ON SEMISIMPLE HOPF ALGEBRAS WITH FEW REPRESENTATIONS OF DIMENSION GREATER THAN ONE

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ABSTRACT. In the paper we consider semisimple Hopf algebras $H$ with the following property: irreducible $H$-modules of the same dimension $> 1$ are isomorphic. Suppose that there exists an irreducible $H$-module $M$ of dimension $> 1$ such that its endomorphism ring is a Hopf ideal in $H$. Then $M$ is the unique irreducible $H$-module of dimension $> 1$.

INTRODUCTION

Let $H$ be a semisimple Hopf algebra over an algebraically closed field $k$. It is assumed that either char $k = 0$ or char $k > \text{dim } H$. Semisimplicity of $H$ means that $H$ is a semisimple left $H$-module. In that case $H$ has finite dimension.

Throughout the paper we shall keep to notations from [M]. For example $H^*$ stands for the dual Hopf algebra with the natural pairing $\langle -, - \rangle : H^* \otimes H \to k$. The algebra $H$ is a left and right $H^*$-module algebra with respect to the left and right actions $f \mapsto x$, $x \mapsto f$ of $f \in H^*$ on $x \in H$, defined as follows, [M, Example 4.1.10]: if

$$\Delta(x) = \sum_x x_1 \otimes x_2$$

then

$$f \mapsto x = \sum_x x_1 \langle f, x_2 \rangle, \quad x \mapsto f = \sum_x \langle f, x_1 \rangle x_2.$$ 

Denote by $G = G(H^*)$ the group of group-like elements in $H^*$. Elements of $G$ are algebra homomorphisms $H \to k$. Hence $H$ as a $k$-algebra has a semisimple direct decomposition

$$H = \left( \oplus_{g \in G} k e_g \right) \oplus \left( \oplus_{j=1}^n \text{Mat}(d_j, k) \right),$$

where $\{e_g, g \in G\}$ is a system of central orthogonal idempotents in $H$ corresponding to one-dimensional direct summands. If $h \in H$ and $g \in G$ then

$$he_g = e_g h = \langle g, h \rangle e_g.$$ 

The counit $\varepsilon : H \to k$ has the form

$$\varepsilon(x) = \begin{cases} \delta_{g,1}, & x = e_g, \\ 0, & x \in \text{Mat}(d_i, k). \end{cases}$$

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Each one-dimensional $H$-module $E_g = ku_g$ related to $g \in G$ has a base $u_g$ and

$$hu_g = \langle g, h \rangle u_g, \quad h \in H.$$  \hspace{1cm} (4)

In this paper we consider the case when

$$1 < d_1 < d_2 < \cdots < d_n.$$  \hspace{1cm} (5)

It just means that irreducible $H$-modules of the same dimension $> 1$ are isomorphic. The main result of the paper is

**Theorem 0.1.** Let $H$ be a semisimple Hopf algebra with decomposition (2), $n \geq 1$, such that (5) holds. Suppose that at least one single matrix constituent is a Hopf ideal in $H$. Then $n = 1$.

Throughout the paper we shall use the following notations. By $E^{(i)}$ we shall denote the identity matrix from $\text{Mat}(d_i, k)$ and by $E^{(i)}_{rs} \in \text{Mat}(d_i, k)$ we shall denote corresponding matrix unit.

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Note that the similar classes of Hopf algebras were considered from another point of view in [N2, §3.4] and in [Ma], [N1], [N2], [T], [TY].

1. The category of modules

Let $H$ be a semisimple Hopf algebra with direct sum decomposition (2) such that (5) is satisfied. The category $H\mathcal{M}$ of left $H$-modules is a monoidal category with respect to tensor products of modules. Since $H$ is semisimple the abelian category is completely defined up to an isomorphism by its Grothendieck group $K_0(H) = K_0(H\mathcal{M})$. Recall that the Abelian $K$-group $K_0(H)$ is a free additive Abelian group with a base consisting of ismorphism classes of irreducible left $H$-modules $E_g$, $g \in G$, and modules $M_i$, $i = 1, \ldots, n$, of dimension $d_i > 1$ corresponding to matrix constituents in (2). As we have already mentioned one-dimensional $H$-modules $E_g$ correspond to one-dimensional simple direct summands of $H$ such that (4) is satisfied and elements $g \in G$.

If $M, N$ are left $H$-modules then $M \otimes N$ is also a left $H$-module under the action

$$h(x \otimes y) = \sum_h h^{(1)}x \otimes h^{(2)}y, \quad h \in H, \quad x \in M, \quad y \in N.$$  \hspace{1cm} (6)

Tensor multiplication induces a structure of an associative ring on $K_0(H)$.

The next fact is well known.

**Proposition 1.1.** Let $M$ be an irreducible $H$-module and $E_g$ a one-dimensional $H$-module. Then $M \otimes E_g$ and $E_g \otimes M$ are irreducible $H$-modules. In particular if $f, g \in G$ then $E_f \otimes E_g \simeq E_{fg}$. Under assumption (5) we have $M \otimes E_g \simeq E_g \otimes M$. \hfill \Box

The module $M \otimes E_g$ is identified with $M$ in which elements $h \in H$ act as $g \rightarrow h$. An isomorphism $M \simeq M \otimes E_g$ means that there exists an invertible linear operator $\Phi_M(g)$ in $M$ such that $\Phi_M(g)hx = (g \rightarrow h)\Phi_M(g)x$ for all $h \in H$ and for all
x ∈ M. Note that we can identify the endomorphism algebra End M with an ideal in H. Namely, if M ∼= M_i then End M ∼= Mat(d_i, k). If g ∈ G then the action g −→ is an endomorphism of H. Because of the assumption (5) the ideal End M is stable under the action g −→. Thus g −→ h = Φ M(g)hΦ M(g)^{-1} in H. Note that we can always assume that Φ M(1) = 1 and Φ M(g^{-1}) = Φ M(g)^{-1} for all g ∈ G.

We denote by S the antipode of H.

**Proposition 1.2.** Let M be an irreducible H-module of dimension > 1. The map Φ M : G → PGL(M) is a projective representation of the group G if M is an irreducible H-module. Moreover [Φ M(g), S(Φ M(v))] = 1 in PGL(M) for all g, v ∈ G.

**Proof.** Since g −→ (v −→ h) = gv −→ h, conjugations by Φ M(g)Φ M(v) and by Φ M(gv) coincide in GL(M). Hence Φ M(gv)^{-1}Φ M(g)Φ M(v) is a scalar matrix and therefore Φ M is a projective representation of G in M. By the assumption (5) the algebra End M is an ideal which is stable not only under the action g −→ where g ∈ G but also under the action −→ g and under the antipode S since S is an involution of the algebra H.

Similarly E_g ⊗ M can be identified with the vector space M in which h ∈ H acts as

\[ h −→ g = S(S(g) −→ S(h)) = S(Φ M(g^{-1}) S(h)Φ M(g)) = S(Φ M(g)) S(Φ M(g^{-1})). \]

In order to prove the last statement it is necessary to recall that the actions −→, −→ commute.

**Proposition 1.3.** Let M, N be irreducible H-modules and g ∈ G. Then

\[ \text{Hom}_H(M ⊗ N, E_g) ≃ \text{Hom}_H(M ⊗ N, E_ε). \]

**Proof.** Let φ ∈ Hom(M ⊗ N, E_g). By Proposition 1.1 there is an isomorphism of H-modules ω : M ⊗ N → M ⊗ N ⊗ E_{g^{-1}}. Hence φ induces by Proposition 1.1 a homomorphism of H-modules

\[ M ⊗ N \xrightarrow{ω^{-1}} M ⊗ N ⊗ E_{g^{-1}} \xrightarrow{φ ⊗ 1} E_g ⊗ E_{g^{-1}} \xrightarrow{1} E_ε. \]

Denote by ŵ this composition. It is easy to see that the correspondence φ → ŵ is an isomorphism between Hom_H(M ⊗ N, E_g) and Hom_H(M ⊗ N, E_ε).

**Corollary 1.4.** If M, N are irreducible H-modules of dimensions > 1, then the multiplicity of each one-dimensional H-module E_g, g ∈ G, in M ⊗ N does not depend upon g.

If M is an irreducible left H-module then H* = Hom_k(M, k) is again an irreducible left H-module. So we have

**Proposition 1.5.** Let M_i, be the unique irreducible left H-module of dimension d_i > 1. Then M_i* ≃ M_i. In particular there exists a non-degenerate bilinear function \( (x, y) \) on M_i such that \( ⟨hx, y⟩ = ⟨x, S(h)y⟩ \) for all \( x, y ∈ M_i \) and for any \( h ∈ H \).
If $e_* = (e_1, \ldots, e_d)$ is a base of $M_i$ as a vector space then we get an identification of End $M_i$ with the full matrix algebra $\text{Mat}(d_i, k)$ which is an ideal in $H$. Denote by $U_i$ the square matrix of size $d_i$ whose $(\alpha, \beta)$th entry $u_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle$. It means that $U_i$ is the Gram matrix of the bilinear function $\langle x, y \rangle$.

**Theorem 1.6.** The bilinear function $\langle x, y \rangle$ is (skew-)symmetric. If $x \in \text{Mat}(d_i, k)$ from decomposition (2), then $S(x) = U_i^t x U_i^{-1}$. Using a canonical form of a Gram matrix of a (skew-)symmetric bilinear function we can always choose a base in $M_i$ such that $U_i$ is either an identity matrix in the symmetric case or the matrix $U_i = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, in skew-symmetric (hyperbolic) case where $E$ is the identity matrix of size $d_i^2$. In both cases $U_i^{-1} = \pm U_i$.

**Proof.** Without loss of generality we can assume that $h, S(h) \in \text{Mat}(d_i, k)$ have entries $h_{\zeta\beta}, r_{\gamma\lambda}$, respectively. Then

$$\langle he_\alpha, e_\beta \rangle = \sum_\zeta \langle e_\zeta, e_\beta \rangle h_{\alpha\zeta} = \sum_\zeta u_{\zeta\beta} h_{\alpha\zeta};$$

$$\langle e_\alpha, S(h)e_\beta \rangle = \sum_\gamma r_{\beta\gamma} \langle e_\alpha, e_\gamma \rangle = \sum_\gamma r_{\beta\gamma} u_{\alpha\gamma}.$$ 

The equality $\langle hx, y \rangle = \langle x, S(h)y \rangle$ means that $^tU_i \cdot ^t h = S(h)^t U_i$.

Finally since the antipode $S$ has order 2, the matrix $U_i$ and the form $\langle x, y \rangle$ are (skew)-symmetric [LR]. So we can replace $^tU_i$ by $U_i$. □

We now study some properties of irreducible $H$-modules. The first one is a well-known, see [DNR, Lemma 7.5.10, p. 322]. Recall that $M_i$ is an irreducible $H$-module of dimension $d_i$, $1 \leq i \leq n$.

**Lemma 1.7.** If $1 \leq i, j \leq n$, then $\text{dim} \text{Hom}_H(M_i \otimes M_j, E_\varepsilon) = \delta_{ij}$. □

Set $A = \bigoplus_{g \in G} E_g$.

**Proposition 1.8.** There is a direct sum decomposition

$$M_i \otimes M_j = \delta_{ij} A \oplus \bigoplus_{t=1}^n m_{ij}^t M_t,$$

$$m_{ij}^t = \text{dim}_k \text{Hom}_H(M_i \otimes M_j, M_t) \geq 0. \quad (7)$$

Hence

$$d_id_j = \delta_{ij} |G| + \sum_{t=1}^n m_{ij}^t d_t. \quad (8)$$

In particular $|G| \leq d_1^2$.

**Proof.** Apply Lemma 1.7, Corollary 1.4 and compare dimensions of modules in (7). This gives (7), (8). We prove the last claim. Each coefficient $m_{pq}^i$ estimated by (8) as

$$0 \leq m_{pq}^i \leq \frac{1}{d_i} d_pd_q - \delta_{pq} |G|.$$
In particular if \( p = q \) then \( 0 \leq d_p^2 - |G| \). □

Proposition 1.5 implies that \( A \otimes M_s = M_s \otimes A = |G| M_s \). Using the associativity conditions in the Grothendieck ring we obtain

**Theorem 1.9.** The multiplicities \( m_{ij}^l \) satisfy the equation (8) and the equations

\[
m_{ij}^s = m_{js}^i, \quad \delta_{ij} \delta_{ls} |G| + \sum_{t=1}^{n} m_{ij}^l m_{ts}^l = \delta_{js} \delta_{li} |G| + \sum_{t=1}^{n} m_{js}^l m_{it}^l
\]

for all \( i, j, s, l = 1, \ldots, n \). In particular \( m_{ij}^s = m_{js}^i = m_{si}^j \) and

\[
\delta_{ij} \delta_{ls} |G| + \sum_{t=1}^{n} m_{ti}^s m_{ts}^l = \delta_{js} \delta_{li} |G| + \sum_{t=1}^{n} m_{si}^j m_{it}^l.
\]

□

**Corollary 1.10.** If \( i, j, p = 1, \ldots, n \), then \( m_{ij}^p \leq d_{\min(i,j,p)} \).

**Proof.** Since \( m_{ij}^p = m_{jp}^i = m_{pj}^i \) we can assume that \( p \geq i, j \). Then by (8) we get

\[
m_{ij}^p \leq \frac{d_i d_j}{d_p} \leq \min(d_i, d_j) = d_{\min(i,j)} = d_{\min(i,j,p)}.
\]

□

2. **Special matrices**

Consider special elements

\[
\mathcal{T}_i = \sum_{\alpha, \beta = 1}^{d_i} E_{\alpha \beta}^{(i)} \otimes E_{\alpha \beta}^{(i)}, \quad \mathcal{R}_i = \frac{1}{d_i} \sum_{\alpha, \beta = 1}^{d_i} E_{\alpha \beta}^{(i)} \otimes E_{\beta \alpha}^{(i)},
\]

\[
\mathcal{D}_i = (1 \otimes S) \mathcal{R}_i
\]

in \( \text{Mat}(d_i, k)^{\otimes 2} \). These elements will be used in the rest of the paper.

Direct calculations prove the first equality in

**Proposition 2.1.** Let \( F, H \in \text{Mat}(d_i, k) \). Then

\[
(F \otimes E^{(i)}) \mathcal{T}_i (E^{(i)} \otimes H) = (E^{(i)} \otimes ^t F) \mathcal{T}_i (E^{(i)} \otimes ^t H),
\]

\[
(F \otimes H) \mathcal{R}_i = \mathcal{R}_i (H \otimes F).
\]

Here \( E^{(i)} \) stands for the identity matrix of the size \( d_i \). Moreover

\[
(1 \otimes S) \mathcal{T}_i = (S \otimes 1) \mathcal{T}_i, \quad \mathcal{D}_i = (1 \otimes S) \mathcal{R}_i = (S \otimes 1) \mathcal{R}_i.
\]

In particular for \( F, H \) as above

\[
(F \otimes 1) \cdot \mathcal{D}_t \cdot (1 \otimes S(H)) = (1 \otimes S(F)) \cdot \mathcal{D}_t \cdot (H \otimes 1).
\]
Proof. The second equality was proved in [ACh, Proposition 1.1].

In order to prove the next two equalities we shall use the first equality, (skew-)symmetry of $U_i$ and the property $U_i^{-1} = \pm U_i$. Hence

\[
(1 \otimes S)\mathcal{T}_i = d_i \left( E^{(i)} \otimes U_i \right) \mathcal{R}_i \left( E^{(i)} \otimes U_i^{-1} \right) \\
= d_i \left( U_i \otimes E^{(i)} \right) \mathcal{R}_i \left( U_i^{-1} \otimes E^{(i)} \right) = (S \otimes 1) \mathcal{T}_i;
\]

\[
(1 \otimes S)\mathcal{R}_i = \frac{1}{d_i} \left( E^{(i)} \otimes U_i \right) \mathcal{T}_i \left( E^{(i)} \otimes U_i^{-1} \right) \\
= \frac{1}{d_i} \left( U_i \otimes E^{(i)} \right) \mathcal{T}_i \left( U_i^{-1} \otimes E^{(i)} \right) = (S \otimes 1)\mathcal{R}_i = \mathcal{D}_i.
\]

In order to prove the last equality apply $(1 \otimes S)$ to (9) and use previous equalities.

\[\square\]

3. From module category to Hopf algebra

In this section we shall apply previous results to a presentation of an explicit form of comultiplication and antipode in $H$. Let $H$ have the direct sum decomposition (2) such that (5) holds.

Denote by $\pi_g$, $g \in G$, and by $\pi_i$, $1 \leq i \leq n$, the projections of $H$ onto $E_g$ and onto Mat($d_i, k$), respectively. Then the isomorphism $E_f \otimes E_g \simeq E_{fg}$ in Proposition 1.1 means that

\[
(\pi_f \otimes \pi_g)\Delta(h) = (fg, h)(e_f \otimes e_g);
\]

\[
(\pi_g \otimes \pi_i)\Delta(h) = e_g \otimes (\pi_i(h) \leftarrow g),
\]

\[
(\pi_i \otimes \pi_g)\Delta(h) = (g \rightarrow \pi_i(h)) \otimes e_g.
\]

Applying Proposition 1.8 we obtain

\[
(\pi_i \otimes \pi_j)\Delta(\pi_g(h)) = 0, \quad 1 \leq i \neq j \leq n, \quad g \in G.
\]

Using (10), (11) we see that

\[
\Delta(e_g) = \sum_{f \in G} e_f \otimes e_{f^{-1}g} + \sum_{i=1, \ldots, n} \mathcal{D}_{g,i},
\]

where $\{\mathcal{D}_{g,i}, \; g \in G, \; 1 \leq i \leq n\}$, is a system of orthogonal idempotents in Mat($d_i, k \otimes^2$. Moreover

\[
\mathcal{D}_{g,i} \in \text{Mat}(d_i, k) \otimes \text{Mat}(d_i, k) \simeq \text{Mat}(d_i^2, k) \simeq \text{End}_k(M_i \otimes M_i).
\]

Lemma 3.1. Let $g \in G$, and $\mathcal{D}_i$ from (9), $i = 1, \ldots, n$. Then the $H$-module $\mathcal{D}_{g,i}(M_i \otimes M_i)$ is isomorphic to $E_g$ and

\[
\mathcal{D}_{g,i} = (1 \otimes (g^{-1} \leftarrow)) \mathcal{D}_i.
\]

Proof. Let $h \in H$ and $x, y \in M_i$. Applying (3), (4), (6) and (12) we obtain

\[
h\mathcal{D}_{g,i}(x \otimes y) = \Delta(h)\mathcal{D}_{g,i}(x \otimes y) \\
= \Delta(h)\Delta(e_g)(x \otimes y) = \Delta(he_g)(x \otimes y) = (g, h)\mathcal{D}_{g,i}(x \otimes y).
\]

Thus (4) implies that $\mathcal{D}_{g,i}(M_i \otimes M_i) \simeq E_g$. 

By (7) and previous considerations it suffices to show that the element \((1 \otimes (g^{-1} \rightarrow)) D_i\) is an idempotent and
\[
\Delta(h) (1 \otimes (g^{-1} \rightarrow)) D_i = \langle g, h \rangle (1 \otimes (g^{-1} \rightarrow)) D_i.
\]

Direct calculations show that
\[
\begin{align*}
D_i^2 &= \frac{1}{d_i^2} \sum_{\alpha, \beta, \gamma, \xi} E^{(i)}_{\alpha \beta} E^{(i)}_{\gamma \xi} \otimes S \left( E^{(i)}_{\beta \alpha} \right) S \left( E^{(i)}_{\xi \gamma} \right) \\
&= \frac{1}{d_i^2} \sum_{\alpha, \beta, \gamma, \xi} E^{(i)}_{\alpha \beta} E^{(i)}_{\gamma \xi} \otimes \left( E^{(i)}_{\xi \gamma} \right) \left( E^{(i)}_{\beta \alpha} \right) \\
&= \frac{1}{d_i^2} \sum_{\alpha, \beta, \gamma, \xi} E^{(i)}_{\alpha \beta} \otimes S \left( E^{(i)}_{\xi \alpha} \right) D_i.
\end{align*}
\]

Since \(g \rightarrow\) is an algebra automorphism of \(H\) by Proposition 2.1 we have
\[
\begin{align*}
\Delta(h) (1 \otimes (g^{-1} \rightarrow) S) R_i &= \sum_h (h_{(1)} \otimes h_{(2)} (g^{-1} \rightarrow) S) R_i \\
&= \sum_h \left( (h_{(1)} \otimes (g^{-1} \rightarrow) (g \rightarrow h_{(2)}) S) R_i \right) \\
&= \sum_h \left( (h_{(1)} \otimes (g^{-1} \rightarrow) S) \left( R_i S (g \rightarrow h_{(2)}) \right) \right) \\
&= (1 \otimes (g^{-1} \rightarrow) S) \left[ \sum_h \left( h_{(1)} \otimes 1 \right) R_i \left( 1 \otimes (S (g \rightarrow h_{(2)})) \right) \right] \\
&= (1 \otimes (g^{-1} \rightarrow) S) \left[ R_i \left( 1 \otimes \sum_h \left( h_{(1)} S (g \rightarrow h_{(2)}) \right) \right) \right]
\end{align*}
\]

Finally
\[
\sum_h h_{(1)} S (g \rightarrow h_{(2)}) = \sum_h h_{(1)} S (h_{(2)} \langle g, h_{(3)} \rangle) \\
= \sum_h h_{(1)} S (h_{(2)}) \langle g, h_{(3)} \rangle = \sum_h \varepsilon (h_{(1)}) \langle g, h_{(2)} \rangle \\
= \langle g, \sum_h \varepsilon (h_{(1)}) h_{(2)} \rangle = \langle g, h \rangle
\]

Applying (10) for any \(x \in \text{Mat}(d_r, k)\) we get
\[
\Delta(x) = \sum_{g \in G} [(g \rightarrow x) \otimes e_g + e_g \otimes (x \leftarrow g)] + \sum_{i,j=1}^n \Delta_{ij}^r(x),
\]  
where \(\Delta_{ij}^r(x) \in \text{Mat}(d_i, k) \otimes \text{Mat}(d_j, k)\).

Note that \(\Delta_{ij}^r(x) = (\pi_i \otimes \pi_j) \Delta(x)\). It means that
\[
\Delta_{ij}^r : \text{Mat}(d_r, k) \rightarrow \text{Mat}(d_i, k) \otimes \text{Mat}(d_j, k)
\]
is an algebra homomorphism not necessarily preserving the unit element.

**Lemma 3.2.** Let \( x \in \text{Mat}(d_i, k) \), \( g \in G \) and \( \tau \) is the twist \( \tau(a \otimes b) = b \otimes a \) for all \( a \in \text{Mat}(d_q, k) \), \( b \in \text{Mat}(d_p, k) \). Then

\[
\Delta^i_{pq}(x) = (S \otimes S) \left( \Delta^i_{qp} \right)^T (S(x)),
\]

\[
S(e_g) = e_{g^{-1}}, \quad \mu(1 \otimes S) \Delta^i_{pq}(x) = \mu(S \otimes 1) \Delta^i_{pq}(x) = 0,
\]

\[
\mu(S \otimes 1) D_{g,i} = \mu(1 \otimes S) D_{g,i} = \delta_{g,1} E^{(i)}.
\]

Here \( \mu : H \otimes H \to H \) is the multiplication map.

**Proof.** The antipode \( S \) is an algebra and coalgebra involution. Hence each ideal \( \text{Mat}(d_i, k) \) in \( H \) is \( S \)-stable and hence (14) holds. The equalities (15) were proved in [A]. \( \square \)

Note that \( D_{g,i} \) and \( \Delta^i_{ij}(E_t) \) are in fact projections of \( M_i \otimes M_j \) onto \( E_g \) by Lemma 3.1 and onto \( m^i_{ij}, M_t \), respectively. Hence \( D_{g,i} \) and \( \Delta^i_{ij}(E_t) \) are matrices in \( \text{Mat}(d_i d_j, k) \) whose ranks are equal to \( \dim E_g = 1 \) and to \( m^i_{ij} \dim M_t \), respectively. In particular, (14) implies \( m^i_{pq} = m^i_{qp} \) for all \( i, p, q \).

Since \( \Delta \) is an algebra homomorphism for all \( i, j, p, q, r, s = 1, \ldots, m \) and \( x \in \text{Mat}(d_r, x) \), \( y \in \text{Mat}(d_s, k) \) we have

\[
\begin{align*}
D_{g,i} \cdot D_{h,i} &= \delta_{gh} \delta_{ij} D_{g,i}, \\
D_{g,i} \cdot \Delta^r_{pq}(x) &= \Delta^r_{pq}(x) \cdot D_{g,i} = 0, \\
\Delta^r_{ij}(x) \cdot \Delta^s_{pq}(y) &= \delta_{ip} \delta_{jq} \delta_{rs} \Delta^r_{ij}(xy).
\end{align*}
\]

### 4. Coassociativity conditions

In this section we shall consider coassociativity conditions of comultiplication \( \Delta \) from (12), (13). Multiplying the equalities from [A, §2, Proposition 2.1] by \( \Delta^i_{ij}(E^{(r)}) \otimes 1 \) from the left, by \( E^{(j)} \otimes \Delta^r_{ij}(E^{(r)}) \) from the right and the last equation by \( \Delta^r_{jp}(E^{(i)}) \otimes E^{(ij)} \) from the left and by \( E^{(ij)} \otimes \Delta^r_{pq}(E^{(r)}) \) from the right, respectively, we obtain using (16) that for any \( i, j, p, q = 1, \ldots, n \) and all \( x \in \text{Mat}(d_i, k) \) the following five groups of identities are satisfied.

The first group:

\[
\begin{align*}
[1 \otimes (g \twoheadrightarrow)] D_{t,i} &= D_{tg^{-1},i}; \\
[(\leftarrow g) \otimes 1] D_{t,i} &= D_{g^{-1},t,i}; \\
[1 \otimes (\leftarrow g)] D_{t,i} &= [(g \twoheadrightarrow) \otimes 1] D_{t,i}.
\end{align*}
\]

The second group:

\[
\begin{align*}
[(\leftarrow g) \otimes 1] \Delta^i_{pq}(x) &= \Delta^i_{pq}(x \leftarrow g); \\
[1 \otimes (g \twoheadrightarrow)] \Delta^i_{pq}(x) &= \Delta^i_{pq}(g \twoheadrightarrow x); \\
[1 \otimes (\leftarrow g)] \Delta^i_{pq}(x) &= [(g \twoheadrightarrow) \otimes 1] \Delta^i_{pq}(x).
\end{align*}
\]
The third group:
\[
(1 \otimes \Delta^i_{p_j}) D_i = \left( \Delta^j_{ip} \otimes 1 \right) D_j;
\]
\[
(D_i \otimes E^{(i)}) [x \otimes D_i] = [D_i \otimes x] \left( E^{(i)} \otimes D_i \right).
\] (19)

The fourth group:
\[
\left[ \Delta^i_{r_j} (E^{(r)}) \otimes E^{(j)} \right] \left[ (1 \otimes \Delta^i_{r_j} (x)) (E^{(i)} \otimes D_j) \right];
\]
\[
(D_j \otimes E^{(i)}) \left[ (1 \otimes \Delta^i_{j_i}) \Delta^i_{j_r} (x) \right] = [D_j \otimes x] \left[ E^{(j)} \otimes \Delta^i_{j_i} (E^{(r)}) \right].
\] (20)

The last fifth group:
\[
\left[ \Delta^i_{j_p} (E^{(i)} \otimes E^{(q)}) \left[ (1 \otimes \Delta^i_{p_q}) \Delta^i_{j_q} (x) \right]
\right.
\]
\[
\left. = \left[ (\Delta^i_{j_p} \otimes 1) \Delta^i_{q_j} (x) \right] \left[ E^{(j)} \otimes \Delta^i_{p_q} (E^{(r)}) \right].\right)
\] (21)

Also the equations (16) and (15) are satisfied.

We shall apply the identities (17) — (21) to the study of the comultiplication and the antipode of \( H \).

**Proposition 4.1.** Let
\[
\Delta^i_{r_j} (x) = \sum_{\tau} A_{\tau} (x) \otimes A'_{\tau} (x), \quad \Delta^i_{j_r} (x) = \sum_{\tau} B'_{\tau} (x) \otimes B_{\tau} (x),
\] (22)

where \( A_{\tau}, B_{\tau} \in \text{Mat}(d_{\tau}, k) \) and \( A'_{\tau}, B'_{\tau} \in \text{Mat}(d_j, k) \). Then the equalities (20) are equivalent to

\[
\Delta^i_{r_j} (E^{(r)}) \left( x \otimes E^{(j)} \right) = \sum_{\tau} \Delta^i_{r_j} (A_{\tau} (x)) \left[ E^{(i)} \otimes S (A'_{\tau} (x)) \right];
\]
\[
\left( E^{(j)} \otimes x \right) \Delta^i_{r_j} (E^{(r)}) = \sum_{\tau} \left[ S (B'_{\tau} (x)) \otimes E^{(i)} \right] \Delta^i_{j_r} (B_{\tau} (x))
\]

for all \( x \in \text{Mat}(d_i, k) \).

**Proof.** In the notations (22) using Proposition 2.1 we obtain in the right hand part of the first equation in (20)
\[
\left[ (\Delta^i_{r_j} \otimes 1) \Delta^i_{r_j} (x) \right] \left( E^{(i)} \otimes D_j \right)
\]
\[
= \left[ \sum_{\tau} \Delta^i_{r_j} (A_{\tau} (x)) \left( E^{(i)} \otimes S (A'_{\tau} (x)) \right) \otimes E^{(j)} \right] \left[ E^{(i)} \otimes D_j \right].
\]

By Proposition 2.1 the first equation in (20) has the form
\[
\sum_{\alpha, \beta} \Delta^i_{r_j} \left( E^{(r)} \right) \left( x \otimes E^{(j)}_{\alpha \beta} \right) \otimes S \left( E^{(j)}_{\alpha \beta} \right)
\]
\[
= \sum_{\tau, \alpha, \beta} \Delta^i_{r_j} (A_{\tau} (x)) \left( E^{(i)} \otimes S (A'_{\tau} (x)) E^{(j)}_{\alpha \beta} \right) \otimes S \left( E^{(j)}_{\beta \alpha} \right)
\]
which is equivalent to

\[ \sum_{\alpha,\beta} \Delta^{r}_{ij} \left( E^{(r)} \right) \left( x \otimes E^{(j)}_{\alpha\beta} \right) = \sum_{\tau,\alpha,\beta} \Delta^{r}_{ij} \left( A_{\tau}(x) \right) \left[ E^{(i)} \otimes S \left( A'_{\tau}(x) \right) E^{(j)}_{\alpha\beta} \right] \]

for any \( \alpha, \beta \). Identifying \( \alpha = \beta \) and taking a sum over all \( \alpha \) we obtain the first required equality. The second equality is obtained in a similar way from the second equality in (20).

5. Hopf ideals

The next statement follows from the definition of a Hopf ideal and Theorem 1.9.

**Proposition 5.1.** For a matrix constituent \( \text{Mat}(d_t, k) \) in \( H \) from direct decomposition (2) the following are equivalent:

(i) \( \text{Mat}(d_t, k) \) is a Hopf ideal in \( H \);
(ii) \( \Delta^{t}_{ij} = 0 \) for \( i, j \neq t \);
(iii) \( m^{t}_{ij} = 0 \) for \( i, j \neq t \);
(iv) if \( i, j \neq t \) then there are \( H \)-module isomorphisms

\[
M_i \otimes M_j \simeq \delta_{ij} (\oplus_{g \in G} E_g) \oplus [\oplus_{l \neq t} m^{l}_{ij} M_l]
\]

\[
M_j \otimes M_t \simeq d_j M_t \simeq M_t \otimes M_j,
\]

\[
M_t \otimes M_t \simeq (\oplus_{g \in G} E_g) \oplus [\oplus_{l=1}^{n} m^{l}_{tt} M_l].
\]

**Proposition 5.2.** If \( 1 \leq i, j, p \leq n \) then the following are equivalent:

(i) \( \Delta^{i}_{pj} \neq 0 \),
(ii) \( \Delta^{p}_{ji} \neq 0 \),
(iii) \( \Delta^{j}_{ip} \neq 0 \).

**Proof.** Let \( \Delta^{i}_{pj} \neq 0 \). Then \( m^{i}_{pj} \neq 0 \) by Proposition 5.1. But \( m^{i}_{pj} = m^{p}_{ji} = m^{j}_{ip} \). Hence \( \Delta^{p}_{ji}, \Delta^{j}_{ip} \neq 0 \).

**Proposition 5.3.** Assume that \( \text{Mat}(d_t, k) \) is a Hopf ideal in \( H \). If \( q \neq t \), then

\[
\Delta^{t}_{iq}(E^{(t)}) = E^{(t)} \otimes E^{(q)}, \quad \Delta^{t}_{qt}(E^{(t)}) = E^{(q)} \otimes E^{(t)},
\]

and \( q < t \). Thus \( t = n \).

**Proof.** Suppose that there exists an index \( j \neq t \) such that \( \Delta^{j}_{tq} \neq 0 \). By Proposition 5.2 we have \( \Delta^{j}_{tq} \neq 0 \) which contradicts Proposition 5.1. Hence

\[ E^{(t)} \otimes E^{(q)} = \sum_{j} \Delta^{j}_{tq} \left( E^{(j)} \right) = \Delta^{t}_{tq} \left( E^{(t)} \right). \]

The case \( \Delta^{t}_{qt} \) is similar.

The equalities mean that \( m^{t}_{tq} = m^{t}_{qt} = d_q \). By Corollary 1.10 that \( d_q = m^{t}_{tq} \leq d_{\min(t, q)} \) and therefore \( q \leq \min(t, q) \leq t \).
Suppose that \( t < n \). By (8) we get \( d_t d_{t+1} = \sum_i m_{i,t+1}^i d_i \). But \( m_{i,t+1}^i = m_{i+1,t}^i \neq 0 \) implies \( i = t \) since \( \text{Mat}(d_t, k) \) is a Hopf ideal. Hence \( d_t d_{t+1} = m_{t,t+1}^t d_t \) and \( m_{i,t+1}^t = d_{t+1} \). On the other from previous considerations it follows that \( m_{i,t+1}^i = 0 \). □

Consider in each matrix ideal \( \text{Mat}(d_i, k) \) the bilinear form \( \langle x, y \rangle \) such that \( \langle E_{a\beta}^{(i)}, E_{\gamma\tau}^{(i)} \rangle = \delta_{a\gamma} \delta_{\beta\tau} \). Then the dual space \( \text{Mat}(d_i, k)^* \) can be identified with itself via this bilinear function.

**Proposition 5.4.** Let \( \text{Mat}(d_n, k) \) be a Hopf ideal in \( H \). Suppose that \( x \in \text{Mat}(d_n, k) \) and \( q < n \). Then

\[
\Delta_{nq}^n(x) = \sum_{i,j=1}^{d_q} (E_{ij}^{(q)} \rightarrow x) \otimes E_{ij}^{(q)}, \quad \Delta_{qn}^n(x) = \sum_{i,j=1}^{d_q} E_{ij}^{(q)} \otimes (x \leftarrow E_{ij}^{(q)}).
\]

**Proof.** Let

\[
\Delta_{nq}^n(x) = \sum_{i,j=1}^{d_q} a_{ij}(x) \otimes E_{ij}^{(q)}, \quad a_{ij}(x) \in \text{Mat}(d_n, k).
\]

Then \( E_{ij}^{(q)} \rightarrow x = \sum_{i,j=1}^{d_q} a_{ij}(x) \langle E_{rs}^{(q)}, E_{ij}^{(q)} \rangle = a_{rs}(x) \). The second case is similar. □

**Proposition 5.5.** Let \( i < n \) and \( x \in \text{Mat}(d_i, k) \). Then

\[
\Delta_{nn}^i(x) = \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} \left[ E_{qp}^{(n)} \leftarrow S \left( \sum_{a,b=1}^{d_i} E_{ab}^{(i)} \right) \right] \otimes S \left( E_{pq}^{(n)} \right)
\]

\[
= \frac{d_i}{d_n} \left[ \left( \sum_{a,b=1}^{d_i} E_{ab}^{(i)} \right) \otimes 1 \right] D_n.
\]

**Proof.** The first identity in (19) with \( p = j = n \) has the form

\[
(1 \otimes \Delta_{nn}^i) D_i = (\Delta_{nn}^i \otimes 1) D_n.
\]

Applying Proposition 2.1 and Proposition 5.4 in the left and in the right hands sides of (23) we get the following

\[
(1 \otimes \Delta_{nn}^i) \left[ \frac{1}{d_i} \sum_{a,b=1}^{d_i} E_{ab}^{(i)} \otimes S \left( E_{ba}^{(i)} \right) \right] = \frac{1}{d_i} \sum_{a,b=1}^{d_i} E_{ab}^{(i)} \otimes \Delta_{nn}^i \left[ S \left( E_{ba}^{(i)} \right) \right];
\]

\[
(\Delta_{nn}^i \otimes 1) \left[ \frac{1}{d_n} \sum_{p,q=1}^{d_n} E_{pq}^{(n)} \otimes S \left( E_{qp}^{(n)} \right) \right] = \frac{1}{d_n} \sum_{p,q=1}^{d_n} \Delta_{nn}^i \left( E_{pq}^{(n)} \right) \otimes S \left( E_{qp}^{(n)} \right)
\]

\[
= \frac{1}{d_n} \sum_{p,q=1}^{d_n} \sum_{r,s=1}^{d_i} E_{rs}^{(i)} \otimes \left( E_{pq}^{(n)} \leftarrow E_{rs}^{(i)} \right) \otimes S \left( E_{qp}^{(n)} \right).
\]

Hence (23) is equivalent to

\[
\Delta_{nn}^i \left[ S \left( E_{ba}^{(i)} \right) \right] = \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} \left( E_{pq}^{(n)} \leftarrow E_{ab}^{(i)} \right) \otimes S \left( E_{qp}^{(n)} \right).
\]
If \( x = \sum_{a,b} x_{ab} S \left( E_{ba}^{(i)} \right) \in \text{Mat}(d_i, k) \) then

\[
\Delta^{i}_{n}(x) = \frac{d_i}{d_n} \sum_{a,b,\gamma,\lambda} \left[ E_{\gamma\lambda}^{(n)} \leftarrow \left( x_{ab} E_{ab}^{(n)} \right) \right] \otimes S \left( E_{\lambda\gamma}^{(n)} \right) 
= \frac{d_i}{d_n} \sum_{a,b,\gamma,\lambda} \left[ E_{\gamma\lambda}^{(n)} \leftarrow S \left( t x \right) \right] \otimes S \left( E_{\lambda\gamma}^{(n)} \right)
\]

**Proposition 5.6.** The equations \((20)\) with \( r = i = n, \ j < n \) hold in \( H \) if and only if \( E^{(j)} \rightarrow x = x \leftarrow E^{(j)} \) for all \( x \in \text{Mat}(d_n, k) \).

**Proof.** Note first that \( \Delta^{n}_{pq} \left( E^{(n)} \right) = E^{(n)} \otimes E^{(q)} \) by Proposition 5.3. Using Propositions 5.4, 4.1 and 2.1 we can rewrite the first equation in \((20)\) with \( r = i = n, \ j < n \) in the form

\[
x \otimes E^{(j)} = \sum_{\alpha,\beta} \Delta^{n}_{n j} \left( E_{\alpha\beta}^{(j)} \rightarrow x \right) \left[ E^{(n)} \otimes S \left( E_{\alpha\beta}^{(j)} \right) \right] \\
= \sum_{\alpha,\beta,\gamma,\lambda} \left( E_{\gamma\lambda}^{(j)} * E_{\alpha\beta}^{(j)} \rightarrow x \right) \otimes E_{\gamma\lambda}^{(j)} S \left( E_{\alpha\beta}^{(j)} \right) \\
= \sum_{\alpha,\beta,\gamma,\lambda} \left[ \left( E_{\gamma\lambda}^{(j)} * S \left( E_{\alpha\beta}^{(j)} \right) \right) \rightarrow x \right] \otimes E_{\gamma\lambda}^{(j)} E_{\alpha\beta}^{(j)} \\
= \sum_{\alpha,\beta,\gamma,\lambda} \left[ \left( E_{\gamma\lambda}^{(j)} * S \left( E_{\alpha\beta}^{(j)} \right) \right) \rightarrow x \right] \otimes E_{\gamma\beta}^{(j)} \\
= \sum_{\beta,\gamma} \left[ \delta_{\gamma\beta} E^{(j)} \rightarrow x \right] \otimes E_{\gamma\beta}^{(j)} = \left( E^{(j)} \rightarrow x \right) \otimes E^{(j)}
\]

for any \( x \in \text{Mat}(d_n, k) \). Thus \((24)\) means that \( E^{(q)} \rightarrow x = x \) for all \( x \).

Similarly the second equality \( x \leftarrow E^{(q)} \) is equivalent to the second one from \((20)\). \( \square \)

**Proof of Theorem 0.1.** Suppose that \( n > 1 \). Then \( \text{Mat}(d_n, k) \) is the unique component which is a Hopf ideal, Proposition 5.3.

Let \( i < n \). By Proposition 5.5 and Proposition 5.6 we have

\[
\Delta^{i}_{n n} \left( E^{(i)} \right) = \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} E_{qp}^{(n)} \otimes \left[ S \left( t E^{(i)} \right) \rightarrow S \left( E_{pq}^{(n)} \right) \right] \\
= \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} E_{qp}^{(n)} \otimes \left[ E^{(i)} \rightarrow S \left( E_{pq}^{(n)} \right) \right] = \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} E_{qp}^{(n)} \otimes S \left( E_{pq}^{(n)} \right) = d_i D_n.
\]

Suppose now that \( n \geq 2 \). Then \( e_1 \cdot E^{(1)} = 0 \) in \( H \) and therefore

\[
D_n \cdot \Delta^{1}_{n n} \left( E^{(1)} \right) = 0.
\]

On the other hand \( D_n \cdot \Delta^{1}_{n n} \left( E^{(1)} \right) = d_1 D_n^2 = d_1 D_n \neq 0 \), a contradiction. \( \square \)
6. The case $n = 1$ with $\Delta_{11}^1 = 0$

We shall remind some results in the case $n = 1$. For simplicity we put $E^{(1)} = E$, $E_{ij}^{(1)} = E_{ij}$, $d_1 = d$, $\Phi_1 = \Phi$, $U = U_1$.

**Theorem 6.1.** [A, ACh] Suppose that $n = 1$ and $\text{char } k$ is not a divisor of $2d^2$. Let $G = G(H^*)$ be the group of group-like elements of the dual Hopf algebra $H^*$. Assume that $\Delta_{11}^1 = 0$ in (13), or equivalently $|G| = d^2$. Taking an isomorphic copy of $H$ we can assume that the comultiplication in $H$ has the form:

$$\Delta(e_g) = \sum_{h \in G} e_h \otimes e_{h^{-1}g} + \frac{1}{d} \sum_{i,j=1}^d E_{ij} \otimes (g^{-1} \mapsto S(E_{ji}));$$

$$\Delta(x) = \sum_{g \in G} [(g \mapsto x) \otimes e_g + e_g \otimes (x \mapsto g)];$$

$$\varepsilon(e_g) = \delta_{g,1}, \quad \varepsilon(x) = 0; \quad S(x) = U^txU^{-1}$$

for all $x \in \text{Mat}(d,k)$, where $U \in \text{GL}(d,k)$ is a (skew-)symmetric matrix. There exists a faithful irreducible projective representation $\Phi : G \rightarrow \text{PGL}(d,k)$ such that

$$g \mapsto x = \Phi(g)x\Phi(g)^{-1}, \quad x \mapsto g = S(\Phi(g))xS(\Phi(g))^{-1} \quad (25)$$

for any $x \in \text{Mat}(d,k)$ and

$$[\Phi(g), S(\Phi(v))] = 1 \quad (27)$$

in $\text{PGL}(d,k)$ for all $g, v \in G$.

It is shown in [F, J2] that $G \simeq A \times A$ is a direct product of two copies of an Abelian group $A$ of order $d$. The next theorem shows the converse. If we start with an irreducible projective representation of a group $G$ from Theorem 6.1 we can obtain Hopf algebra structure.

**Theorem 6.2.** [ACh] Suppose that $G$ is Abelian group of order $d^2$ with direct decomposition $G \simeq A \times A$ for some Abelian group $A$ of order $d$. The group $G$ has a faithful irreducible projective representation $\Phi$ of degree $d$. There exists a (skew-)symmetric matrix $U \in \text{GL}(d,k)$ such that (26), (27) holds where $S(x) = U^txU^{-1}$ for any $x \in \text{Mat}(d,k)$. Then an algebra $H$ with direct decomposition (2) admits Hopf algebra structure defined in (26), (25).

Each element $g \in G$ can be identified with an element $g \in H^*$ such that if

$$a = \sum_{h \in G} \alpha_h e_h + x \in H, \quad \alpha_h \in k, \quad x \in \text{Mat}(d,k),$$

then $\langle g, a \rangle = \alpha_g$. The identification is an isomorphism of the group $G$ and the group $G(H^*)$ of group-like elements in $H^*$.

If $d > 2$, then these Hopf algebras are neither commutative nor cocommutative.

Note that semisimple Hopf algebras of dimension $2p^2$ for an odd prime $p$ in the same case as in Theorem 6.1 were classified in [Ma] in other terms. All Hopf algebras of dimension $2p^2$ for an odd prime $p$ were classified in [N0]. All Hopf algebra of dimension $2p^2$ for an odd prime $p$ were classified in [HN]. In Theorem 6.1, 6.2
we expand these results of an arbitrary $n$ using the language of faithful projective representations of the group $G = G(H^*)$.

Note that by [J1], [J2] all projective faithful irreducible representations of the group $G \simeq A \times A$ as in Theorems 6.1, 6.2 have dimension $d$ and obtained one from the other by a group automorphism of $G$. Any of these representations is monomial and it is induced by a one-dimensional representation of $A$.

References


[J1] Jmud E., Simplectic geometry on finite abelian groups, Mat. Sbornik, 86 (1971), 9–34. 6

[J2] Jmud E., Simplectic geometry and projective representations of finite abelian groups, Mat. Sbornik, 87 (1972), 3–17. 6, 6


[N1] Natale S., On group theoretical algebras and exact factorizations of finite groups, J. Algebra, 270 (2003), 190–211. 5


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