REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 51, Número 1, 2010, Páginas 193–210

TRIANGULAR STRUCTURES OF HOPF ALGEBRAS AND TENSOR MORITA EQUIVALENCES

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Dedicated to Professor Hans-J. Schneider on the occasion of his 65th birthday

Abstract. In this paper, the triangular structures of a Hopf algebra A are discussed as a tensor Morita invariant. It is shown by many examples that triangular structures are useful for detecting whether module categories are monoidally equivalent or not. By counting and comparing the numbers of triangular structures, we give simple proofs of some results obtained in [25] without polynomial invariants.

1. Introduction

Since Tambara and Yamagami [24] have classified semisimple tensor categories whose fusion rules among simple objects are the same as a fixed fusion algebra, many researchers have studied on module categories of Hopf algebras as abstract tensor categories. New tensor invariants have recently been found and investigated, for example, the higher Frobenius-Schur indicators due to Ng and Schauenburg [18], and invariants based on braid group representations arising from the Drinfel'd double of a Hopf algebra due to Shimizu [21]. The author [25] also introduced a new family of tensor invariants of the module category of a finite-dimensional semisimple and cosemisimple Hopf algebra A by using the quasitriangular structures of A, or equivalently the braiding structures of the module category over A. As noticeable facts, the invariants are able to recognize the difference of module categories with the same Grothendieck ring, and the coefficients of the polynomials are integers under suitable assumption for A. However, it is hard to compute the polynomial invariants since we need to know all braiding structures.

In this paper, as a tensor invariant of the module category over a Hopf algebra A we will focus attention on the triangular structures of A (or equivalently, the symmetric braiding structures for the module category over A). We report on a result that the triangular structures of a Hopf algebra are useful for detecting whether module categories are monoidally equivalent or not. For example, the module categories over the three non-commutative and semisimple Hopf algebras of dimension 8 are distinguished by the numbers of triangular structures, that the result itself has already proved by Tambara and Yamagami [24], Masuoka [14], and Ng and Schauenburg [19] by different methods. Moreover, by counting and

2000 *Mathematics Subject Classification.* 16T05, 18D10, 19A49.

comparing the numbers of triangular structures, we can give simple proofs of some results obtained in [25] without polynomial invariants.

Given a Hopf algebra A, it is fundamental to know whether A has a (quasi-)triangular structure, and to determine all the (quasi-)triangular structures of A. As a trivial example of Hopf algebras having a triangular structure, one may take the group Hopf algebra $\boldsymbol{k}[G]$ of a finite group G over a field $\boldsymbol{k}$. If G is not commutative, then the dual Hopf algebra $(\boldsymbol{k}[G])^*$ has no quasitriangular structure. A non-trivial family of Hopf algebras which are not (quasi-)triangular and not co(quasi-)triangular is found by Masuoka [14, Proposition 2.5], [15, Corollary 2]. Another examples are given by Suzuki [22]. He introduced a family of cosemisimple Hopf algebras $A_{NL}^{\nu\lambda}$ parametrized by integers $N \geq 1,\ L \geq 2$ and $\lambda, \nu = \pm$, and succeeded in determining all braidings of each of them, and determining when a braiding is symmetric, though it is so hard to do this in general. In [25], based on representation theory of cyclic groups, the author also determined the quasitriangular structures of some group Hopf algebras $\boldsymbol{k}[G_{NL}]$, which are closely related to Suzuki's Hopf algebra $A_{NL} := A_{NL}^{\nu\lambda}$ in the case when N is odd. In this paper, we determine the triangular structures of $\boldsymbol{k}[G_{NL}]$, and show that for an odd integer N and an even integer L, the Hopf algebra A_{NL} has no triangular structure by combining Suzuki's results [22, Proposition 3.10] and the result [25, Corollary A.6] on self-duality. By Etingof and Gelaki[6], this means that the Hopf algebra A_{NL} can not be obtained from any group Hopf algebra by a twist. In addition, it is interesting to note that the numbers of quasitriangular structures of Hopf algebras $\boldsymbol{k}[G_{NL}]$ and A_{NL} are the same as shown in [25], however the numbers of triangular structures are nevertheless distinct. So, the author would like to make a suggestion that the triangular structures of a Hopf algebra should be examined in more detail as an tensor Morita invariant. Besides, there are many important contributions to classification on (minimal) triangular Hopf algebras, which are raised by Etingof and Gelaki [6], Gelaki [7, 8], Andruskiewitsch, Etingof and Gelaki [2], Masuoka [16], et al.

This paper is organized as follows. In Section 2 we recall some definitions of a (quasi-)triangular Hopf algebra and a (symmetric) braided Hopf algebra, and review some basic facts on the monoidal equivalence between module categories over Hopf algebras. In Section 3 the triangular structures are determined for several families of concretely given Hopf algebras. As an application, simple proofs for some results in [25] are given without polynomial invariants.

Throughout this paper, for a Hopf algebra A, denoted by Δ, ε and S are the comultiplication, the counit and the antipode of A, respectively. We use Sweedler's notation such as $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for $x \in A$. Denoted by ${}_A\mathbb{M}$ is the monoidal category whose objects are left A-modules, and morphisms are A-linear maps between them. The cyclic group of order m and the dihedral group of order $2m$ are denoted by C_m and D_{2m}, respectively. For general facts on Hopf algebras, refer to Abe's book [1], Montgomery's book [17] and Sweedler's book [23], and for general facts on monoidal categories, refer to MacLane's book [11] and Kassel's book [10].

2. Preliminaries

In this section, we recall some definitions of a (quasi-)triangular Hopf algebra and a (symmetric) braided Hopf algebra, and review some basic facts on the monoidal equivalence between module categories over Hopf algebras.

Let A be a Hopf algebra over a field $\boldsymbol{k}$, and R be an invertible element in $A\otimes A$. The pair (A,R) is said to be a *quasitriangular Hopf algebra*, and R is said to be a *universal R-matrix* of A [4] if the following conditions are satisfied:

(QT.1) $\Delta^{\mathrm{cop}}(a)\cdot R = R\cdot\Delta(a)$ for all $a\in A$,
(QT.2) $(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}$,
(QT.3) $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$.

Here, $\Delta^{\mathrm{cop}} = T_{A,A}\circ\Delta$ is the opposite comultiplication, and $R_{ij}\in A\otimes A\otimes A$ is given by $R_{12}=R\otimes 1, R_{23}=1\otimes R$ and $R_{13}=(T_{A,A}\otimes\mathrm{id})(R_{23})$, and $T_{A,A}$ is the linear transformation on $A\otimes A$ such that $T_{A,A}(a\otimes b)=b\otimes a$ for all $a,b\in A$. It is easy to see that a universal R-matrix R of A satisfies

(QT.4) $(\varepsilon\otimes\mathrm{id})(R)=1$,
(QT.5) $(\mathrm{id}\otimes\varepsilon)(R)=1$.

A quasitriangular Hopf algebra (A,R) is called *triangular* if $R_{21}R=1\otimes 1$, where $R_{21}=T_{A,A}(R)$.

Any universal R-matrix $R=\sum_i \alpha_i\otimes\beta_i$ of A induces a braiding $c=c_R$ of the monoidal category ${}_A\mathbb{M}$ as follows: $c=\{c_{X,Y}: X\otimes Y\longrightarrow Y\otimes X\}_{X,Y\in{}_A\mathbb{M}}$, where

$$c_{X,Y}(x\otimes y)=\sum_i \beta_i y\otimes\alpha_i x \qquad (x\in X,\ y\in Y).$$

A braiding c of a monoidal category $\mathcal{V}$ is called *symmetric* if the equation $c_{Y,X}\circ c_{X,Y}=\mathrm{id}_{X\otimes Y}$ holds for all objects X,Y of $\mathcal{V}$. A quasitriangular Hopf algebra (A,R) is triangular if and only if the braiding c_R is symmetric.

Lemma 2.1. *Let A be a Hopf algebra over a field $\boldsymbol{k}$. Let $\underline{Braid}(A)$ denote the set of the universal R-matrices of A, and let $\underline{Braid}({}_A\mathbb{M})$ denote the set of the braidings of the monoidal category ${}_A\mathbb{M}$. Then, the maps*

$$\underline{Braid}(A)\longrightarrow\underline{Braid}({}_A\mathbb{M}),\quad R\longmapsto c_R,$$
$$\underline{Braid}({}_A\mathbb{M})\longrightarrow\underline{Braid}(A),\quad c\longmapsto R=(T_{A,A}\circ c_{A,A})(1\otimes 1)$$

are inverse of each other. Furthermore, the above bijection $\underline{Braid}(A)\longrightarrow\underline{Braid}({}_A\mathbb{M})$ gives rise to a bijection between the set of the triangular structures of A, which is denoted by $\underline{Braid}_0(A)$, and the set of the symmetric braidings of ${}_A\mathbb{M}$. □

See [17, THEOREM 10.4.2] for a proof of the above lemma.

Let $\mathcal{V}=(\mathcal{C},\otimes,\mathbb{I},a,l,r)$, $\mathcal{M}=(\mathcal{D},\otimes,\mathbb{I}',a',l',r')$ be two monoidal categories. By a *monoidal (covariant) functor* from $\mathcal{V}$ to $\mathcal{M}$ we mean a triple (F,Φ,ω) consisting of

- a (covariant) functor $F:\mathcal{C}\longrightarrow\mathcal{D}$,
- a natural equivalence $\Phi=\{\Phi_{X,Y}: F(X)\otimes F(Y)\longrightarrow F(X\otimes Y)\}_{X,Y\in\mathcal{C}}$,

- an isomorphism $\omega : \mathbb{I}' \longrightarrow F(\mathbb{I})$,

such that the following three diagrams commute for all $X, Y, Z \in \mathcal{C}$:

$$\begin{tikzcd}
(F(X) \otimes F(Y)) \otimes F(Z) \arrow[r, "\Phi_{X,Y} \otimes \mathrm{id}_Z"] \arrow[d, "a'"'] & F(X \otimes Y) \otimes F(Z) \arrow[r, "\Phi_{X \otimes Y, Z}"] & F((X \otimes Y) \otimes Z) \arrow[d, "F(a)"] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\mathrm{id}_X \otimes \Phi_{Y,Z}"] & F(X) \otimes F(Y \otimes Z) \arrow[r, "\Phi_{X, Y \otimes Z}"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{I}' \otimes F(X) \arrow[r, "\omega \otimes \mathrm{id}"] \arrow[d, "l'"'] & F(\mathbb{I}) \otimes F(X) \arrow[d, "\Phi"] \\
F(X) & F(\mathbb{I} \otimes X) \arrow[l, "F(l)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \otimes \mathbb{I}' \arrow[r, "\mathrm{id} \otimes \omega"] \arrow[d, "r'"'] & F(X) \otimes F(\mathbb{I}) \arrow[d, "\Phi"] \\
F(X) & F(X \otimes \mathbb{I}) \arrow[l, "F(r)"']
\end{tikzcd}$$

Let $F : \mathcal{V} \longrightarrow \mathcal{M}$ and $G : \mathcal{M} \longrightarrow \mathcal{V}$ be two covariant functors. By a *monoidal natural equivalence* from $F \circ G$ to the identity monoidal functor $1_{\mathcal{M}}$ we mean a natural equivalence $\varphi = \{\varphi(P) : F(G(P)) \longrightarrow P\}_{P \in \mathcal{D}}$ such that for all objects P, Q in $\mathcal{D}$ the following two diagrams commute:

$$\begin{tikzcd}
F(G(P)) \otimes F(G(Q)) \arrow[r, "\varphi(P) \otimes \varphi(Q)"] \arrow[d, "\Phi_{G(P),G(Q)}"'] & P \otimes Q \\
F(G(P) \otimes G(Q)) \arrow[r, "F(\Phi'_{P,Q})"] & F(G(P \otimes Q)) \arrow[u, "\varphi(P \otimes Q)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}' \arrow[r, "\omega"] \arrow[d, "\mathrm{id}_{\mathbb{I}'}"'] & F(\mathbb{I}) \arrow[d, "F(\omega')"] \\
\mathbb{I}' & F(G(\mathbb{I}')) \arrow[l, "\varphi(\mathbb{I}')"']
\end{tikzcd}$$

Such a φ is written as $\varphi : (F, \Phi, \omega) \circ (G, \Phi', \omega') \Longrightarrow 1_{\mathcal{M}}$. Two monoidal categories $\mathcal{V}$ and $\mathcal{M}$ are called *monoidally equivalent* if there exist monoidal covariant functors $F : \mathcal{V} \longrightarrow \mathcal{M}$ and $G : \mathcal{M} \longrightarrow \mathcal{V}$, and exist monoidal natural isomorphisms $\varphi : (F, \Phi, \omega) \circ (G, \Phi', \omega') \Longrightarrow 1_{\mathcal{M}}$ and $\psi : (G, \Phi', \omega') \circ (F, \Phi, \omega) \Longrightarrow 1_{\mathcal{V}}$.

Lemma 2.2. *Let $\mathcal{V}$ and $\mathcal{M}$ be two monoidal categories. If $\mathcal{V}$ and $\mathcal{M}$ are monoidally equivalent, then there exists a bijection between the sets of the braidings of $\mathcal{V}$ and $\mathcal{M}$. This bijection induces a bijection between the sets of the symmetric braidings of $\mathcal{V}$ and $\mathcal{M}$.*

Proof. By assumption, there exist monoidal functors $(F, \Phi, \omega) : \mathcal{V} \longrightarrow \mathcal{M}$, $(G, \Phi', \omega') : \mathcal{M} \longrightarrow \mathcal{V}$ and monoidal natural equivalences $\varphi : (F, \Phi, \omega) \circ (G, \Phi', \omega') \Longrightarrow 1_{\mathcal{M}}$, $\psi : (G, \Phi', \omega') \circ (F, \Phi, \omega) \Longrightarrow 1_{\mathcal{V}}$.

Let c be a braiding of $\mathcal{V}$. Then, for $P, Q \in \mathcal{D}$ a morphism $c'_{P,Q}$ in $\mathcal{D}$ defined by which the following diagram commutes.

$$\begin{tikzcd}
P \otimes Q \arrow[rr, "c'_{P,Q}"] & & Q \otimes P \\
F(G(P)) \otimes F(G(Q)) \arrow[u, "\varphi(P)\otimes\varphi(Q)"] \arrow[d, "\Phi_{G(P),G(Q)}"'] & & F(G(Q)) \otimes F(G(P)) \arrow[u, "\varphi(Q)\otimes\varphi(P)"'] \arrow[d, "\Phi_{G(Q),G(P)}"] \\
F(G(P) \otimes G(Q)) \arrow[rr, "F(c_{G(P),G(Q)})"] & & F(G(Q) \otimes G(P)).
\end{tikzcd} \tag{2.1}$$

Then, c' is a braiding of $\mathcal{M}$. In this way, we have a map $f : \{$ the braidings of $\mathcal{V}$ $\} \longrightarrow \{$the braidings of $\mathcal{M}\}$, $c \longmapsto c'$. Similarly, we have a map $g : \{$ the braidings of $\mathcal{M}\} \longrightarrow \{$the braidings of $\mathcal{V}\}$. Then, it can be easily shown that the last two maps are inverse of each other. If c is a symmetric braiding of $\mathcal{V}$, then the corresponding braiding c' of $\mathcal{M}$ is also symmetric. Therefore, the bijection f induces a bijection between the sets of the symmetric braidings of $\mathcal{V}$ and $\mathcal{M}$. $\square$

Remark 2.3. Let c be a braiding of $\mathcal{V}$, and c' be the braiding of $\mathcal{M}$ induced from c as in the proof of Lemma 2.2. Then, the monoidal functor $(F, \Phi, \omega) : \mathcal{V} \longrightarrow \mathcal{M}$ is a braided monoidal functor from $(\mathcal{V}, c)$ to $(\mathcal{M}, c')$, and the monoidal functor $(G, \Phi', \omega') : \mathcal{M} \longrightarrow \mathcal{V}$ is a braided monoidal functor from $(\mathcal{M}, c')$ to $(\mathcal{V}, c)$.

Two Hopf algebras A and B are called *tensor Morita equivalent* if the module categories ${}_A\mathbb{M}$ and ${}_B\mathbb{M}$ are monoidally equivalent. By Lemmas 2.1 and 2.2 we have the following:

Proposition 2.4. *Let A and B be two Hopf algebras over a field $\boldsymbol{k}$. If A and B are tensor Morita equivalent, then the cardinalities of $\underline{Braid}(A)$ and $\underline{Braid}(B)$ are equal, and the same result holds for $\underline{Braid}_0(A)$ and $\underline{Braid}_0(B)$.* $\square$

The dual notion of a quasitriangular Hopf algebra is a braided Hopf algebra, which was introduced by Doi [3]. Let A be a Hopf algebra over a field $\boldsymbol{k}$, and let $\sigma : A \otimes A \longrightarrow \boldsymbol{k}$ be a $\boldsymbol{k}$-linear map that is invertible with respect to the convolution product. The pair (A, σ) is said to be a *braided Hopf algebra*, and σ is said to be a *braiding* of A if the following conditions are satisfied: for all $x, y, z \in A$

(B1) $\sum \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = \sum \sigma(x_{(2)}, y_{(2)})y_{(1)}x_{(1)}$,
(B2) $\sigma(xy, z) = \sum \sigma(x, z_{(1)})\sigma(y, z_{(2)})$,
(B3) $\sigma(x, yz) = \sum \sigma(x_{(1)}, z)\sigma(x_{(2)}, y)$.

Any braiding σ of A satisfies

(B4) $\sigma(1_A, x) = \sigma(x, 1_A) = \varepsilon(x)$ for all $x \in A$.

A braiding σ of A is said to be *symmetric* if the equation

(SB) $\sum \sigma(y_{(1)}, x_{(1)})\sigma(x_{(2)}, y_{(2)}) = \varepsilon(x)\varepsilon(y)$

is satisfied for all $x, y \in A$. If A is generated by a subcoalgebra C as an algebra, then by the conditions (B2), (B3) we see that the braiding σ is determined by the values on C.

Let $\iota : A^* \otimes A^* \longrightarrow (A \otimes A)^*$ be the natural linear isomorphism defined by

$$\langle \iota(p \otimes q),\ a \otimes b \rangle = p(a)q(b) \qquad (p, q \in A^*,\ a, b \in A),$$

where $\langle\ ,\ \rangle$ stands for the natural pairing between $(A \otimes A)^*$ and $A \otimes A$. Denoted by $\underline{\mathrm{braid}}(A)$ is the set of the braidings of A, and denoted by $\underline{\mathrm{braid}}_0(A)$ is the set of the symmetric braidings of A. If A is finite-dimensional, then, the map

$$\underline{\mathrm{braid}}(A) \longrightarrow \underline{\mathrm{Braid}}(A^*), \qquad \sigma \longmapsto \iota^{-1}(\sigma)$$

is bijective, and this map induces a bijection $\underline{\mathrm{braid}}_0(A) \longrightarrow \underline{\mathrm{Braid}}_0(A^*)$.

3. Examples of triangular structures

3.1. **Triangular structures of Suzuki's Hopf algebras.** Suzuki introduced a family of cosemisimple Hopf algebras $A_{NL}^{\nu\lambda}$ parametrized by integers $N \geq 1,\ L \geq 2$ and $\lambda, \nu = \pm$, and investigated various properties and structures of them [22]. In particular, he determined all braidings of his Hopf algebras, and determined when a braiding is symmetric. Throughout this subsection we assume that $\boldsymbol{k}$ is an algebraically closed field whose characteristic is not 2.

As an algebra Suzuki's Hopf algebra $A_{NL}^{\nu\lambda}$ is generated by a subcoalgebra C, which is isomorphic to the the dual coalgebra of the 2×2-matrix algebra $\mathrm{M}_2(\boldsymbol{k})$. More precisely, the Hopf algebra $A_{NL}^{\nu\lambda}$ is generated by $x_{11}, x_{12}, x_{21}, x_{22}$ subject to the relations:

$$\begin{aligned} x_{11}^2 = x_{22}^2, \quad x_{12}^2 = x_{21}^2, \quad x_{ij}x_{lm} = 0\ (i - j \equiv l - m \pmod 2), \\ x_{11}^{2N} + \nu x_{12}^{2N} = 1, \quad \chi_{11}^L = \chi_{22}^L, \quad \chi_{21}^L = \lambda \chi_{12}^L, \end{aligned}$$

where we use the following notation for $i, j = 1, 2,\ m \geq 1$.

$$\chi_{11}^m := \underbrace{x_{11}x_{22}x_{11}\cdots\cdots}_{m}, \quad \chi_{22}^m := \underbrace{x_{22}x_{11}x_{22}\cdots\cdots}_{m},$$
$$\chi_{12}^m := \underbrace{x_{12}x_{21}x_{12}\cdots\cdots}_{m}, \quad \chi_{21}^m := \underbrace{x_{21}x_{12}x_{21}\cdots\cdots}_{m}.$$

The Hopf algebra structure of $A_{NL}^{\nu\lambda}$ is given by

$$\Delta(x_{ij}) = x_{i1} \otimes x_{1j} + x_{i2} \otimes x_{2j}, \quad \varepsilon(x_{ij}) = \delta_{ij}, \quad {}^*S(x_{ij}) = x_{ji}^{4N-1},$$

and the set

$$\{\ x_{11}^s \chi_{22}^t,\ x_{12}^s \chi_{21}^t \mid 1 \leq s \leq 2N,\ 0 \leq t \leq L - 1\ \}$$

is a basis of it over $\boldsymbol{k}$. So, $\dim A_{NL}^{\nu\lambda} = 4NL$, and $A_{NL}^{\nu\lambda}$ is generated by the subcoalgebra $C = \bigoplus_{i,j=1,2} \boldsymbol{k} x_{ij}$. By construction, $A_{NL}^{\nu\lambda}$ is always cosemisimple, and if the

*the description of the antipode of $A_{Nn}^{\nu\lambda}$ in [25] is wrong in the case when $\nu = -$.

characteristic $\mathrm{ch}(\boldsymbol{k})$ does not divide NL, then $A_{NL}^{\nu\lambda}$ is semisimple [22, Theorem 3.1 vii)]. We note that

$$\Delta(\chi_{ij}^m) = \chi_{i1}^m \otimes \chi_{1j}^m + \chi_{i2}^m \otimes \chi_{2j}^m$$

for all $i, j = 1, 2$ and all positive integers m.

The Hopf algebra A_{12}^{+-} is the unique non-commutative and non-cocommutative semisimple Hopf algebra of dimension 8, so-called the Kac-Paljutkin algebra [9], [13]. We will denote it by K_8.

The braidings of the Hopf algebra $A_{NL}^{\nu\lambda}$ are given as follows.

Theorem 3.1 (**Suzuki** [22])**.** *(1) For $\alpha, \beta \in \boldsymbol{k}$, let $\sigma_{\alpha\beta}$ be the $\boldsymbol{k}$-linear functional on $C \otimes C$ such that the values $\sigma_{\alpha\beta}(x_{ij}, x_{kl})$ $(i, j, k, l = 1, 2)$ are given as in the following table.*

$x \diagdown y$	x_{11}	x_{12}	x_{21}	x_{22}
x_{11}	0	0	0	0
x_{12}	0	α	β	0
x_{21}	0	β	α	0
x_{22}	0	0	0	0

Then, $\sigma_{\alpha\beta}$ is extended to a braiding of $A_{NL}^{\nu\lambda}$ if and only if $\alpha, \beta \in \boldsymbol{k}^\times$, $(\alpha\beta)^N = \nu$, $(\alpha\beta^{-1})^L = \lambda$.

(2) Suppose that $L = 2$. For $\gamma, \delta \in \boldsymbol{k}$, let $\tau_{\gamma\delta}$ be the $\boldsymbol{k}$-linear functional on $C \otimes C$ such that the values $\tau_{\gamma\delta}(x_{ij}, x_{kl})$ $(i, j, k, l = 1, 2)$ are given as in the following table.

$x \diagdown y$	x_{11}	x_{12}	x_{21}	x_{22}
x_{11}	γ	0	0	δ
x_{12}	0	0	0	0
x_{21}	0	0	0	0
x_{22}	$\lambda\delta$	0	0	γ

Then, $\tau_{\gamma\delta}$ is extended to a braiding of $A_{N2}^{\nu\lambda}$ if and only if $\gamma, \delta \in \boldsymbol{k}^\times$, $\gamma^2 = \delta^2$, $\gamma^{2N} = 1$.

(3) If $L \geq 3$, then the set of all the braidings of $A_{NL}^{\nu\lambda}$ is given by

$$\{\ \sigma_{\alpha\beta} \mid \alpha, \beta \in \boldsymbol{k}^\times,\ (\alpha\beta)^N = \nu,\ (\alpha\beta^{-1})^L = \lambda\ \}.$$

If $L = 2$, then the set of all the braidings of $A_{N2}^{\nu\lambda}$ is given by

$$\{\ \sigma_{\alpha\beta} \mid \alpha, \beta \in \boldsymbol{k}^\times,\ (\alpha\beta)^N = \nu,\ (\alpha\beta^{-1})^2 = \lambda\ \} \cup \{\ \tau_{\gamma\delta} \mid \gamma, \delta \in \boldsymbol{k}^\times,\ \gamma^2 = \delta^2,\ \gamma^{2N} = 1\ \}.$$

If a Hopf algebra A is generated by a subcoalgebra C as an algebra, then a braiding σ of A is symmetric if and only if the condition (SB) for σ is satisfied for all $x, y \in C$. By using this, one can easily determine when the braidings $\sigma_{\alpha\beta}$ and $\tau_{\gamma\delta}$ are symmetric.

Lemma 3.2 (**Suzuki** [22, Proposition 3.10]). *(1) For the braiding $\sigma_{\alpha\beta}$ of $A_{NL}^{\nu\lambda}$,*

$$\sigma_{\alpha\beta} \text{ is symmetric} \iff (\alpha,\beta) = \begin{cases} (1,1),\ (1,-1),\ (-1,1),\ (-1,-1) \\ \qquad \text{if } (\nu,\lambda)=(+,+),\ (N,L)=(\text{even},\text{even}), \\ (1,1),\ (-1,-1) \\ \qquad \text{if } \begin{pmatrix} (\nu,\lambda)=(+,+),\ (N,L)\neq(\text{even},\text{even}) \text{ or} \\ (\nu,\lambda)=(+,-),\ (N,L)=(\text{even},\text{odd}) \text{ or} \\ (\nu,\lambda)=(-,+),\ (N,L)=(\text{odd},\text{even}) \text{ or} \\ (\nu,\lambda)=(-,-),\ (N,L)=(\text{odd},\text{odd}) \end{pmatrix}. \end{cases}$$

(2) For the braiding $\tau_{\gamma\delta}$ of $A_{N2}^{\nu\lambda}$,

$$\tau_{\gamma\delta} \text{ is symmetric} \iff \lambda=+ \text{ and } (\gamma,\delta)=(1,1),\ (1,-1),\ (-1,1),\ (-1,-1). \qquad \square$$

From here to the end of this subsection, we suppose that the characteristic of $\boldsymbol{k}$ does not divide $2LN$. Let N be an odd integer. Then, as shown in [25, Corollary A.6], in the case when $\lambda = -$ or $(L,\lambda)=(\text{odd},+)$, the Hopf algebra $A_{NL}^{+\lambda}$ is self-dual. Thus, by Lemma 3.2, we have:

Corollary 3.3. *Suppose that N is odd, and $\nu=+$.*
If $L\geq 3$, then,

$$\sharp \underline{Braid}_0(A_{NL}^{+\lambda}) = \begin{cases} 2 & (\lambda=+), \\ 0 & (\lambda=-). \end{cases}$$

If $L=2$, then

$$\sharp \underline{Braid}_0(A_{N2}^{+\lambda}) = \begin{cases} 6 & (\lambda=+), \\ 0 & (\lambda=-). \end{cases}$$

If N is odd, then $A_{NL}^{+\lambda}$ is isomorphic to the group algebra of the finite group

$$G=\langle h,\ t,\ w \mid t^2=h^{2N}=1,\ w^n=h^{(L+\frac{\lambda-1}{2})N},\ tw=w^{-1}t,\ ht=th,\ hw=wh\rangle$$

as an algebra [25]. The order of G is $4LN$. If $(L,\lambda)=(\text{even},+)$ or $(L,\lambda)=(\text{odd},-)$, then the group G is isomorphic to the direct product $D_{2L}\times C_{2N}$. If $(L,\lambda)=(\text{even},-)$ or $(L,\lambda)=(\text{odd},+)$, then G coincides with the group

$$G_{NL}=\langle h,\ t,\ w \mid t^2=h^{2N}=1,\ w^L=h^N,\ tw=w^{-1}t,\ ht=th,\ hw=wh\rangle.$$

The group G_{NL} is isomorphic to the direct product $C_N\times D_{4L}$ since it is decomposed as $G_{NL}=\langle h^2\rangle\langle t,w\rangle$ by the normal subgroups $\langle h^2\rangle$, $\langle t,w\rangle$ of G_{NL} satisfying $\langle h^2\rangle\cap\langle t,w\rangle=\{1\}$.

For an odd integer $N\geq 1$ and an integer $L\geq 2$, we set

$$A_{NL}=\begin{cases} A_{NL}^{++} & \text{if } L \text{ is odd}, \\ A_{NL}^{+-} & \text{if } L \text{ is even}. \end{cases}$$

We note that the Grothendieck rings of A_{NL} and $\boldsymbol{k}[G_{NL}]$ are isomorphic as rings with $*$-structure [25, Proposition A.3]. Since every group Hopf algebra has at least one triangular structure, by Proposition 2.4 and Corollary 3.3, we have the following result, which has obtained by comparing polynomial invariants [25, Corollary 5.18].

Corollary 3.4. *Let $N \geq 1$ be an odd integer, and $L \geq 2$ be an even integer. Then two Hopf algebras A_{NL} and $\boldsymbol{k}[G_{NL}]$ are not tensor Morita equivalent meanwhile their Grothendieck rings are isomorphic.*

By Etingof and Gelaki [6, Corollary 6.2], a finite-dimensional semisimple and cosemisimple Hopf algebra over an algebraically closed field is obtained from some group Hopf algebra by a twist. Since $\underline{\mathrm{Braid}}_0(A_{NL}^{+-}) = \varnothing$ by Corollary 3.3 we also have:

Corollary 3.5. *Let $N \geq 1$ be an odd integer, and $L \geq 2$ an even integer. Then the Hopf algebra A_{NL} can not be obtained from a group Hopf algebra by a twist.*

A similar result with the above corollary is also obtained by Masuoka [14, Proposition 2.5] by an argument of Hopf algebra extensions.

In contrast to Corollary 3.4, if L is odd, then the following holds.

Proposition 3.6 (Shimizu)**.** *Let $N \geq 1$ be an odd integer, and $L \geq 3$ be an odd integer. Then two Hopf algebras A_{NL} and $\boldsymbol{k}[G_{NL}]$ are tensor Morita equivalent.*

Proof. The idea of the proof is due to Shimizu. By [22, Corollary 3.7 (iii)] the Hopf algebra A_{NL}^{++} is isomorphic to the Hopf algebra $\boldsymbol{k}[C_N] \otimes A_{1L}^{++}$. By [14, Theorem 4.1 (2)] and the self-duality of A_{1L}^{++}, we see that the Hopf algebras A_{1L}^{++} and $\boldsymbol{k}[D_{4L}]$ are tensor Morita equivalent. So, $A_{NL}^{++} \cong \boldsymbol{k}[C_N] \otimes A_{1L}^{++}$ and $\boldsymbol{k}[G_{NL}] \cong \boldsymbol{k}[C_N] \otimes \boldsymbol{k}[D_{4L}]$ are tensor Morita equivalent since two finite-dimensional Hopf algebras over a field which are tensor Morita equivalent are obtained by a twist of each other [20]. $\square$

3.2. **Triangular structures of group Hopf algebras.** In this subsection, we determine the triangular structures of some group Hopf algebras, including cyclic groups, dihedral groups, generalized quaternion groups, and the groups G_{NL}. Note that for any finite group G and any universal R-matrix R of the group Hopf algebra $\boldsymbol{k}[G]$, there exists a maximal abelian normal subgroup H of G such that R is a universal R-matrix of $\boldsymbol{k}[H]$ (for a proof see [25, Lemma 5.1]).

Let G be a finite abelian group, and m be the exponent of G, that is, m is the smallest positive integer such that $g^m = 1$ for all $g \in G$. Let $\boldsymbol{k}$ be a field which contains a primitive m-th root of unity. Then, the characteristic of $\boldsymbol{k}$ does not divide m, and it is known that the universal R-matrices of the group Hopf algebra $\boldsymbol{k}[G]$ are characterized by bicharacters of the character group $\widehat{G} = \{\ \chi : G \longrightarrow \boldsymbol{k}^\times \mid \chi(gh) = \chi(g)\chi(h) \text{ for all } g, h \in G\ \}$, where $\boldsymbol{k}^\times = \boldsymbol{k} - \{0\}$ (See Lemma 3.7 below). Here, by a bicharacter of $\widehat{G}$ we mean a function $b : \widehat{G} \times \widehat{G} \longrightarrow \boldsymbol{k}$ which satisfies the following three conditions: for all $\chi, \chi_1, \chi_2 \in \widehat{G}$

(Bich.1) $b(\chi_1\chi_2, \chi) = b(\chi_1, \chi)b(\chi_2, \chi)$,
(Bich.2) $b(\chi, \chi_1\chi_2) = b(\chi, \chi_1)b(\chi, \chi_2)$,

(Bich.3) $b(\chi, 1) = b(1, \chi) = 1$.

The above three conditions are equivalent to that the linear map $\tilde{b} : \boldsymbol{k}[\widehat{G}] \otimes \boldsymbol{k}[\widehat{G}] \longrightarrow \boldsymbol{k}$ induced from b is a braiding of $\boldsymbol{k}[\widehat{G}]$. Denoted by $\underline{\mathrm{Bich}}(\widehat{G})$ is the set of the bicharacters of $\widehat{G}$. Then by using the orthogonality relations of characters we have:

Lemma 3.7. *For each $\chi \in \widehat{G}$ we define an element $e(\chi)$ of $\boldsymbol{k}[G]$ by*

$$e(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

Then, the map $\Phi : \underline{Bich}(\widehat{G}) \longrightarrow \underline{Braid}(\boldsymbol{k}[G])$ defined by

$$\Phi(b) = \sum_{\chi, \eta \in \widehat{G}} b(\chi, \eta) e(\chi) \otimes e(\eta)$$

is a bijection. □

See [12, Corollary 1.5.6 & Example 2.1.18] for a proof of the above lemma. We note that $\{ e(\chi) \mid \chi \in \widehat{G} \}$ is the primitive orthogonal idempotents of $\boldsymbol{k}[G]$.

The (quasi-)triangular structures of the group Hopf algebras of a cyclic group and a direct product of two cyclic groups are given as in the following two lemmas.

Lemma 3.8. *Let m be a positive integer, and $\omega \in \boldsymbol{k}$ be a primitive m-th root of unity. Let s be a generator of the cyclic group C_m. Then any universal R-matrix of the group Hopf algebra $\boldsymbol{k}[C_m]$ is given by the formula*

$$R_d = \frac{1}{m} \sum_{k,i=0}^{m-1} \omega^{-ik} s^k \otimes s^{di} \qquad (d \in \{0, 1, \dots, m-1\}).$$

Furthermore,

$$\underline{Braid}_0(\boldsymbol{k}[C_m]) = \begin{cases} \{R_0\} & \text{if } m \text{ is odd,} \\ \{R_0, R_{m/2}\} & \text{if } m \text{ is even.} \end{cases}$$

□

Lemma 3.9. *Let G be the direct product of the cyclic groups $C_m = \langle g \rangle$ and $C_n = \langle h \rangle$, and let $\omega \in \boldsymbol{k}$ be a primitive mn-th root of unity. We set $X(m,n) = \{d \in \{0, 1, \dots, m-1\} \mid dn \equiv 0 \pmod{m}\}$. Then any universal R-matrix of $\boldsymbol{k}[G]$ is given by the formula*

$$R_{pqrs}^{\boldsymbol{k}[G]} = \sum_{i,j=0}^{m-1} \sum_{k,l=0}^{n-1} \omega^{n(pij+rkj)+m(skl+qil)} E_{ik} \otimes E_{jl},$$

where $p \in X(m,m), q \in X(n,m), r \in X(m,n), s \in X(n,n)$, and $E_{ik} = \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{l=0}^{n-1} \omega^{-ijn-klm} g^j h^l$. Furthermore,

$$\underline{Braid}_0(\boldsymbol{k}[G]) = \left\{ R_{pqrs}^{\boldsymbol{k}[G]} \;\middle|\; \begin{array}{c} p, r \in \{0, 1, \dots, m-1\},\ q, s \in \{0, 1, \dots, m-1\}, \\ rn \equiv 2p \equiv 0 \ (\mathrm{mod}\, m),\ qm \equiv 2s \equiv 0 \ (\mathrm{mod}\, n),\ rn + qm \equiv 0 \ (\mathrm{mod}\, mn) \end{array} \right\}.$$

Proof. The character group of G is given by $\widehat{G} = \{\chi_{ik} \mid i = 0, 1, \ldots, m-1,\ k = 0, 1, \ldots, n-1\}$, where the function χ_{ik} is given by $\chi_{ik}(g^j h^l) = \omega^{ijn+klm}$ for all integers j, l. We note that $E_{ik} = e(\chi_{ik})$, and the set $\{\ E_{ik} \mid i = 0, 1, \ldots, m-1,\ k = 0, 1, \ldots, n-1\}$ is the primitive orthogonal idempotents of $\boldsymbol{k}[G]$.

Let R be a universal R-matrix of $\boldsymbol{k}[G]$, and write R in the form

$$R = \sum_{i,j=0}^{m-1} \sum_{k,l=0}^{n-1} R_{jl}^{ik} E_{ik} \otimes E_{jl} \quad (R_{jl}^{ik} \in \boldsymbol{k}).$$

Then $\{R_{jl}^{ik}\}$ is determined by a bicharacter of $\widehat{G}$. Thus, the following equations hold:

$$\begin{cases} R_{j\ \ \ \ l}^{u+x\ v+y} = R_{j\ l}^{u\ v} R_{j\ l}^{x\ y} & \text{for } \ u, x, j = 0, 1, \ldots, m-1,\ v, y, l = 0, 1, \ldots, n-1, \\ R_{u+x\ v+y}^{i\ \ \ \ \ k} = R_{xy}^{i\ k} R_{uv}^{i\ k} & \text{for } \ u, x, i = 0, 1, \ldots, m-1,\ v, y, k = 0, 1, \ldots, n-1, \\ R_{i\ k}^{0\ 0} = R_{0\ 0}^{i\ k} = 1 & \text{for } i = 0, 1, \ldots, m-1,\ k = 0, 1, \ldots, n-1, \end{cases}$$

where the indices i, j of R_{jl}^{ik} are treated as modulo m and the indices k, l of R_{jl}^{ik} are treated as modulo n. From the above equations we obtain

$$R_{j\ l}^{i\ k} = (R_{1\ 0}^{0\ 1})^{jk} (R_{1\ 0}^{1\ 0})^{ji} (R_{0\ 1}^{0\ 1})^{lk} (R_{0\ 1}^{1\ 0})^{li}$$

for $i, j = 0, 1, \ldots, m-1$ and $k, l = 0, 1, \ldots, n-1$. Thus $R_{1\ 0}^{1\ 0}$ and $R_{1\ 0}^{0\ 1}$ are need to be m-th roots of unity, and $R_{0\ 1}^{0\ 1}$ and $R_{0\ 1}^{1\ 0}$ are need to be n-th roots of unity. So, there are integers $p \in \{0, 1, \ldots, m-1\}$, $q \in X(n, m)$, $r \in X(m, n)$ and $s \in \{0, 1, \ldots, n-1\}$ such that $R_{1\ 0}^{1\ 0} = \omega^{pn}$, $R_{0\ 1}^{1\ 0} = \omega^{qm}$, $R_{1\ 0}^{0\ 1} = \omega^{rn}$ and $R_{0\ 1}^{0\ 1} = \omega^{sm}$. Thus R is expressed by

$$R = \sum_{i,j=0}^{m-1} \sum_{k,l=0}^{n-1} \omega^{n(pij+rjk)+m(skl+qli)} E_{ik} \otimes E_{jl}.$$

Conversely, it is immediately checked that the above R is a universal R-matrix of $\boldsymbol{k}[G]$.

For the universal R-matrix $R = R_{pqrs}^{\boldsymbol{k}[G]}$ of $\boldsymbol{k}[G]$, we have

$$R_{21}R = \sum_{i,j=0}^{m-1} \sum_{k,l=0}^{n-1} \omega^{n(2pij+rli+rkj)+m(2skl+qjk+qli)} E_{ik} \otimes E_{jl}.$$

Thus, $(\boldsymbol{k}[G], R)$ is triangular if and only if $\omega^{n(2pij+rli+rkj)+m(2skl+qjk+qli)} = 1$ for all $i, j = 0, 1, \ldots, m-1$ and $k, l = 0, 1, \ldots, n-1$. This condition is equivalent to that $2ms, 2pn, nr + mq$ are all 0 modulo mn, that is, $rn \equiv 2p \equiv 0 \pmod{m}$, $qm \equiv 2s \equiv 0 \pmod{n}$, $rn + qm \equiv 0 \pmod{mn}$. □

Next, let us consider the finite group

$$D_{m,n} = \langle s,\ t \mid s^m = 1,\ t^2 = s^n,\ t^{-1}st = s^{-1} \rangle$$

for integers $m \geq 3$ and $n \geq 1$. If $m = n$, then $D_{m,n}$ is the dihedral group D_{2m} of order $2m$, and if $m = 2n$, then it is the generalized quaternion group Q_{2m} of order

$2m$. We determine the (quasi-)triangular structures of the group Hopf algebra $\boldsymbol{k}[D_{m,n}]$ in the case when the order of $D_{m,n}$ is $2m$.

Lemma 3.10. *Suppose that the order of $D_{m,n}$ is $2m$.*

(1) If $m \neq 4$, then $\underline{\mathit{Braid}}(\boldsymbol{k}[D_{m,n}]) = \underline{\mathit{Braid}}(\boldsymbol{k}[\langle s\rangle])$ and $\underline{\mathit{Braid}}_0(\boldsymbol{k}[D_{m,n}]) = \underline{\mathit{Braid}}_0(\boldsymbol{k}[\langle s\rangle])$.

(2) If $m = 4$, then n is even, and

$$\underline{\mathit{Braid}}(\boldsymbol{k}[D_{4,n}]) = \begin{cases} \underline{\mathit{Braid}}(\boldsymbol{k}[\langle s\rangle]) \cup \{R_1^{\boldsymbol{k}[\langle t\rangle]}, R_3^{\boldsymbol{k}[\langle t\rangle]}, R_1^{\boldsymbol{k}[\langle ts\rangle]}, R_3^{\boldsymbol{k}[\langle ts\rangle]}\} & \text{if } n = 2, \\ \underline{\mathit{Braid}}(\boldsymbol{k}[\langle s\rangle]) \cup \{R_{0110}^{\boldsymbol{k}[\langle t,s^2\rangle]}, R_{0111}^{\boldsymbol{k}[\langle t,s^2\rangle]}, R_{0110}^{\boldsymbol{k}[\langle ts,s^2\rangle]}, R_{0111}^{\boldsymbol{k}[\langle ts,s^2\rangle]}\} & \text{if } n = 4, \end{cases}$$

$$\underline{\mathit{Braid}}_0(\boldsymbol{k}[D_{4,n}]) = \begin{cases} \underline{\mathit{Braid}}_0(\boldsymbol{k}[\langle s\rangle]) & \text{if } n = 2, \\ \underline{\mathit{Braid}}_0(\boldsymbol{k}[\langle s\rangle]) \cup \{R_{0110}^{\boldsymbol{k}[\langle t,s^2\rangle]}, R_{0111}^{\boldsymbol{k}[\langle t,s^2\rangle]}, R_{0110}^{\boldsymbol{k}[\langle ts,s^2\rangle]}, R_{0111}^{\boldsymbol{k}[\langle ts,s^2\rangle]}\} & \text{if } n = 4. \end{cases}$$

Proof. By using $t^{-1}st = s^{-1}$, it is easy to check that $R_d^{\boldsymbol{k}[\langle s\rangle]}$ is a universal R-matrix of $\boldsymbol{k}[D_{m,n}]$. If a maximal commutative normal subgroup K of $D_{m,n}$ contains ts^i for some i, then $t \in K$ or $ts \in K$ occurs since $s(ts^i)s^{-1} = ts^{i-2}$. Since $s^2 = s^{-i+2}t^{-1} \cdot ts^i \in K$, we see that $ts^2,\ ts^{-2} = s^2t \in K$ or $ts^3 = (ts)s^2,\ ts^{-1} = s^2(ts) \in K$. It follows that $s^4 = 1$. Thus, if $m \neq 4$, then the subgroup $H = \langle s\rangle$ is a unique maximal commutative normal subgroup of $D_{m,n}$, and whence, by [25, Lemma 5.1], any universal R-matrix of $\boldsymbol{k}[D_{m,n}]$ is a universal R-matrix of $\boldsymbol{k}[H]$. This implies that $\underline{\mathrm{Braid}}(\boldsymbol{k}[D_{m,n}]) = \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle s\rangle])$ and $\underline{\mathrm{Braid}}_0(\boldsymbol{k}[D_{m,n}]) = \underline{\mathrm{Braid}}_0(\boldsymbol{k}[\langle s\rangle])$.

Let $m = 4$. If n is odd, then $t^2 = s^{\pm 1}$, and it follows from $t^{-1}st = s^{-1}$ that $s^2 = 1$, this is a contradiction with $\sharp D_{4,n} = 8$. Thus, n is need to be even, and there are essentially two choices for n, that is, $n = 2$ and $n = 4$. In the case of $n = 2$, then the finite group $D_{4,2}$ $(= Q_8)$ has exactly three maximal commutative normal subgroups, which are given by $\langle s\rangle$, $\langle t\rangle$, $\langle ts\rangle$. Thus, $\underline{\mathrm{Braid}}(\boldsymbol{k}[D_{4,2}]) \subset \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle s\rangle]) \cup \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle t\rangle]) \cup \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle ts\rangle])$. Since any universal R-matrix of the group Hopf algebras of $\langle s\rangle$, $\langle t\rangle$, $\langle ts\rangle$ is a universal R-matrix of $\boldsymbol{k}[D_{4,2}]$, we see that $\underline{\mathrm{Braid}}(\boldsymbol{k}[D_{4,2}]) = \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle s\rangle]) \cup \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle t\rangle]) \cup \underline{\mathrm{Braid}}(\boldsymbol{k}[\langle ts\rangle])$. We note that $R_{2d}^{\boldsymbol{k}[\langle t\rangle]} = R_{2d}^{\boldsymbol{k}[\langle ts\rangle]} = R_{2d}^{\boldsymbol{k}[\langle s\rangle]}$ for $d = 0, 1$, while by Lemma 3.8 the universal R-matrices $R_{2d+1}^{\boldsymbol{k}[\langle t\rangle]} = R_{2d+1}^{\boldsymbol{k}[\langle ts\rangle]}$ are not triangular, thus $\underline{\mathrm{Braid}}_0(\boldsymbol{k}[D_{4,2}]) = \underline{\mathrm{Braid}}_0(\boldsymbol{k}[\langle s\rangle])$.

In the case of $n = 4$, then the finite group $D_{4,4}$ $(= D_8)$ has exactly three maximal commutative normal subgroups, which are given by $\langle s\rangle$, $\langle t, s^2\rangle$, $\langle ts, s^2\rangle$. As the same manner for $D_{4,2}$, one can determine the (quasi-)triangular structures of the group Hopf algebra $\boldsymbol{k}[D_{4,4}]$. $\square$

By Lemmas 3.2 and 3.10 we have

$$\begin{aligned} \sharp\underline{\mathrm{Braid}}_0(\boldsymbol{k}[D_8]) &= \sharp\underline{\mathrm{Braid}}_0(\boldsymbol{k}[D_{4,4}]) = 6, \\ \sharp\underline{\mathrm{Braid}}_0(\boldsymbol{k}[Q_8]) &= \sharp\underline{\mathrm{Braid}}_0(\boldsymbol{k}[D_{4,2}]) = 2, \\ \sharp\underline{\mathrm{Braid}}_0(K_8) &= \sharp\underline{\mathrm{Braid}}_0(\boldsymbol{k}[A_{1,2}^{+-}]) = 0. \end{aligned}$$

This shows that the Hopf algebras $\boldsymbol{k}[D_8], \boldsymbol{k}[Q_8], K_8$ are not mutually tensor Morita equivalent, that the result itself has already proved by Tambara and Yamagami [24], Masuoka [14], and Ng and Schauenburg [19].

In what follows, N is an odd integer, and $\omega \in \boldsymbol{k}$ is a primitive $4NL$-th root of unity. Let us determine the triangular structures of the group Hopf algebra of

$$G_{NL} = \langle h,\ t,\ w \mid t^2 = h^{2N} = 1,\ w^L = h^N,\ tw = w^{-1}t,\ ht = th,\ hw = wh\rangle.$$

The universal R-matrices of the group Hopf algebra $\boldsymbol{k}[G_{NL}]$ are as follows.

Proposition 3.11 ([25, Proposition 5.14]). *Let H be the commutative subgroup of G_{NL} defined by $H = \langle h,\ w \mid h^{2N} = 1,\ w^L = h^N,\ hw = wh\rangle$. Let $\{E_{ik}\}$ be the set of the primitive idempotents of $\boldsymbol{k}[H]$, where*

$$E_{ik} = \frac{1}{2LN} \sum_{j=0}^{L-1} \sum_{l=0}^{2N-1} \omega^{-2j(k+2i)N-2klL} w^j h^l.$$

Then for any $\nu \in \{\pm\},\ a \in \{0,1,\dots,L-1\},\ q \in \{0,1,\dots,N-1\}$,

$$R_{aq\nu} := \sum_{i,j=0}^{L-1} \sum_{k,l=0}^{2N-1} \nu^{kl} \omega^{2aN(2i+k)(2j+l)+2qklL}\, E_{ik} \otimes E_{jl} \tag{3.1}$$

is a universal R-matrix of the group Hopf algebra $\boldsymbol{k}[G_{NL}]$. Furthermore,

- *$R_{aq\nu}$ is a universal R-matrix of the group Hopf algebra $\boldsymbol{k}[\langle h\rangle]$ if and only if $a = 0$, and*
- *if $L \geq 3$, then any universal R-matrix is given by the above form, therefore, the number of the universal R-matrices of $\boldsymbol{k}[G_{NL}]$ is $2LN$.* $\square$

By using the following relations for all $i, j, k, l \in \mathbb{Z}$

- $E_{i+L,k} = E_{ik},\ E_{i-N,k+2N} = E_{ik}$,
- $E_{ik}E_{jl} = \delta_{kl}^{(2N)}\delta_{k+2i,l+2j}^{(2L)}E_{jl} = \delta_{kl}^{(2N)}\delta_{k+2i,l+2j}^{(2L)}E_{ik}$,

  where $\delta_{kl}^{(m)} = \begin{cases} 1 & (k \equiv l \pmod{m}) \\ 0 & (k \not\equiv l \pmod{m}) \end{cases}$ for $m = 2N$ or $m = 2L$.

one can list up the pairs $(\boldsymbol{k}[G_{NL}], R_{aq\nu})$ that are triangular.

Corollary 3.12. *For $a \in \{\ 0, 1, \dots, L-1\ \},\ q \in \{\ 0, 1, \dots, N-1\ \},\ \nu = \pm$, the quasitriangular Hopf algebra $(\boldsymbol{k}[G_{NL}], R_{aq\nu})$ is triangular if and any if $(a, q, \nu) = (0, 0, \pm)$.*

Proof. We set $R = R_{aq\nu}$. Since

$$R_{21}R = \sum_{i,j=0}^{L-1} \sum_{k,l=0}^{2N-1} \omega^{4a(2i+k)(2j+l)N+4qklL}\, E_{ik} \otimes E_{jl},$$

we have

$$\begin{aligned}
R_{21}R = 1\otimes 1 \iff & \omega^{4a(2i+k)(2j+l)N+4qklL} = 1 \\
& \qquad\qquad (0\le i,j\le L-1,\ 0\le k,l\le 2N-1) \\
\iff & a(2i+k)(2j+l)N + qklL \equiv 0 \pmod{NL} \\
& \qquad\qquad (0\le i,j\le L-1,\ 0\le k,l\le 2N-1) \\
\iff & \begin{cases} aN+qL\equiv 0 & \pmod{NL}, \\ 2aN\equiv 0 & \pmod{NL}. \end{cases}
\end{aligned}$$

Since $2aN \equiv 0 \pmod{NL}$ is equivalent to $2a \equiv 0 \pmod{L}$, if L is odd, then $a = 0$, and if L is even, then there are two possibilities for a, that is, $a = 0, L/2$.

In the case of $a = 0$, from the equation $aN + qL \equiv 0 \pmod{NL}$, we have $q \equiv 0 \pmod{N}$. This congruence equation has a unique solution in $\{0, 1, \dots, N-1\}$, namely, $q = 0$.

In the case when L is even, and $a = L/2$, from the equation $aN + qL \equiv 0 \pmod{NL}$ we have $NL/2 + qL \equiv 0 \pmod{NL}$. This implies that $NL/2 + qL = NLm$ for some $m \in \mathbb{Z}$. This is a contradiction with N to be even. Therefore, if $R = R_{aq\nu}$ satisfies $R_{21}R = 1 \otimes 1$, then $a = q = 0$ is required. □

Let us consider the case of $L = 2$. In this case, there are universal R-matrices other than $R_{aq\nu}$ ($a \in \{0, 1\}$, $q \in \{0, 1, \dots, N-1\}$, $\nu = \pm$).

Proposition 3.13 ([25, Proposition 5.15])**.** *The number of universal R-matrices of the group Hopf algebra $\boldsymbol{k}[G_{N2}]$ is $8N$, and they are given by the list below.*

- $R_d^{\boldsymbol{k}[\langle h\rangle]}$ $(d = 0, 1, \dots, 2N-1)$,
- $R_{1q\nu}$ $(q = 0, 1, \dots, N-1,\ \nu = \pm)$ *(the universal R-matrix given in Proposition 3.11),*
- $R_d^{(1)} := R_{0N1d}^{\boldsymbol{k}[\langle t,h\rangle]}$ $(d = 0, 1, \dots, 2N-1)$,
- $R_d^{(2)} := R_{0N1d}^{\boldsymbol{k}[\langle tw,h\rangle]}$ $(d = 0, 1, \dots, 2N-1)$.

Corollary 3.14. $\underline{\mathrm{Braid}}_0(\boldsymbol{k}[G_{N2}]) = \{R_0 = 1\otimes 1,\ R_N,\ R_0^{(1)},\ R_N^{(1)},\ R_0^{(2)},\ R_N^{(2)}\}$.

Proof. By Lemma 3.8, $\underline{\mathrm{Braid}}_0(\boldsymbol{k}[\langle h\rangle]) = \{R_0, R_N\}$. By Corollary 3.12, any quasi-triangular Hopf algebra $(\boldsymbol{k}[G_{N2}], R_{1q\nu})$ is not triangular. By Lemma 3.9 we have

$$\begin{aligned}
\underline{\mathrm{Braid}}_0(\boldsymbol{k}[C_2\times C_{2N}]) &= \left\{ R_{pqrs}^{\boldsymbol{k}[C_2\times C_{2N}]} \;\middle|\; \begin{array}{c} p,r\in\{0,1\},\ q,s\in\{0,1,\dots,2N-1\}, \\ 2Nr\equiv 2p\equiv 0 \pmod 2,\ 2q\equiv 2s\equiv 0 \pmod{2N}, \\ 2Nr+2q\equiv 0 \pmod{4N} \end{array} \right\} \\
&= \{R_{pqrs}^{\boldsymbol{k}[C_2\times C_{2N}]} \mid p,r\in\{0,1\},\ q,s\in\{0,N\}, Nr+q\equiv 0 \pmod{2N}\} \\
&= \{R_{p00s}^{\boldsymbol{k}[C_2\times C_{2N}]},\ R_{pN1s}^{\boldsymbol{k}[C_2\times C_{2N}]} \mid p\in\{0,1\},\ s\in\{0,N\}\}.
\end{aligned}$$

So, for $s \in \{0, 1, \ldots, 2N-1\}$ the quasitriangular Hopf algebra $(\boldsymbol{k}[G_{N2}], R_s^{(1)})$ is triangular if and only if $s = 0$ or $s = N$. By the same argument we see that the quasitriangular Hopf algebra $(\boldsymbol{k}[G_{N2}], R_s^{(2)})$ is triangular if and only if $s = 0$ or $s = N$. □

Concluding Remarks.

In [25] the author introduced a family of polynomial invariants of a semisimple and cosemisimple Hopf algebra of finite dimension. Let us recall the definition. Let A be a semisimple and cosemisimple Hopf algebra of finite dimension over a field $\boldsymbol{k}$, and R be a universal R-matrix of A. For a finite-dimensional left A-module M, we denote by $\underline{\dim}_R M$ the trace of the left action of the Drinfel'd element u on M. By Etingof and Gelaki [5] it is known that the set $\underline{\mathrm{Braid}}(A)$ is finite, and $(\dim M)1_{\boldsymbol{k}} \neq 0$ for any absolutely simple left A-module M. Thus, for each absolutely simple left A-module M a polynomial $P_{A,M}(x) \in \boldsymbol{k}[x]$ can be defined by

$$P_{A,M}(x) := \prod_{R \in \underline{\mathrm{Braid}}(A)} \Big(x - \frac{\underline{\dim}_R M}{\dim M} \Big).$$

Furthermore, for each positive integer d a polynomial $P_A^{(d)}(x) \in \boldsymbol{k}[x]$ is defined by

$$P_A^{(d)}(x) := \prod_{i=1}^{t} P_{A,M_i}(x),$$

where $\{M_1, \ldots, M_t\}$ is a full set of non-isomorphic absolutely simple left A-modules of dimension d. Here, if there is no absolutely simple left A-module of dimension d, then we set $P_A^{(d)}(x) = 1$. By definition the polynomial $P_A^{(d)}(x)$ is a tensor Morita invariant of A, that is, if the monoidal categories ${}_A\mathbb{M}$ and ${}_B\mathbb{M}$ are equivalent as $\boldsymbol{k}$-linear monoidal categories, then $P_A^{(d)}(x) = P_B^{(d)}(x)$. It is known that the polynomial invariants are useful for detecting the difference of module categories of Hopf algebras whose Grothendieck rings are isomorphic [25]. For example, for an odd integer N and an integer $L \geq 2$ the polynomial invariants of the Hopf algebras $\boldsymbol{k}[G_{NL}]$, A_{NL}^{+-}, A_{NL}^{++} (where L is assumed to be odd for A_{NL}^{++}) are given as follows, provided that $\boldsymbol{k}$ contains a primitive $4NL$-th root of unity $\omega \in \boldsymbol{k}$.

$$P_{\boldsymbol{k}[G_{NL}]}^{(1)}(x) = \begin{cases} \displaystyle\prod_{k,s=0}^{N-1} (x - \omega^{-16sk^2})^{32} & \text{if } L = 2, \\ \displaystyle\prod_{s,q=0}^{N-1} (x - \omega^{-8qs^2L})^{8L} & \text{if } L \geq 4 \text{ is even}, \\ \displaystyle\prod_{s,q=0}^{N-1} (x - \omega^{-8qs^2L})^{4L} (x^2 - \omega^{-4q(2s+1)^2L})^{2L} & \text{if } L \text{ is odd}. \end{cases}$$

$$P^{(2)}_{\boldsymbol{k}[G_{NL}]}(x) = \begin{cases} \prod_{k,s=0}^{N-1} (x^2 - \omega^{-8s(2k+1)^2})^2 (x^4 - \omega^{-16s(2k+1)^2}) & \text{if } L = 2, \\ \prod_{s,q=0}^{N-1} \prod_{t=1}^{\frac{L-\epsilon(L)}{2}} \prod_{a=0}^{L-1} (x^2 - \omega^{-4(q(2s+1)^2L + a(2t-1)^2N)}) & \\ \qquad \times \prod_{s,q=0}^{N-1} \prod_{t=1}^{\frac{L-2+\epsilon(L)}{2}} \prod_{a=0}^{L-1} (x - \omega^{-8(qs^2L + at^2N)})^2 & \text{if } L \geq 3. \end{cases}$$

$$P^{(1)}_{A^{+-}_{NL}}(x) = \begin{cases} \prod_{s,i=0}^{N-1} (x^2 - \omega^{-32is^2})^{16} & \text{if } L = 2, \\ \prod_{s,i=0}^{N-1} (x - \omega^{-8is^2L})^{4L} (x - \omega^{-8is^2L}(-1)^{\frac{L}{2}})^{4L} & \text{if } L \geq 4 \text{ is even}, \\ \prod_{s,i=0}^{N-1} (x - \omega^{-8is^2L})^{4L} (x^2 + \omega^{-4i(2s+1)^2L})^{2L} & \text{if } L \text{ is odd}. \end{cases}$$

$$P^{(2)}_{A^{+-}_{NL}}(x) = \begin{cases} \prod_{s=0}^{N-1} \prod_{i=0}^{N-1} (x^4 + \omega^{-16i(2s+1)^2})(x^2 - \omega^{-8i(2s+1)^2})^2 & \text{if } L = 2, \\ \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{L-\epsilon(L)}{2}} \prod_{j=0}^{L-1} (x^2 - \omega^{-4i(2s+1)^2L + 2(2j+1)(2t-1)^2N}) & \\ \qquad \cdot \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{L-2+\epsilon(L)}{2}} \prod_{j=0}^{L-1} (x - \omega^{-8is^2L + 4(2j+1)t^2N})^2 & \text{if } L \geq 3. \end{cases}$$

$$P^{(1)}_{A^{++}_{NL}}(x) = \prod_{s,i=0}^{N-1} (x - \omega^{-8is^2L})^{4L} (x^2 - \omega^{-4i(2s+1)^2L})^{2L} \qquad \text{if } L \text{ is odd},$$

$$P^{(2)}_{A^{++}_{NL}}(x) = \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{L-1}{2}} \prod_{j=0}^{L-1} (x^2 - \omega^{-4i(2s+1)^2L - 4j(2t-1)^2N})$$
$$\cdot \prod_{s,i=0}^{N-1} \prod_{t=1}^{\frac{L-1}{2}} \prod_{j=0}^{L-1} (x - \omega^{-8is^2L + 8jt^2N})^2 \qquad \text{if } L \text{ is odd}.$$

Here,

$$\epsilon(L) = \begin{cases} 0 & L \text{ is even}, \\ 1 & L \text{ is odd}. \end{cases}$$

The author confesses to missing up on the computational result of $P^{(1)}_{A^{+-}_{N2}}(x)$ in [25, Proposition 5.10], though $P^{(1)}_{A_{N2}}(x)$ for $N = 1, 3, 5$ in the table of [25, Example 5.19] are correct. This mistake came from $P_{A^*, \boldsymbol{k}g}(x)$ in the case of $n = 2$. The correct result of the polynomial is $P_{A^*, \boldsymbol{k}g}(x) = \prod_{i=0}^{N=1} (x - \omega^{-8(s+1)^2(2i + \frac{1-\nu}{2})}(-1)^{\frac{1-\lambda}{2}})^4 \cdot \prod_{i=0}^{N-1} (x - \omega^{-16i(s+1)^2}\lambda)^4$.

Now, we will go back to the main subject. As an analogy of $P_A^{(d)}(x)$ one can define another polynomial invariants $\check{P}_A^{(d)}(x)$ by replacing $\underline{\mathrm{Braid}}(A)$ by $\underline{\mathrm{Braid}}_0(A)$, namely,

$$\check{P}_A^{(d)}(x) = \prod_{i=1}^{t} \prod_{R\in\underline{\mathrm{Braid}}_0(A)} \Big(x - \frac{\dim_R M_i}{\dim M_i}\Big).$$

The polynomial invariants $\check{P}_A^{(d)}(x)$ are also tensor invariants of the module category ${}_A\mathbb{M}$, and have similar properties to that of $P_A^{(d)}(x)$, including integer property. The polynomial invariants $\check{P}_A^{(d)}(x)$ are very simple when compared with $P_A^{(d)}(x)$. For example, for an odd integer N we have:

$$\check{P}^{(1)}_{\boldsymbol{k}[G_{NL}]}(x) = \begin{cases} (x-1)^{24N} & \text{if } L=2,\\ (x-1)^{8N} & \text{if } L\geq 3 \text{ is even},\\ (x-1)^{6N}(x+1)^{2N} & \text{if } L\geq 3 \text{ is odd}.\end{cases}$$

$$\check{P}^{(2)}_{\boldsymbol{k}[G_{NL}]}(x) = \begin{cases} (x-1)^{3N}(x+1)^{3N} & \text{if } L=2,\\ (x-1)^{N(\frac{3L+\epsilon(L)}{2}-2)}(x+1)^{\frac{N(L-\epsilon(L))}{2}} & \text{if } L\geq 3.\end{cases}$$

$$\check{P}^{(1)}_{A^{+-}_{NL}}(x) = \check{P}^{(2)}_{A^{+-}_{NL}}(x) = 1,$$

$$\check{P}^{(1)}_{A^{++}_{NL}}(x) = \check{P}^{(1)}_{\boldsymbol{k}[G_{NL}]}(x) = (x-1)^{6N}(x+1)^{2N}, \qquad \text{if } L\geq 3 \text{ is odd},$$

$$\check{P}^{(2)}_{A^{++}_{NL}}(x) = \check{P}^{(2)}_{\boldsymbol{k}[G_{NL}]}(x) = (x-1)^{\frac{3N(L-1)}{2}}(x+1)^{\frac{N(L-1)}{2}} \quad \text{if } L\geq 3 \text{ is odd}.$$

So far, we have no example of a pair of Hopf algebras A and B such that the number of the triangular structures of them are the same, and the Grethendieck rings are isomorphic, but $\check{P}_A^{(d)}(x)$ and $\check{P}_B^{(d)}(x)$ (or $P_A^{(d)}(x)$ and $P_B^{(d)}(x)$) are distinct.

Acknowledgements

The author would like to thank Professors Akira Masuoka, Siu-Hung Ng and Kenichi Shimizu for helpful advice and discussions. The author is also most grateful to the organizers of the colloquium and the students of Professor Nicolás Andruskiewitsch for all of their warm hospitality and for all of their kind assistance during his stay in La Falda and Córdoba city, Argentina. Thanks are also due to the referee for his or her thoughtful comments helped to improve the paper.

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Recibido: 30 de marzo de 2010
Aceptado: 20 de mayo de 2010