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ON SEMISIMPLE HOPF ALGEBRAS WITH FEW REPRESENTATIONS OF DIMENSION GREATER THAN ONE

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Abstract. In the paper we consider semisimple Hopf algebras H with the following property: irreducible H-modules of the same dimension > 1 are isomorphic. Suppose that there exists an irreducible H-module M of dimension > 1 such that its endomorphism ring is a Hopf ideal in H. Then M is the unique irreducible H-module of dimension > 1.

Introduction

Let H be a semisimple Hopf algebra over an algebraically closed field k. It is assumed that either $\operatorname{char} k = 0$ or $\operatorname{char} k > \dim H$. Semisimplicity of H means that H is a semisimple left H-module. In that case H has finite dimension.

Throughout the paper we shall keep to notations from [M]. For example H^* stands for the dual Hopf algebra with the natural pairing $\langle -, - \rangle : H^* \otimes H \to k$. The algebra H is a left and right H^*-module algebra with respect to the left and right actions $f \rightharpoonup x$, $x \leftharpoonup f$ of $f \in H^*$ on $x \in H$, defined as follows, [M, Example 4.1.10]: if

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)} \tag{1}$$

then

$$f \rightharpoonup x = \sum_x x_{(1)} \langle f, x_{(2)} \rangle, \quad x \leftharpoonup f = \sum_x \langle f, x_{(1)} \rangle x_{(2)}.$$

Denote by $G = G(H^*)$ the group of group-like elements in H^*. Elements of G are algebra homomorphisms $H \to k$. Hence H as a k-algebra has a semisimple direct decomposition

$$H = (\oplus_{g \in G} k e_g) \oplus \left(\oplus_{j=1}^{n} \operatorname{Mat}(d_j, k) \right), \tag{2}$$

where $\{e_g,\, g \in G\}$ is a system of central orthogonal idempotents in H corresponding to one-dimensional direct summands. If $h \in H$ and $g \in G$ then

$$h e_g = e_g h = \langle g, h \rangle e_g. \tag{3}$$

The counit $\varepsilon : H \to k$ has the form

$$\varepsilon(x) = \begin{cases} \delta_{g,1}, & x = e_g, \\ 0, & x \in \operatorname{Mat}(d_i, k). \end{cases}$$

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Each one-dimensional H-module $E_g = ku_g$ related to $g \in G$ has a base u_g and

$$hu_g = \langle g, h\rangle u_g, \quad h \in H. \tag{4}$$

In this paper we consider the case when

$$1 < d_1 < d_2 < \cdots < d_n. \tag{5}$$

It just means that irreducible H-modules of the same dimension > 1 are isomorphic. The main result of the paper is

Theorem 0.1. *Let H be a semisimple Hopf algebra with decomposition (2), $n \geqslant 1$, such that (5) holds. Suppose that at least one single matrix constituent is a Hopf ideal in H. Then $n = 1$.*

Throughout the paper we shall use the following notations. By $E^{(i)}$ we shall denote the identity matrix from $\mathrm{Mat}(d_i, k)$ and by $E^{(i)}_{rs} \in \mathrm{Mat}(d_i, k)$ we shall denote corresponding matrix unit.

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Note that the similar classes of Hopf algebras were considered from another point of view in [N2, §3.4] and in [Ma], [N1], [N2], [T], [TY].

1. The category of modules

Let H be a semisimple Hopf algebra with direct sum decomposition (2) such that (5) is satisfied. The category ${}_H\mathcal{M}$ of left H-modules is a monoidal category with respect to tensor products of modules. Since H is semisimple the abelian category is completely defined up to an isomorphism by its Grothendieck group $K_0(H) = K_0({}_H\mathcal{M})$. Recall that the Abelian K-group $K_0(H)$ is a free additive Abelian group with a base consisting of ismorphism clases of irreducible left H-modules E_g, $g \in G$, and modules M_i, $i = 1, \dots, n$, of dimension $d_i > 1$ corresponding to matrix constituents in (2). As we have already mentioned one-dimensional H-modules E_g correspond to one-dimensional simple direct summands of H such that (4) is satisfied and elements $g \in G$.

If M, N are left H-modules then $M \otimes N$ is also a left H-module under the action

$$h(x \otimes y) = \sum_h h_{(1)}x \otimes h_{(2)}y, \quad h \in H, \quad x \in M,\ y \in N. \tag{6}$$

Tensor multiplication induces a structure of an associative ring on $K_0(H)$.

The next fact is well known.

Proposition 1.1. *Let M be an irreducible H-module and E_g a one-dimensional H-module. Then $M \otimes E_g$ and $E_g \otimes M$ are irreducible H-modules. In particular if $f, g \in G$ then $E_f \otimes E_g \simeq E_{fg}$. Under assumption (5) we have $M \otimes E_g \simeq E_g \otimes M$.* □

The module $M \otimes E_g$ is identified with M in which elements $h \in H$ act as $g \rightharpoonup h$. An isomorphism $M \simeq M \otimes E_g$ means that there exists an invertible linear operator $\Phi_M(g)$ in M such that $\Phi_M(g)hx = (g \rightharpoonup h)\Phi_M(g)x$ for all $h \in H$ and for all

$x \in M$. Note that we can identify the endomorphism algebra $\operatorname{End} M$ with an ideal in H. Namely, if $M \simeq M_i$ then $\operatorname{End} M \simeq \operatorname{Mat}(d_i, k)$. If $g \in G$ then the action $g \rightharpoonup$ is an endomorphism of H. Because of the assumption (5) the ideal $\operatorname{End} M$ is stable under the action $g \rightharpoonup$. Thus $g \rightharpoonup h = \Phi_M(g) h \Phi_M(g)^{-1}$ in H. Note that we can always assume that $\Phi_M(1) = 1$ and $\Phi_M\left(g^{-1}\right) = \Phi_M(g)^{-1}$ for all $g \in G$.

We denote by S the antipode of H.

Proposition 1.2. *Let M be an irreducible H-module of dimension > 1. The map $\Phi_M : G \to \operatorname{PGL}(M)$ is a projective representation of the group G if M is an irreducible H-module. Moreover $[\Phi_M(g), S(\Phi_M(v))] = 1$ in $\operatorname{PGL}(M)$ for all $g, v \in G$.*

Proof. Since $g \rightharpoonup (v \rightharpoonup h) = gv \rightharpoonup h$, conjugations by $\Phi_M(g)\Phi_M(v)$ and by $\Phi_M(gv)$ coincide in $GL(M)$. Hence $\Phi_M(gv)^{-1}\Phi_M(g)\Phi_M(v)$ is a scalar matrix and therefore Φ_M is a projective representation of G in M. By the assumption (5) the algebra $\operatorname{End} M$ is an ideal which is stable not only under the action $g \rightharpoonup$ where $g \in G$ but also under the action $\leftharpoonup g$ and under the antipode S since S is an involution of the algebra H.

Similarly $E_g \otimes M$ can be identified with the vector space M in which $h \in H$ acts as

$$\begin{aligned} h \leftharpoonup g &= S\left(S(g) \rightharpoonup S(h)\right) = S\left(\Phi_M\left(g^{-1}\right) S(h) \Phi_M(g)\right) \\ &= S\left(\Phi_M(g)\right) h S\left(\Phi_M\left(g^{-1}\right)\right). \end{aligned}$$

In order to prove the last statement it is necessary to recall that the actions $\leftharpoonup, \rightharpoonup$ commute. □

Proposition 1.3. *Let M, N be irreducible H-modules and $g \in G$. Then*

$$\operatorname{Hom}_H(M \otimes N, E_g) \simeq \operatorname{Hom}_H(M \otimes N, E_\varepsilon).$$

Proof. Let $\phi \in \operatorname{Hom}(M \otimes N, E_g)$. By Proposition 1.1 there is an isomorphism of H-modules $\omega : M \otimes N \to M \otimes N \otimes E_{g^{-1}}$. Hence ϕ induces by Proposition 1.1 a homomorphism of H-modules

$$M \otimes N \xrightarrow{\ \omega^{-1}\ } M \otimes N \otimes E_{g^{-1}} \xrightarrow{\ \phi \otimes 1\ } E_g \otimes E_{g^{-1}} \longrightarrow E_\varepsilon.$$

Denote by $\tilde{\phi}$ this composition. It is easy to see that the correspondence $\phi \mapsto \tilde{\phi}$ is an isomorphism between $\operatorname{Hom}_H(M \otimes N, E_g)$ and $\operatorname{Hom}_H(M \otimes N, E_\varepsilon)$. □

Corollary 1.4. *If M, N are irreducible H-modules of dimensions > 1, then the multiplicity of each one-dimensional H-module E_g, $g \in G$, in $M \otimes N$ does not depend upon g.* □

If M is an irreducible left H-module then $H^* = \operatorname{Hom}_k(M, k)$ is again an irreducible left H-module. So we have

Proposition 1.5. *Let M_i, be the unique irreducible left H-module of dimension $d_i > 1$. Then $M_i^* \simeq M_i$. In particular there exists a non-degenerate bilinear function $\langle x, y \rangle$ on M_i such that $\langle hx, y \rangle = \langle x, S(h)y \rangle$ for all $x, y \in M_i$ and for any $h \in H$.* □

If $\mathbf{e}_* = (e_1, \dots, e_{d_i})$ is a base of M_i as a vector space then we get an identification of $\operatorname{End} M_i$ with the full matrix algebra $\mathrm{Mat}(d_i, k)$ which is an ideal in H. Denote by U_i the square matrix of size d_i whose (α, β)th entry $u_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle$. It means that U_i is the Gram matrix of the bilinear function $\langle x, y\rangle$.

Theorem 1.6. *The bilinear function $\langle x, y\rangle$ is (skew-)symmetric. If $x \in \mathrm{Mat}(d_i, k)$ from decomposition (2), then $S(x) = U_i\, {}^t x U_i^{-1}$. Using a canonical form of a Gram matrix of a (skew-)symmetric bilinear function we can always choose a base in M_i such that U_i is either an identity matrix in the symmetric case or the matrix*

$$U_i = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix},$$

in skew-symmetric (hyperbolic) case where E is the identity matrix of size $\frac{d_i}{2}$. In both cases $U_i^{-1} = \pm {}^t U_i$.

Proof. Without loss of generality we can assume that $h, S(h) \in \mathrm{Mat}(d_i, k)$ have entries $h_{\zeta\beta}, r_{\gamma\lambda}$, respectively. Then

$$\langle he_\alpha, e_\beta\rangle = \sum_\zeta \langle e_\zeta, e_\beta\rangle h_{\alpha\zeta} = \sum_\zeta u_{\zeta\beta} h_{\alpha\zeta};$$
$$\langle e_\alpha, S(h)e_\beta\rangle = \sum_\gamma r_{\beta\gamma}\langle e_\alpha, e_\gamma\rangle = \sum_\gamma r_{\beta\gamma} u_{\alpha\gamma}.$$

The equality $\langle hx, y\rangle = \langle x, S(h)y\rangle$ means that ${}^tU_i \cdot {}^th = S(h)\, {}^tU_i$.

Finally since the antipode S has order 2, the matrix U_i and the form $\langle x, y\rangle$ are (skew)-symmetric [LR]. So we can replace ${}^tU_i$ by U_i. □

We now study some properties of irreducible H-modules. The first one is a well-known, see [DNR, Lemma 7.5.10, p. 322]. Recall that M_i is an irreducible H-module of dimension d_i, $1 \le i \le n$.

Lemma 1.7. *If $1 \le i, j \le n$, then $\dim \mathrm{Hom}_H(M_i \otimes M_j, E_\varepsilon) = \delta_{ij}$.* □

Set $A = \oplus_{g\in G} E_g$.

Proposition 1.8. *There is a direct sum decomposition*

$$\begin{aligned} M_i \otimes M_j &= \delta_{ij} A \oplus \left(\oplus_{t=1}^n m_{ij}^t M_t\right), \\ m_{ij}^t &= \dim_k \mathrm{Hom}_H(M_i \otimes M_j, M_t) \geqslant 0. \end{aligned} \tag{7}$$

Hence

$$d_i d_j = \delta_{ij}|G| + \sum_{t=1}^n m_{ij}^t d_t. \tag{8}$$

In particular $|G| \leqslant d_1^2$.

Proof. Apply Lemma 1.7, Corollary 1.4 and compare dimensions of modules in (7). This gives (7), (8). We prove the last claim. Each coefficient m_{pq}^i estimated by (8) as

$$0 \leqslant m_{pq}^i \leqslant \frac{1}{d_i} d_p d_q - \delta_{pq}|G|.$$

In particular if $p = q$ then $0 \leqslant d_p^2 - |G|$. □

Proposition 1.5 implies that $A \otimes M_s = M_s \otimes A = |G| M_s$. Using the associativity conditions in the Grothendieck ring we obtain

Theorem 1.9. *The multiplicities m_{ij}^t satisfy the equation (8) and the equations*

$$m_{ij}^s = m_{js}^i, \quad \delta_{ij}\delta_{ls}|G| + \sum_{t=1}^n m_{ij}^t m_{ts}^l = \delta_{js}\delta_{li}|G| + \sum_{t=1}^n m_{js}^t m_{it}^l$$

for all $i, j, s, l = 1, \dots, n$. In particular $m_{ij}^s = m_{js}^i = m_{si}^j$ and

$$\delta_{ij}\delta_{ls}|G| + \sum_{t=1}^n m_{ti}^j m_{ts}^l = \delta_{js}\delta_{li}|G| + \sum_{t=1}^n m_{st}^j m_{it}^l.$$

□

Corollary 1.10. *If $i, j, p = 1, \dots, n$, then $m_{ij}^p \leqslant d_{\min(i,j,p)}$.*

Proof. Since $m_{ij}^p = m_{jp}^i = m_{pi}^j$ we can assume that $p \geqslant i, j$. Then by (8) we get $m_{ij}^p d_p \leqslant d_i d_j$ or

$$m_{ij}^p \leqslant \frac{d_i d_j}{d_p} \leqslant \min(d_i, d_j) = d_{\min(i,j)} = d_{\min(i,j,p)}.$$

□

2. Special matrices

Consider special elements

$$\mathcal{T}_i = \sum_{\alpha,\beta=1}^{d_i} E_{\alpha\beta}^{(i)} \otimes E_{\alpha\beta}^{(i)}, \quad \mathcal{R}_i = \frac{1}{d_i} \sum_{\alpha,\beta=1}^{d_i} E_{\alpha\beta}^{(i)} \otimes E_{\beta\alpha}^{(i)}, \qquad (9)$$
$$\mathcal{D}_i = (1 \otimes S)\, \mathcal{R}_i$$

in $\mathrm{Mat}(d_i, k)^{\otimes 2}$. These elements will be used in the rest of the paper.
Direct calculations prove the first equality in

Proposition 2.1. *Let $F, H \in \mathrm{Mat}(d_i, k)$. Then*

$$\left(F \otimes E^{(i)}\right) \mathcal{T}_i \left(E^{(i)} \otimes H\right) = \left(E^{(i)} \otimes {}^tF\right) \mathcal{T}_i \left(E^{(i)} \otimes {}^tH\right),$$
$$(F \otimes H)\mathcal{R}_i = \mathcal{R}_i (H \otimes F).$$

Here $E^{(i)}$ stands for the identity matrix of the size d_i. Moreover

$$(1 \otimes S)\, \mathcal{T}_i = (S \otimes 1)\, \mathcal{T}_i, \quad \mathcal{D}_i = (1 \otimes S)\, \mathcal{R}_i = (S \otimes 1)\, \mathcal{R}_i.$$

In particular for F, H as above

$$(F \otimes 1) \cdot \mathcal{D}_t \cdot (1 \otimes S(H)) = (1 \otimes S(F)) \cdot \mathcal{D}_t \cdot (H \otimes 1).$$

Proof. The second equality was proved in [ACh, Proposition 1.1].

In order to prove the nest two equalities we shall use the first equality, (skew-) symmetry of U_i and the property $U_i^{-1} = \pm U_i$. Hence

$$\begin{aligned}(1\otimes S)\mathcal{T}_i &= d_i\left(E^{(i)}\otimes U_i\right)\mathcal{R}_i\left(E^{(i)}\otimes U_i^{-1}\right)\\ &= d_i\left(U_i\otimes E^{(i)}\right)\mathcal{R}_i\left(U_i^{-1}\otimes E^{(i)}\right) = (S\otimes 1)\mathcal{T}_i;\\ (1\otimes S)\mathcal{R}_i &= \frac{1}{d_t}\left(E^{(i)}\otimes U_i\right)\mathcal{T}_i\left(E^{(i)}\otimes U_i^{-1}\right)\\ &= \frac{1}{d_t}\left(U_i\otimes E^{(i)}\right)\mathcal{T}_i\left(U_i^{-1}\otimes E^{(i)}\right) = (S\otimes 1)\mathcal{R}_i = \mathcal{D}_i.\end{aligned}$$

In order to prove the last equality apply $(1\otimes S)$ to (9) and use previous equalities. □

3. From module category to Hopf algebra

In this section we shall apply previous results to a presentation of an explicit form of comultiplication and antipode in H. Let H have the direct sum decomposition (2) such that (5) holds.

Denote by π_g, $g\in G$, and by π_i, $1\leqslant i\leqslant n$, the projections of H onto E_g and onto $\mathrm{Mat}(d_i,k)$, respectively. Then the isomorphism $E_f\otimes E_g\simeq E_{fg}$ in Proposition 1.1 means that

$$\begin{aligned}(\pi_f\otimes\pi_g)\Delta(h) &= \langle fg,h\rangle(e_f\otimes e_g);\\ (\pi_g\otimes\pi_i)\Delta(h) &= e_g\otimes(\pi_i(h)\leftharpoonup g),\\ (\pi_i\otimes\pi_g)\Delta(h) &= (g\rightharpoonup\pi_i(h))\otimes e_g.\end{aligned}\tag{10}$$

Applying Proposition 1.8 we obtain

$$(\pi_i\otimes\pi_j)\Delta(\pi_g(h)) = 0,\qquad 1\leqslant i\neq j\leqslant n,\quad g\in G.\tag{11}$$

Using (10), (11) we see that

$$\Delta(e_g) = \sum_{f\in G} e_f\otimes e_{f^{-1}g} + \sum_{i=1,\dots,n}\mathcal{D}_{g,i},\tag{12}$$

where $\{\mathcal{D}_{g,i},\ g\in G, 1\leqslant i\leqslant n\}$, is a system of orthogonal idempotents in $\mathrm{Mat}(d_i,k)^{\otimes 2}$. Moreover

$$\mathcal{D}_{g,i}\in\mathrm{Mat}(d_i,k)\otimes\mathrm{Mat}(d_i,k)\simeq\mathrm{Mat}(d_i^2,k)\simeq\mathrm{End}_k(M_i\otimes M_i).$$

Lemma 3.1. *Let $g\in G$, and $\mathcal{D}_i$ from (9), $i=1,\dots,n$. Then the H-module $\mathcal{D}_{g,i}(M_i\otimes M_i)$ is isomorphic to E_g and*

$$\mathcal{D}_{g,i} = \left(1\otimes\left(g^{-1}\rightharpoonup\right)\right)\mathcal{D}_i.$$

Proof. Let $h\in H$ and $x,y\in M_i$. Applying (3), (4), (6) and (12) we obtain

$$\begin{aligned}h\mathcal{D}_{g,i}(x\otimes y) &= \Delta(h)\mathcal{D}_{g,i}(x\otimes y)\\ &= \Delta(h)\Delta(e_g)(x\otimes y) = \Delta(he_g)(x\otimes y) = \langle g,h\rangle\mathcal{D}_{g,i}(x\otimes y).\end{aligned}$$

Thus (4) implies that $\mathcal{D}_{g,i}(M_i\otimes M_i)\simeq E_g$.

By (7) and previous considerations it suffices to show that the element $(1\otimes (g^{-1}\rightharpoonup))\,\mathcal{D}_i$ is an idempotent and

$$\Delta(h)\left(1\otimes\left(g^{-1}\rightharpoonup\right)\right)\mathcal{D}_i=\langle g,\,h\rangle\left(1\otimes\left(g^{-1}\rightharpoonup\right)\right)\mathcal{D}_i.$$

Direct calculations show that

$$\begin{aligned}\mathcal{D}_i^2&=\frac{1}{d_i^2}\sum_{\alpha,\beta,\gamma,\xi}E^{(i)}_{\alpha\beta}E^{(i)}_{\gamma\xi}\otimes S\left(E^{(i)}_{\beta\alpha}\right)S\left(E^{(i)}_{\xi\gamma}\right)\\&=\frac{1}{d_i^2}\sum_{\alpha,\beta,\gamma,\xi}E^{(i)}_{\alpha\beta}E^{(i)}_{\gamma\xi}\otimes S\left(E^{(i)}_{\xi\gamma}E^{(i)}_{\beta\alpha}\right)\\&=\frac{1}{d_i^2}d_i\sum_{\alpha,\beta=\gamma,\xi}E^{(i)}_{\alpha\xi}\otimes S\left(E^{(i)}_{\xi\alpha}\right)=\mathcal{D}_i.\end{aligned}$$

Since $g\rightharpoonup$ is an algebra automorphism of H by Proposition 2.1 we have

$$\begin{aligned}\Delta(h)\left(1\otimes\left(g^{-1}\rightharpoonup\right)S\right)\mathcal{R}_i&=\sum_h\left(h_{(1)}\otimes h_{(2)}\left(g^{-1}\rightharpoonup\right)S\right)\mathcal{R}_i\\&=\sum_h\left(h_{(1)}\otimes\left(g^{-1}\rightharpoonup\right)\left(g\rightharpoonup h_{(2)}\right)S\right)\mathcal{R}_i\\&=\sum_h\left(h_{(1)}\otimes\left(g^{-1}\rightharpoonup\right)S\right)\left(\mathcal{R}_iS\left(g\rightharpoonup h_{(2)}\right)\right)\\&=\left(1\otimes\left(g^{-1}\rightharpoonup\right)S\right)\left[\sum_h\left(h_{1)}\otimes 1\right)\mathcal{R}_i\left(1\otimes\left(S\left(g\rightharpoonup h_{(2)}\right)\right)\right)\right]\\&=\left(1\otimes\left(g^{-1}\rightharpoonup\right)S\right)\left[\mathcal{R}_i\left(1\otimes\sum_h\left(h_{(1)}S\left(g\rightharpoonup h_{(2)}\right)\right)\right)\right]\end{aligned}$$

Finally

$$\begin{aligned}\sum_h h_{(1)}S\left(g\rightharpoonup h_{(2)}\right)&=\sum_h h_{(1)}S\left(h_{(2)}\langle g,\,h_{(3)}\rangle\right)\\&=\sum_h h_{(1)}S\left(h_{(2)}\right)\langle g,\,h_{(3)}\rangle=\sum_h\varepsilon\left(h_{(1)}\right)\langle g,\,h_{(2)}\rangle\\&=\langle g,\,\sum_h\varepsilon\left(h_{(1)}\right)h_{(2)}\rangle=\langle g,\,h\rangle\end{aligned}$$

□

Applying (10) for any $x\in\mathrm{Mat}(d_r,k)$ we get

$$\Delta(x)=\sum_{g\in G}\left[(g\rightharpoonup x)\otimes e_g+e_g\otimes(x\leftharpoonup g)\right]+\sum_{i,j=1}^{n}\Delta^r_{ij}(x),\tag{13}$$

where $\Delta^r_{ij}(x)\in\mathrm{Mat}(d_i,k)\otimes\mathrm{Mat}(d_j,k)$.

Note that $\Delta^r_{ij}(x)=(\pi_i\otimes\pi_j)\Delta(x)$. It means that

$$\Delta^r_{ij}:\mathrm{Mat}(d_r,k)\to\mathrm{Mat}(d_i,k)\otimes\mathrm{Mat}(d_j,k)$$

is an algebra homomorphism not necessarily preserving the unit element.

Lemma 3.2. *Let* $x \in \operatorname{Mat}(d_i, k)$, $g \in G$ *and* τ *is the twist* $\tau(a \otimes b) = b \otimes a$ *for all* $a \in \operatorname{Mat}(d_q, k)$, $b \in \operatorname{Mat}(d_p, k)$. *Then*

$$\Delta^i_{pq}(x) = (S \otimes S)\left(\Delta^i_{qp}\right)^\tau (S(x)), \tag{14}$$

$$\begin{gathered} S(e_g) = e_{g^{-1}}, \quad \mu(1 \otimes S)\Delta^i_{pq}(x) = \mu(S \otimes 1)\Delta^i_{pq}(x) = 0, \\ \mu(S \otimes 1)\mathcal{D}_{g,i} = \mu(1 \otimes S)\mathcal{D}_{g,i} = \delta_{g,1}E^{(i)}. \end{gathered} \tag{15}$$

Here $\mu : H \otimes H \to H$ *is the multiplication map.*

Proof. The antipode S is an algebra and coalgebra involution. Hence each ideal $\operatorname{Mat}(d_i, k)$ in H is S-stable and hence (14) holds. The equalities (15) were proved in [A]. □

Note that $\mathcal{D}_{g,i}$ and $\Delta^t_{ij}(E_t)$ are in fact projections of $M_i \otimes M_j$ onto E_g by Lemma 3.1 and onto $m^t_{ij}M_t$, respectively. Hence $\mathcal{D}_{g,i}$ and $\Delta^t_{ij}(E_t)$ are matrices in $\operatorname{Mat}(d_id_j, k)$ whose ranks are equal to $\dim E_g = 1$ and to $m^t_{ij} \dim M_t$, respectively. In particular, (14) implies $m^i_{pq} = m^i_{qp}$ for all i, p, q.

Since Δ is an algebra homomorphism for all $i, j, p, q, r, s = 1, \ldots, m$ and $x \in \operatorname{Mat}(d_r, x)$, $y \in \operatorname{Mat}(d_s, k)$ we have

$$\begin{gathered} \mathcal{D}_{g,i} \cdot \mathcal{D}_{h,j} = \delta_{gh}\delta_{ij}\mathcal{D}_{g,i}, \\ \mathcal{D}_{g,i} \cdot \Delta^r_{pq}(x) = \Delta^r_{pq}(x) \cdot \Delta_{g,i} = 0, \\ \Delta^r_{ij}(x) \cdot \Delta^s_{pq}(y) = \delta_{ip}\delta_{jq}\delta_{rs}\Delta^r_{ij}(xy). \end{gathered} \tag{16}$$

4. **Coassociativity conditions**

In this section we shall consider coassociativity conditions of comultiplication Δ from (12), (13). Multiplying the equalities from [A, §2, Proposition 2.1] by $\Delta^r_{ij}(E^{(r)}) \otimes 1$ from the left, by $E^{(j)} \otimes \Delta^r_{ij}(E^{(r)})$ from the right and the last equation by $\Delta^t_{jp}(E^{(t)}) \otimes E^{(q)}$ from the left and by $E^{(j)} \otimes \Delta^r_{pq}(E^{(r)})$ from the right, respectively, we obtain using (16) that for any $i, j, p, q = 1, \ldots, n$ and all $x \in \operatorname{Mat}(d_i, k)$ the following five groups of identities are satisfied.

The first group:

$$\begin{aligned} &[1 \otimes (g \rightharpoonup)]\,\mathcal{D}_{t,i} = \mathcal{D}_{tg^{-1},i}; \\ &[(\leftharpoonup g) \otimes 1]\,\mathcal{D}_{t,i} = \mathcal{D}_{g^{-1}t,i}; \\ &[1 \otimes (\leftharpoonup g)]\,\mathcal{D}_{t,i} = [(g \rightharpoonup) \otimes 1]\,\mathcal{D}_{t,i}. \end{aligned} \tag{17}$$

The second group:

$$\begin{aligned} &[(\leftharpoonup g) \otimes 1]\,\Delta^i_{pq}(x) = \Delta^i_{pq}(x \leftharpoonup g); \\ &[1 \otimes (g \rightharpoonup)]\,\Delta^i_{pq}(x) = \Delta^i_{pq}(g \rightharpoonup x); \\ &[1 \otimes (\leftharpoonup g)]\,\Delta^i_{pq}(x) = [(g \rightharpoonup) \otimes 1]\,\Delta^i_{pq}(x). \end{aligned} \tag{18}$$

The third group:

$$\begin{aligned} \left(1\otimes\Delta_{pj}^{i}\right)\mathcal{D}_i &= \left(\Delta_{ip}^{j}\otimes 1\right)\mathcal{D}_j;\\ (\mathcal{D}_i\otimes E^{(i)})\left[x\otimes\mathcal{D}_i\right] &= \left[\mathcal{D}_i\otimes x\right]\left(E^{(i)}\otimes\mathcal{D}_i\right). \end{aligned} \tag{19}$$

The fourth group:

$$\begin{aligned} \left[\Delta_{ij}^{r}(E^{(r)})\otimes E^{(j)}\right]\left[x\otimes\mathcal{D}_j\right] &= \left[\left(\Delta_{ij}^{r}\otimes 1\right)\Delta_{rj}^{i}(x)\right]\left(E^{(i)}\otimes\mathcal{D}_j\right);\\ \left(\mathcal{D}_j\otimes E^{(i)}\right)\left[(1\otimes\Delta_{ji}^{r})\Delta_{jr}^{i}(x)\right] &= \left[\mathcal{D}_j\otimes x\right]\left[E^{(j)}\otimes\Delta_{ji}^{r}(E^{(r)})\right]. \end{aligned} \tag{20}$$

The last fifth group:

$$\begin{aligned} &\left[\Delta_{jp}^{t}(E^{(t)})\otimes E^{(q)}\right]\left[\left(1\otimes\Delta_{pq}^{r}\right)\Delta_{jr}^{i}(x)\right]\\ &= \left[\left(\Delta_{jp}^{t}\otimes 1\right)\Delta_{tq}^{i}(x)\right]\left[E^{(j)}\otimes\Delta_{pq}^{r}(E^{(r)})\right]. \end{aligned} \tag{21}$$

Also the equations (16) and (15) are satisfied.

We shall apply the identities (17) — (21) to the study of the comultiplication and the antipode of H.

Proposition 4.1. *Let*

$$\Delta_{rj}^{i}(x)=\sum_{\tau}A_\tau(x)\otimes A'_\tau(x),\quad \Delta_{jr}^{i}(x)=\sum_{\tau}B'_\tau(x)\otimes B_\tau(x), \tag{22}$$

where $A_\tau, B_\tau\in\mathrm{Mat}(d_r,k)$ *and* $A'_\tau, B'_\tau\in\mathrm{Mat}(d_j,k)$*. Then the equalities* (20) *are equivalent to*

$$\begin{aligned} \Delta_{ij}^{r}\left(E^{(r)}\right)\left(x\otimes E^{(j)}\right) &= \sum_{\tau}\Delta_{ij}^{r}\left(A_\tau(x)\right)\left[E^{(i)}\otimes S\left(A'_\tau(x)\right)\right];\\ \left(E^{(j)}\otimes x\right)\Delta_{ij}^{r}\left(E^{(r)}\right) &= \sum_{\tau}\left[S\left(B'_\tau(x)\right)\otimes E^{(i)}\right]\Delta_{ji}^{r}\left(B_\tau(x)\right) \end{aligned}$$

for all $x\in\mathrm{Mat}(d_i,k)$.

Proof. In the notations (22) using Proposition 2.1 we obtain in the right hand part of the first equation in (20)

$$\begin{aligned} &\left[\left(\Delta_{ij}^{r}\otimes 1\right)\Delta_{rj}^{i}(x)\right]\left(E^{(i)}\otimes\mathcal{D}_j\right)\\ &= \left[\sum_{\tau}\Delta_{ij}^{r}\left(A_\tau(x)\right)\left(E^{(i)}\otimes S\left(A'_\tau(x)\right)\right)\otimes E^{(j)}\right]\left[E^{(i)}\otimes\mathcal{D}_j\right]. \end{aligned}$$

By Proposition 2.1 the first equation in (20) has the form

$$\begin{aligned} &\sum_{\alpha,\beta}\left[\Delta_{ij}^{r}\left(E^{(r)}\right)\left(x\otimes E_{\alpha\beta}^{(j)}\right)\otimes S\left(E_{\alpha\beta}^{(j)}\right)\right]\\ &= \sum_{\tau,\alpha,\beta}\Delta_{ij}^{r}\left(A_\tau(x)\right)\left(E^{(i)}\otimes S\left(A'_\tau(x)\right)E_{\alpha\beta}^{(j)}\right)\otimes S\left(E_{\beta\alpha}^{(j)}\right) \end{aligned}$$

which is equivalent to

$$\sum_{\alpha,\beta} \Delta_{ij}^{r}\left(E^{(r)}\right)\left(x \otimes E_{\alpha\beta}^{(j)}\right) \quad = \quad \sum_{\tau,\alpha,\beta} \Delta_{ij}^{r}\left(A_{\tau}(x)\right)\left[E^{(i)} \otimes S\left(A'_{\tau}(x)\right) E_{\alpha\beta}^{(j)}\right]$$

for any α, β. Identifying $\alpha = \beta$ and taking a sum over all α we obtain the first required equality. The second equality is obtained in a similar way from the second equality in (20). □

5. Hopf ideals

The next statement follows from the definition of a Hopf ideal and Theorem 1.9.

Proposition 5.1. *For a matrix constituent* $\mathrm{Mat}(d_t, k)$ *in* H *from direct decomposition* (2) *the following are equivalent:*

(i) $\mathrm{Mat}(d_t, k)$ *is a Hopf ideal in* H*;*
(ii) $\Delta_{ij}^{t} = 0$ *for* $i, j \neq t$*;*
(iii) $m_{ij}^{t} = 0$ *for* $i, j \neq t$*;*
(iv) *if* $i, j \neq t$ *then there are* H*-module isomorphisms*

$$\begin{aligned} M_i \otimes M_j &\simeq \delta_{ij}\left(\oplus_{g\in G} E_g\right) \oplus \left[\oplus_{l\neq t} m_{ij}^{l} M_l\right] \\ M_j \otimes M_t &\simeq d_j M_t \simeq M_t \otimes M_j, \\ M_t \otimes M_t &\simeq \left(\oplus_{g\in G} E_g\right) \oplus \left[\oplus_{l=1}^{n} m_{tt}^{l} M_l\right]. \end{aligned}$$

□

Proposition 5.2. *If* $1 \leqslant i, j, p \leqslant n$ *then the following are equivalent:*

(i) $\Delta_{pj}^{i} \neq 0$,
(ii) $\Delta_{ji}^{p} \neq 0$,
(iii) $\Delta_{ip}^{j} \neq 0$.

Proof. Let $\Delta_{pj}^{i} \neq 0$. Then $m_{pj}^{i} \neq 0$ by Proposition 5.1. But $m_{pj}^{i} = m_{ji}^{p} = m_{ip}^{j}$. Hence $\Delta_{ji}^{p}, \Delta_{ip}^{j} \neq 0$. □

Proposition 5.3. *Assume that* $\mathrm{Mat}(d_t, k)$ *is a Hopf ideal in* H*. If* $q \neq t$*, then*

$$\Delta_{tq}^{t}(E^{(t)}) = E^{(t)} \otimes E^{(q)}, \quad \Delta_{qt}^{t}(E^{(t)}) = E^{(q)} \otimes E^{(t)},$$

and $q < t$*. Thus* $t = n$*.*

Proof. Suppose that there exists an index $j \neq t$ such that $\Delta_{tq}^{j} \neq 0$. By Proposition 5.2 we have $\Delta_{qj}^{t} \neq 0$ which contradicts Proposition 5.1. Hence

$$E^{(t)} \otimes E^{(q)} = \sum_{j} \Delta_{tq}^{j}\left(E^{(j)}\right) = \Delta_{tq}^{t}\left(E^{(t)}\right).$$

The case Δ_{qt}^{t} is similar.

The equalities mean that $m_{tq}^{t} = m_{qt}^{t} = d_q$. By Corollary 1.10 that $d_q = m_{tq}^{t} \leqslant d_{\min(t,q)}$ and therefore $q \leqslant \min(t, q) \leqslant t$.

Suppose that $t < n$. By (8) we get $d_t d_{t+1} = \sum_i m^i_{t,t+1} d_i$. But $m^i_{t,t+1} = m^t_{t+1,i} \neq 0$ implies $i = t$ since $\mathrm{Mat}(d_t, k)$ is a Hopf ideal. Hence $d_t d_{t+1} = m^t_{t,t+1} d_t$ and $m^t_{t,t+1} = d_{t+1}$. On the other from previous considerations it follows that $m^t_{t,t+1} = 0$. □

Consider in each matrix ideal $\mathrm{Mat}(d_i, k)$ the bilinear form $\langle x, y\rangle$ such that $\langle E^{(i)}_{\alpha\beta}, E^{(i)}_{\gamma\tau}\rangle = \delta_{\alpha\gamma}\delta_{\beta\tau}$. Then the dual space $\mathrm{Mat}(d_i, k)^*$ can be identified with itself via this bilinear function.

Proposition 5.4. *Let $\mathrm{Mat}(d_n, k)$ be a Hopf ideal in H. Suppose that $x \in \mathrm{Mat}(d_n, k)$ and $q < n$. Then*

$$\Delta^n_{nq}(x) = \sum_{i,j=1}^{d_q} \left(E^{(q)}_{ij} \rightharpoonup x\right) \otimes E^{(q)}_{ij}, \quad \Delta^n_{qn}(x) = \sum_{i,j=1}^{d_q} E^{(q)}_{ij} \otimes \left(x \leftharpoonup E^{(q)}_{ij}\right).$$

Proof. Let

$$\Delta^n_{nq}(x) = \sum_{i,j=1}^{d_q} a_{ij}(x) \otimes E^{(q)}_{ij}, \quad a_{ij}(x) \in \mathrm{Mat}(d_n, k).$$

Then $E^{(q)}_{rs} \rightharpoonup x = \sum_{i,j=1}^{d_q} a_{ij}(x)\langle E^{(q)}_{rs}, E^{(q)}_{ij}\rangle = a_{rs}(x)$. The second case is similar. □

Proposition 5.5. *Let $i < n$ and $x \in \mathrm{Mat}(d_i, k)$. Then*

$$\begin{aligned}\Delta^i_{nn}(x) &= \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} \left[E^{(n)}_{qp} \leftharpoonup S\left({}^t x\right)\right] \otimes S\left(E^{(n)}_{pq}\right)\\ &= d_i \left[\left(\leftharpoonup S\left({}^t x\right)\right) \otimes 1\right] \mathcal{D}_n.\end{aligned}$$

Proof. The first identity in (19) with $p = j = n$ has the form

$$\left(1 \otimes \Delta^i_{nn}\right) \mathcal{D}_i = \left(\Delta^n_{in} \otimes 1\right) \mathcal{D}_n. \tag{23}$$

Applying Proposition 2.1 and Proposition 5.4 in the left and in the right hands sides of (23) we get the following

$$\left(1 \otimes \Delta^i_{nn}\right) \left[\frac{1}{d_i} \sum_{a,b=1}^{d_i} E^{(i)}_{ab} \otimes S\left(E^{(i)}_{ba}\right)\right] = \frac{1}{d_i} \sum_{a,b=1}^{d_i} E^{(i)}_{ab} \otimes \Delta^i_{nn}\left[S\left(E^{(i)}_{ba}\right)\right];$$

$$\begin{aligned}\left(\Delta^n_{in} \otimes 1\right) \left[\frac{1}{d_n} \sum_{p,q=1}^{d_n} E^{(n)}_{pq} \otimes S\left(E^{(n)}_{qp}\right)\right] &= \frac{1}{d_n} \left[\sum_{p,q=1}^{d_n} \Delta^n_{in}(E^{(n)}_{pq}) \otimes S\left(E^{(n)}_{qp}\right)\right]\\ = \frac{1}{d_n} \sum_{\substack{p,q=1,\dots,d_n\\ r,s=1,\dots,d_i}} E^{(i)}_{rs} \otimes \left(E^{(n)}_{pq} \leftharpoonup E^{(i)}_{rs}\right) \otimes S\left(E^{(n)}_{qp}\right).\end{aligned}$$

Hence (23) is equivalent to

$$\Delta^i_{nn}\left[S\left(E^{(i)}_{ba}\right)\right] = \frac{d_i}{d_n} \sum_{p,q=1}^{d_n} \left(E^{(n)}_{pq} \leftharpoonup E^{(i)}_{ab}\right) \otimes S\left(E^{(n)}_{qp}\right).$$

If $x=\sum_{a,b} x_{ab} S\left(E^{(i)}_{ba}\right)\in \mathrm{Mat}(d_i,k)$ then

$$\Delta^i_{nn}(x)=\frac{d_i}{d_n}\sum_{a,b,\gamma\lambda}\left[E^{(n)}_{\gamma\lambda}\leftharpoondown\left(x_{ab}E^{(n)}_{ab}\right)\right]\otimes S\left(E^{(n)}_{\lambda\gamma}\right)$$
$$=\frac{d_i}{d_n}\sum_{a,b,\gamma\lambda}\left[E^{(n)}_{\gamma\lambda}\leftharpoondown S\left({}^t x\right)\right]\otimes S\left(E^{(n)}_{\lambda\gamma}\right)$$

□

Proposition 5.6. *The equations (20) with $r=i=n$, $j<n$ hold in H if and only if $E^{(j)}\rightharpoonup x=x=x\leftharpoondown E^{(j)}$ for all $x\in\mathrm{Mat}(d_n,k)$.*

Proof. Note first that $\Delta^n_{nq}\left(E^{(n)}\right)=E^{(n)}\otimes E^{(q)}$ by Proposition 5.3. Using Propositions 5.4, 4.1 and 2.1 we can rewite the first equation in (20) with $r=i=n$, $j<n$ in the form

$$\begin{aligned}
x\otimes E^{(j)} &= \sum_{\alpha,\beta}\Delta^n_{nj}\left(E^{(j)}_{\alpha\beta}\rightharpoonup x\right)\left[E^{(n)}\otimes S\left(E^{(j)}_{\alpha\beta}\right)\right]\\
&= \sum_{\alpha,\beta,\gamma,\lambda}\left(E^{(j)}_{\gamma\lambda}*E^{(j)}_{\alpha\beta}\rightharpoonup x\right)\otimes E^{(j)}_{\gamma\lambda}S\left(E^{(j)}_{\alpha\beta}\right)\\
&= \sum_{\alpha,\beta,\gamma,\lambda}\left[\left(E^{(j)}_{\gamma\lambda}*S\left(E^{(j)}_{\alpha\beta}\right)\right)\rightharpoonup x\right]\otimes E^{(j)}_{\gamma\lambda}E^{(j)}_{\alpha\beta}\\
&= \sum_{\alpha,\beta,\gamma}\left[\left(E^{(j)}_{\gamma\alpha}*S\left(E^{(j)}_{\alpha\beta}\right)\right)\rightharpoonup x\right]\otimes E^{(j)}_{\gamma\beta}\\
&= \sum_{\beta,\gamma}\left[\delta_{\gamma\beta}E^{(j)}\rightharpoonup x\right]\otimes E^{(j)}_{\gamma\beta}=\left(E^{(j)}\rightharpoonup x\right)\otimes E^{(j)}
\end{aligned}\tag{24}$$

for any $x\in\mathrm{Mat}(d_n,k)$. Thus (24) means that $E^{(q)}\rightharpoonup x=x$ for all x.

Similarly the second equality $x\leftharpoondown E^{(q)}$ is equivalent to the second one from (20). □

Proof of Theorem 0.1. Suppose that $n>1$. Then $\mathrm{Mat}(d_n,k)$ is the unique component which is a Hopf ideal, Proposition 5.3.

Let $i<n$. By Proposition 5.5 and Proposition 5.6 we have

$$\Delta^i_{nn}(E^{(i)})=\frac{d_i}{d_n}\sum_{p,q=1}^{d_n}E^{(n)}_{qp}\otimes\left[S\left({}^tE^{(i)}\right)\rightharpoonup S\left(E^{(n)}_{pq}\right)\right]$$
$$=\frac{d_i}{d_n}\sum_{p,q=1}^{d_n}E^{(n)}_{qp}\otimes\left[E^{(i)}\rightharpoonup S\left(E^{(n)}_{pq}\right)\right]=\frac{d_i}{d_n}\sum_{p,q=1}^{d_n}E^{(n)}_{qp}\otimes S\left(E^{(n)}_{pq}\right)=d_i\mathcal{D}_n.$$

Suppose now that $n\geqslant 2$. Then $e_1\cdot E^{(1)}=0$ in H and therefore

$$\mathcal{D}_n\cdot\Delta^1_{nn}\left(E^{(1)}\right)=0.$$

On the other hand $\mathcal{D}_n\cdot\Delta^1_{nn}\left(E^{(1)}\right)=d_1\mathcal{D}_n^2=d_1\mathcal{D}_n\neq 0$, a contradiction. □

6. The case $n = 1$ with $\Delta^1_{11} = 0$

We shall remind some results in the case $n = 1$. For simplicity we put $E^{(1)} = E$, $E^{(1)}_{ij} = E_{ij}$, $d_1 = d$, $\Phi_1 = \Phi$, $U = U_1$.

Theorem 6.1. [A, ACh] *Suppose that $n = 1$ and $\operatorname{char} k$ is not a divisor of $2d^2$. Let $G = G(H^*)$ be the group of group-like elements of the dual Hopf algebra H^*. Assume that $\Delta^1_{11} = 0$ in (13), or equivalently $|G| = d^2$. Taking an isomorphic copy of H we can assume that the comultiplication in H has the form:*

$$\begin{aligned} \Delta(e_g) &= \sum_{h\in G} e_h \otimes e_{h^{-1}g} + \frac{1}{d}\sum_{i,j=1}^{d} E_{ij} \otimes \left(g^{-1} \rightharpoonup S\left(E_{ji}\right)\right); \\ \Delta(x) &= \sum_{g\in G} \left[(g \rightharpoonup x) \otimes e_g + e_g \otimes (x \leftharpoonup g)\right]; \\ \varepsilon(e_g) &= \delta_{g,1}, \quad \varepsilon(x) = 0; \quad S(x) = U\,{}^t x U^{-1} \end{aligned} \tag{25}$$

for all $x \in \operatorname{Mat}(d, k)$, where $U \in \operatorname{GL}(d, k)$ is a (skew-)symmetric matrix. There exists a faithful irreducible projective representation $\Phi : G \to \operatorname{PGL}(d, k)$ such that

$$g \rightharpoonup x = \Phi(g)x\Phi(g)^{-1}, \quad x \leftharpoonup g = S\left(\Phi(g)\right) x S\left(\Phi(g)\right)^{-1} \tag{26}$$

for any $x \in \operatorname{Mat}(d, k)$ and

$$[\Phi(g),\, S\left(\Phi(v)\right)] = 1 \tag{27}$$

in $\operatorname{PGL}(d, k)$ for all $g, v \in G$.

It is shown in [F, J2] that $G \simeq A \times A$ is a direct product of two copies of an Abelian group A of order d. The next theorem shows the converse. If we start with a irreducible projective representation of a group G from Theorem 6.1 we can obtain Hopf algebra structure.

Theorem 6.2. [ACh] *Suppose that G is Abelian group of order d^2 with direct decomposition $G \simeq A\times A$ for some Abelian group A of order d. The group G has a faithful irreducible projective representation Φ of degree d. There exists a (skew-) symmetric matrix $U \in \operatorname{GL}(d, k)$ such that (26), (27) holds where $S(x) = U\,{}^t x U^{-1}$ for any $x \in \operatorname{Mat}(d, k)$. Then an algebra H with direct decomposition (2) admits Hopf algebra structure defined in (26), (25).*

Each element $g \in G$ can be identified with an element $g \in H^*$ such that if

$$a = \sum_{h\ inG} \alpha_h e_h + x \in H, \qquad \alpha_h \in k, \quad x \in \operatorname{Mat}(d, k),$$

then $\langle g, a\rangle = \alpha_g$. The identification is an isomorphism of the group G and the group $G(H^*)$ of group-like elements in H^*.

If $d > 2$, then these Hopf algebras are neither commutative nor cocommutative.

Note that semisimple Hopf algebras of dimension $2p^2$ for an odd prime p in the same case as in Theorem 6.1 were classified in [Ma] in other terms. All Hopf algebras of dimension $2p^2$ for an odd prime p were classified in [N0]. All Hopf algebra of dimension $2p^2$ for an odd prime p were classified in [HN]. In Theorem 6.1, 6.2

we expand these results of an arbitrary n using the language of faithful projective representations of the group $G = G(H^*)$.

Note that by [J1], [J2] all projective faithful irreducible representations of the group $G \simeq A \times A$ as in Theorems 6.1, 6.2 have dimension d and obtained one from the other by a group automorphism of G. Any of these representations is monomial and it is induced by a one-dimensional representation of A.

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