#), [134](#), [142](#), [143](#), [149](#), [150](#)
- [263] V. G. Turaev: Homotopy field theory in dimension 3 and crossed group-categories. [math.GT/0005291](#). [146](#)
- [264] V. G. Turaev: *Homotopy TQFTs*. Birkhäuser Verlag, 2010. [146](#)
- [265] V. G. Turaev, O. Viro: State sum invariants of 3-manifolds and quantum  $6j$ -symbols. *Topology* **31**, 865-902 (1992). [128](#), [151](#)
- [266] V. G. Turaev, H. Wenzl: Quantum invariants of 3-manifolds associated with classical simple Lie algebras. *Intern. Journ. Math.* **4**, 323-358 (1993). [143](#)
- [267] V. G. Turaev, H. Wenzl: Semisimple and modular categories from link invariants. *Math. Ann.* **309**, 411-461 (1997). [134](#), [143](#)
- [268] K.-H. Ulbrich: On Hopf algebras and rigid monoidal categories. *Israel J. Math.* **72**, 252-256 (1990) [119](#)
- [269] A. Van Daele: Multiplier Hopf algebras. *Trans. Amer. Math. Soc.* **342**, 917-932 (1994). [119](#)
- [270] A. Van Daele: An algebraic framework for group duality. *Adv. Math.* **140**, 323-366 (1998). [119](#)
- [271] E. Vasselli: Bundles of  $C^*$ -categories. *J. Funct. Anal.* **247**, 351-377 (2007). [111](#)
- [272] E. Vasselli: Group bundle duality, invariants for certain  $C^*$ -algebras, and twisted equivariant K-theory. *In: D. Burghelea, R. Melrose, A. S. Mishchenko, E. V. Troitsky (eds.):  $C^*$ -Algebras and elliptic theory II*, pp. 281-295, Birkhäuser Verlag, 2008. [124](#)
- [273] P. Vogel: Algebraic structure on modules of diagrams. Preprint 1997. [135](#)
- [274] Z. Wang: *Topological quantum computation*. American Mathematical Society, 2010. [150](#)
- [275] A. Wassermann: Operator algebras and conformal field theory III. Fusion of positive energy representations of  $SU(N)$  using bounded operators. *Invent. Math.* **133**, 467-538 (1998). [142](#), [143](#)
- [276] H. Wenzl:  $C^*$ -Tensor categories from quantum groups. *J. Amer. Math. Soc.* **11**, 261-282 (1998). [143](#)
- [277] H. Wenzl: On tensor categories of Lie type  $E_N$ ,  $N \neq 9$ . *Adv. Math.* **177**, 66-104 (2003). [135](#)
- [278] N. J. Wildberger: Finite commutative hypergroups and applications from group theory to conformal field theory. *Contemp. Math.* **183**, 413-434 (1995). (Proceedings volume of the conference *Applications of hypergroups and related measure algebras*, Seattle, 1993. [110](#)
- [279] S. L. Woronowicz: Compact matrix pseudogroups. *Comm. Math. Phys.* **111**, 613-665 (1987). [119](#), [138](#)
- [280] S. L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted  $SU(N)$  groups. *Invent. Math.* **93** (1988) 35-76. [111](#), [119](#)
- [281] S. L. Woronowicz: Compact quantum groups. *In: A. Connes, K. Gawedzki, J. Zinn-Justin (eds.): Symétries quantiques*. North-Holland, Amsterdam, 1998. [119](#)

- [282] F. Xu: New braided endomorphisms from conformal inclusions. *Comm. Math. Phys.* **192**, 349-403 (1998). [143](#)
- [283] F. Xu: Some computations in the cyclic permutations of completely rational nets. *Comm. Math. Phys.* **267**, 757-782 (2006). [150](#)
- [284] S. Yamagami:  $C^*$ -tensor categories and free product bimodules. *J. Funct. Anal.* **197**, 323-346 (2003). [97](#)
- [285] S. Yamagami: Frobenius algebras in tensor categories and bimodule extensions. *In: G. Janelidze, B. Pareigis, W. Tholen (eds.): Galois theory, Hopf algebras, and semiabelian categories.* Fields Inst. Commun. **43**, 551-570 (2004). [128](#)
- [286] S. Yamagami: Frobenius duality in  $C^*$ -tensor categories. *J. Oper. Th.* **52**, 3-20 (2004). [119](#)
- [287] D. N. Yetter: Markov algebras. *In: J. S. Birman, A. Libgober (eds.): Braids.* *Contemp. Math.* **78**, 705-730 (1988). [132](#)
- [288] D. N. Yetter: Quantum groups and representations of monoidal categories. *Math. Proc. Cambridge Philos. Soc.* **108**, 261-290 (1990). [119](#)
- [289] D. N. Yetter: Framed tangles and a theorem of Deligne on braided deformations of tannakian categories. *Contemp. Math.* **134**, 325-350 (1992). [132](#)
- [290] D. N. Yetter: *Functorial knot theory. Categories of tangles, coherence, categorical deformations, and topological invariants.* World Scientific, 2001. [110](#), [133](#), [141](#)
- [291] P. Zito:  $2-C^*$ -categories with non-simple units. *Adv. Math.* **210**, 122-164 (2007). [111](#)

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