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FACTOR CONGRUENCES IN SEMILATTICES

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Abstract. We characterize factor congruences in semilattices by using generalized notions of order ideal and of direct sum of ideals. When the semilattice has a minimum (maximum) element, these generalized ideals turn into ordinary (dual) ideals.

Resumen. En este trabajo damos una caracterización de las congruencias factor en semirretículos usando nociones generalizadas de ideal y suma directa de ideales. Cuando un semiretículo tiene elemento mínimo (máximo), estos ideales generalizados resultan ideales (duales) ordinarios.

1. Introduction

Semilattices are ordered structures that admit an algebraic presentation. Formally, a semilattice may be presented as a partially ordered set $\langle A, \leq\rangle$ such that for every pair $a, b \in A$ there exists the supremum or *join* $a \vee b$ of the set $\{a, b\}$. Equivalently, $\langle A, \vee\rangle$ is a semilattice if $\vee$ is an idempotent commutative semigroup operation. In this case we call A a $\vee$-*semilattice* ("join-semilattice"). We can also define a partial *meet* operation $a \wedge b$ in a $\vee$-semilattice that equals the infimum of $\{a, b\}$ whenever it exists.

The paradigmatic example of a semilattice is given by any family of sets that is closed under (finite) union. Actually, the free semilattice on n generators is given by the powerset of $\{0, \dots, n-1\}$ minus $\varnothing$. A reference for concepts of general algebra and ideals in semilattices is [2].

The aim of this paper is to obtain an inner characterization of direct product decompositions of semilattices akin to those in classical algebra. To attain this goal we represent these decompositions by means of factor congruences. A *factor congruence* is the kernel $\{\langle a, b\rangle \in A : \pi(a) = \pi(b)\}$ of a projection π onto a direct factor of A. Thus, a direct product representation $A \cong A_1 \times A_2$ is determined by the pair of *complementary* factor congruences given by the canonical projections $\pi_i : A \to A_i$.

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We prove that factor congruences and direct representations can be described by generalized notions of order ideal and of direct sum of ideals. When the semilattice has a minimum (maximum) element, these generalized ideals turn into ordinary (dual) ideals.

The main feature of this characterization is that it is completely analogous to similar definitions in the realm of classical algebra, as in ring theory. By using the partial $\wedge$ operation we devise a definition with the same (quantifier) complexity as the reader may encounter in classical algebra, and we obtain an equational relation between each element and its direct summands in a given decomposition.

By the end of the paper we apply the characterization to the bounded case and show that each of the axioms is necessary.

2. Generalized Direct Sums

The key idea in our characterization stems from the fact that in a direct product of join-semilattices there must exist "non-trivial" meets satisfying certain modularity and absorption laws with respect to join. And conversely, the existence of meets in a direct product implies the existence of them in each factor. We state without proof the latter property:

Claim 1. *For every $\vee$-semilattices C, D and elements $c, e \in C$ and $d, f \in D$, if $\langle c, d\rangle \wedge \langle e, f\rangle$ exists in $C \times D$, then $c \wedge e$ and $d \wedge f$ exist (and conversely).*

As an a example, a semilattice A is pictured in Figure 1. A is isomorphic to the direct product of its subsemilattices I_1 and I_2. According to this representation, the pair $\langle x_1, x_2\rangle$ corresponds to x. In spite of x not being expressible in terms of x_1, x_2 and $\vee$, we can recover x as the infimum $(x \vee x_1) \wedge (x \vee x_2)$. Other relations between x, x_1, x_2 and c (the only element in $I_1 \cap I_2$) may be found; next we choose four of them that hold exactly when x corresponds to $\langle x_1, x_2\rangle$ in a direct decomposition.

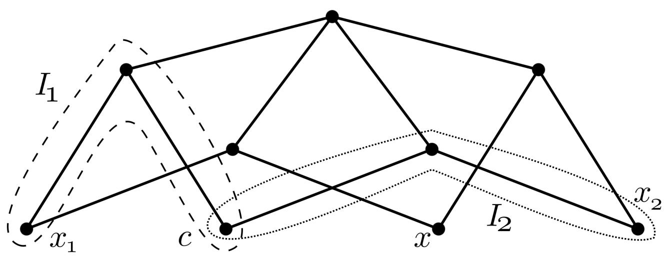

Figure 1. A semilattice A isomorphic to $I_1 \times I_2$.

In the following, we will write formulas in the extended language $\{\vee, \wedge\}$. The formula "$x \wedge y = z$" will be interpreted as "the infimum of $\{x, y\}$ exists and equals z":

$$\exists w : \ w \leq x, y \ \ \& \ (\forall u : u \leq x, y \ \rightarrow \ u \leq w) \ \& \ w = z,$$

unless explicitly indicated; in general, every equation $t_1 = t_2$ involving $\wedge$ will be read as "if either side exists, the other one also does and they are equal". It is easy to see that the associative law holds for the partial $\wedge$ operation in every $\vee$-semilattice, and we will apply it without any further mention.

Heretofore, A will be a $\vee$-semilattice and $c \in A$ arbitrary but fixed. Let $\phi_c(x_1, x_2, x)$ be the conjunction of the following formulas:

dist $x = (x \vee x_1) \wedge (x \vee x_2)$.
p₁ $x_1 = (x \vee x_1) \wedge (c \vee x_1)$.
p₂ $x_2 = (x \vee x_2) \wedge (c \vee x_2)$.
join $x_1 \vee x_2 = x \vee c$.

Definition 2. Assume I_1, I_2 are subsemilattices of A. We will say that A is the *c-direct sum* of I_1 and I_2 (notation: $A = I_1 \oplus_c I_2$) if and only if the following hold:

Mod1 For all $x, y \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \vee c \geq x_1 \vee x_2$ then

$$\big((x \vee x_1) \wedge (x \vee x_2)\big) \vee y = (x \vee y \vee x_1) \wedge (x \vee y \vee x_2).$$

Mod2 For all $x, y \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \leq x_1 \vee x_2$ then

$$\big((x \vee x_i) \wedge (c \vee x_i)\big) \vee y = (x \vee y \vee x_i) \wedge (c \vee y \vee x_i)$$

for $i = 1, 2$.

Abs For all $x_1, y_1 \in I_1$ and $z_2 \in I_2$, we have: $x_1 \wedge (y_1 \vee z_2) = x_1 \wedge (y_1 \vee c)$ (and interchanging I_1 and I_2).
exi $\forall x_1 \in I_1, x_2 \in I_2 \; \exists x \in A : \phi_c(x_1, x_2, x)$.
onto $\forall x \in A \; \exists x_1 \in I_1, x_2 \in I_2 : \; \phi_c(x_1, x_2, x)$.

In order to make notation lighter, we will drop the subindex "c". Let us notice that **exi** implies:

ori $\forall x_1 \in I_1, x_2 \in I_2 \; (x_1 \vee x_2 \geq c)$.

Lemma 3. *Assume $A = I_1 \oplus I_2$. Then $\phi(x_1, x_2, x)$ defines an isomorphism $\langle x_1, x_2\rangle \overset{\phi}{\mapsto} x$ between $I_1 \times I_2$ and A.*

Proof. First we'll see that the function $\langle x_1, x_2\rangle \overset{\phi}{\mapsto} x$ is well defined. Let us suppose $\phi(x_1, x_2, x)$ and $\phi(x_1, x_2, z)$ (there is at least one possible image by **exi**). We operate as follows:

$$\begin{aligned} z &= (z \vee x_1) \wedge (z \vee x_2) && \text{by } \textbf{dist} \text{ for } z \\ &= [z \vee ((x \vee x_1) \wedge (c \vee x_1))] \wedge \\ &\qquad \wedge [z \vee ((x \vee x_2) \wedge (c \vee x_2))] && \text{by } \mathbf{p_1} \text{ and } \mathbf{p_2} \text{for } x. \end{aligned}$$

By **join** for x we may apply **Mod2**:

$$= [(z \vee x \vee x_1) \wedge (\underline{c \vee z} \vee x_1)] \wedge [(z \vee x \vee x_2) \wedge (\underline{c \vee z} \vee x_2)]$$

and by **join** for z,

$$= [(x \vee z \vee x_1) \wedge (x_1 \vee x_2)] \wedge [(x \vee z \vee x_2) \wedge (x_1 \vee x_2)]$$

This last term is symmetric in x and z, hence we obtain $x = z$. The function thus defined by ϕ is surjective by **onto**; let us now prove that it is 1–1. If $\phi(x_1, x_2, x)$ and $\phi(y_1, y_2, x)$, we have:

$$\begin{aligned} x_1 &= x_1 \wedge (x_1 \vee x_2) && \\ &= x_1 \wedge (x \vee c) && \text{by } \textbf{join} \\ &= x_1 \wedge (y_1 \vee y_2) && \text{by } \textbf{join} \text{ again} \\ &= x_1 \wedge (y_1 \vee c) && \text{by } \textbf{Abs} \end{aligned}$$

hence $x_1 \leq y_1 \vee c$. Symmetrically, $y_1 \leq x_1 \vee c$ and in conclusion

$$x_1 \vee c = y_1 \vee c. \tag{1}$$

On the other hand,

$$\begin{aligned} x \vee y_1 &= ((x \vee x_1) \wedge (x \vee x_2)) \vee y_1 && \text{by } \textbf{dist} \\ &= (x \vee y_1 \vee x_1) \wedge (x \vee y_1 \vee x_2) && \text{by } \textbf{Mod1} \\ &= (x \vee y_1 \vee x_1) \wedge (x \vee y_1 \vee x_2 \vee c) && \text{by } \textbf{ori} \\ &= (x \vee y_1 \vee x_1) \wedge (x \vee c) && \text{since } x \vee c \geq x_1, y_1 \\ &= x \vee y_1 \vee x_1 && \text{by the same argument.} \end{aligned}$$

Symmetrically,

$$x \vee x_1 = x \vee y_1 \vee x_1.$$

and hence $x \vee x_1 = x \vee y_1$. Collecting this with (1) and using $\mathbf{p_1}$ we have

$$x_1 = (x \vee x_1) \wedge (c \vee x_1) = (x \vee y_1) \wedge (c \vee y_1) = y_1.$$

By the same reasoning, $x_2 = y_2$.

We will now prove that ϕ preserves $\vee$. Let us suppose that $\phi(x_1, x_2, x)$ and $\phi(z_1, z_2, z)$; since each of I_1, I_2 is a subsemilattice, we know that $x_j \vee z_j \in I_j$ for $j = 1, 2$. We have to see that $\phi(x_1 \vee z_1, x_2 \vee z_2, x \vee z)$. The property **join** is immediate. We now prove **dist**:

$$\begin{aligned} x \vee z &= \big((x \vee x_1) \wedge (x \vee x_2)\big) \vee z && \text{by } \textbf{dist} \text{ for } x \quad (2) \\ &= (x \vee x_1 \vee z) \wedge (x \vee x_2 \vee z) && \text{by } \textbf{Mod1} \quad (3) \\ &= (x \vee z \vee x_1 \vee z_1) \wedge (x \vee z \vee x_1 \vee z_2) \wedge && (4) \\ &\quad \wedge (x \vee z \vee x_2 \vee z_1) \wedge (x \vee z \vee x_2 \vee z_2) && (5) \end{aligned}$$

by **dist** for z and **Mod1**. Note that

$$\begin{aligned} x \vee z \vee x_1 \vee z_2 &= x \vee c \vee z \vee c \vee x_1 \vee z_2 && \text{by } \textbf{ori}, \ x_1 \vee z_2 = x_1 \vee z_2 \vee c \\ &= x_1 \vee x_2 \vee z_1 \vee z_2 \vee x \vee z && \text{by } \textbf{join} \text{ two times} \\ &\geq x \vee z \vee x_1 \vee z_1, && \end{aligned}$$

hence we can eliminate two terms in equation (5) and we obtain **dist** for $x \vee z$ as expected:

$$(x \vee z) = \big((x \vee z) \vee (x_1 \vee z_1)\big) \wedge \big((x \vee z) \vee (x_2 \vee z_2)\big).$$

We may obtain $\mathbf{p_1}$ and $\mathbf{p_2}$ similarly. We prove $\mathbf{p_1}$:

$$\begin{aligned} x_1 \vee z_1 &= ((x \vee x_1) \wedge (c \vee x_1)) \vee z_1 && \text{by } \mathbf{p_1} \text{ for } x \\ &= (x \vee z_1 \vee x_1) \wedge (c \vee z_1 \vee x_1) && \text{by } \mathbf{Mod2} \end{aligned}$$

By $\mathbf{p_1}$ for z followed by **Mod2** in each term of the meet,

$$= (x \vee z \vee x_1 \vee z_1) \wedge (x \vee c \vee x_1 \vee z_1) \wedge (c \vee z \vee x_1 \vee z_1) \wedge (c \vee x_1 \vee z_1) \tag{6}$$

and this equals

$$((x \vee z) \vee (x_1 \vee z_1)) \wedge (c \vee (x_1 \vee z_1)).$$

since the last term in (6) is less than or equal to the previous ones. □

Since $\phi(\cdot,\cdot,\cdot)$ defines an isomorphism (relative to I_1, I_2), there are canonical projections $\pi_j : A \to I_j$ with $j = 1, 2$ such that

$$\forall x \in A, x_1 \in I_1, x_2 \in I_2 \ : \ \phi(x_1, x_2, x) \iff \pi_1(x) = x_1 \text{ and } \pi_2(x) = x_2 \tag{7}$$

holds.

Theorem 4. *Let A be a $\vee$-semilattice and $c \in A$ arbitrary. The mappings*

$$\begin{array}{ccc} \langle \theta, \delta \rangle & \overset{\mathsf{I}}{\longmapsto} & \langle I_\theta, I_\delta \rangle \\ \langle \ker \pi_2, \ker \pi_1 \rangle & \overset{\mathsf{K}}{\longleftarrow\!\shortmid} & \langle I_1, I_2 \rangle \end{array}$$

where $I_\theta := \{a \in A : a\,\theta\,c\}$, are mutually inverse maps defined between the set of pairs of complementary factor congruences of A and the set of pairs of subsemilattices I_1, I_2 of A such that $A = I_1 \oplus_c I_2$.

Proof. We first prove that map I is well defined, i.e. $I_\theta \oplus_c I_\delta = A$ for every pair of complementary factor congruences θ, δ. It's clear that I_θ, I_δ are subsemilattices of A, and we know that the mapping $a \mapsto \langle a/\theta, a/\delta \rangle$ is an isomorphism between A and $A/\delta \times A/\theta$. Under this isomorphism, I_θ corresponds to $\{\langle a', c'' \rangle : a' \in A/\delta\}$ and I_δ to $\{\langle c', a'' \rangle : a'' \in A/\theta\}$, where $c' = c/\delta$ and $c'' = c/\theta$. Henceforth we will identify I_θ and I_δ with their respective isomorphic images and we will check the axioms for $I_\theta \oplus_c I_\delta = A$ in $A/\delta \times A/\theta$.

In order to see **Abs**, let $x_1 = \langle x, c'' \rangle \in I_\theta$, $y_1 = \langle y, c'' \rangle \in I_\theta$ and $z_2 = \langle c', z \rangle \in I_\delta$ as in the hypothesis and suppose that $x_1 \wedge (y_1 \vee z_2)$ exists. That is to say,

$$\langle x, c'' \rangle \wedge [\langle y, c'' \rangle \vee \langle c', z \rangle] = \langle x, c'' \rangle \wedge \langle y \vee c', c'' \vee z \rangle$$

exists. By Claim 1, we know that $x \wedge (y \vee c')$ must exist in A/δ, and in conclusion,

$$\begin{aligned} \langle x, c'' \rangle \wedge [\langle y, c'' \rangle \vee \langle c', z \rangle] &= \langle x \wedge (y \vee c'), c'' \wedge (c'' \vee z) \rangle = \\ &= \langle x \wedge (y \vee c'), c'' \rangle = \langle x, c'' \rangle \wedge [\langle y, c'' \rangle \vee \langle c', c'' \rangle]. \end{aligned}$$

That is to say, the other meet $x_1 \wedge (y_1 \vee c)$ exists and equals the former. The other half is analogous.

To check **Mod1**, we assume $x = \langle x', x'' \rangle$, $y = \langle y', y'' \rangle$, $x_1 = \langle x'_1, c'' \rangle \in I_\theta$ and $x_2 = \langle c', x''_2 \rangle \in I_\delta$. Note that $x \vee c \geq x_1 \vee x_2$ implies $x' \vee c' \geq x' \vee x'_1$ and $x'' \vee c'' \geq$

$x'' \vee x_2''$, and hence we have $x' \vee y' \vee c' \geq x' \vee y' \vee x_1'$ and $x'' \vee y'' \vee c'' \geq x'' \vee y'' \vee x_2''$. By applying Claim 1 we obtain:

$$\begin{aligned}\big((x \vee x_1) \wedge (x \vee x_2)\big) \vee y &= \langle((x' \vee x_1') \wedge (x' \vee c')) \vee y', ((x'' \vee c'') \wedge (x'' \vee x_2'')) \vee y''\rangle \\ &= \langle(x' \vee x_1') \vee y', (x'' \vee x_2'') \vee y''\rangle \\ &= \langle(x' \vee y' \vee x_1') \wedge (x' \vee y' \vee c'), (x'' \vee y'' \vee x_2'')\rangle \\ &= \langle(x' \vee y' \vee x_1') \wedge (x' \vee y' \vee c'), (x'' \vee y'' \vee x_2'') \wedge (x'' \vee y'' \vee x_2'')\rangle \\ &= (x \vee y \vee x_1) \wedge (x \vee y \vee x_2).\end{aligned}$$

The verification of **Mod2** is similar.

Let $x = \langle x', x''\rangle$, $x_1 = \langle x', c''\rangle$ and $x_2 = \langle c', x''\rangle$. The map $I_\theta \times I_\delta \to A/\delta \times A/\theta$ given by $\langle x_1, x_2\rangle \mapsto x$ is a bijection. Hence, to check **exi** and **onto** we just have to check that $\phi_c(x_1, x_2, x)$ holds:

dist We have:

$$\begin{aligned}x &= \langle x', x''\rangle \\ &= \langle x' \vee x', x'' \vee c''\rangle \wedge \langle x' \vee c', x'' \vee x''\rangle \\ &= [\langle x', x''\rangle \vee \langle x', c''\rangle] \wedge [\langle x', x''\rangle \vee \langle c', x''\rangle] \\ &= (x \vee x_1) \wedge (x \vee x_2),\end{aligned}$$

which is what we were looking for.

$\mathbf{p_1}$ We reason as follows

$$\begin{aligned}x_1 &= \langle x', c''\rangle \\ &= \langle x' \vee x', x'' \vee c''\rangle \wedge \langle x' \vee c', c'' \vee c''\rangle \\ &= [\langle x', x''\rangle \vee \langle x', c''\rangle] \wedge [\langle c', c''\rangle \vee \langle x', c''\rangle] \\ &= (x \vee x_1) \wedge (c \vee x_1),\end{aligned}$$

$\mathbf{p_2}$ It's totally analogous.

join Obvious.

Let π_θ and π_δ be the projections from $A/\delta \times A/\theta$ onto I_θ and I_δ, respectively, as defined by (7). Since $\phi_c(\langle x', c''\rangle, \langle c', x''\rangle, \langle x', x''\rangle)$, we have

$$\pi_\theta(\langle x', x''\rangle) = \langle x', c''\rangle \text{ and } \pi_\delta(\langle x', x''\rangle) = \langle c', x''\rangle \tag{8}$$

for all $x' \in A/\delta$ and $x'' \in A/\theta$.

We will prove now that each one of the maps I, K is the inverse of the other.

We first prove $\mathsf{K} \circ \mathsf{I} = Id$. For this we have to show that $\ker \pi_\delta = \theta$ and $\ker \pi_\theta = \delta$. Let $a = \langle a', a''\rangle$ and $b = \langle b', b''\rangle$. By using (8) we have:

$$\ker \pi_\delta = \{\langle a, b\rangle : \pi_\delta(a) = \pi_\delta(b)\} = \{\langle a, b\rangle : \langle c', a''\rangle = \langle c', b''\rangle\} = \{\langle a, b\rangle : a'' = b''\}$$

i.e., $\ker \pi_\delta = \{\langle a, b\rangle : a/\theta = b/\theta\} = \theta$. The proof for $\ker \pi_\theta = \delta$ is entirely analogous.

Now we show $\mathsf{I} \circ \mathsf{K} = Id$. We check $I_{\ker \pi_2} = I_1$. By taking $x = c$ in **onto** we can deduce $c \in I_1 \cap I_2$, $\phi(c, c, c)$ and hence $\pi_1(c) = \pi_2(c) = c$. By inspection it is immediate that $\phi(a, c, a)$ for all a; in particular, for $a \in I_1$. Hence we have $\pi_2(a) = c = \pi_2(c)$ for all $a \in I_1$ and then $I_1 \subseteq I_{\ker \pi_2}$. Now take $a \in I_{\ker \pi_2}$; we have

$\pi_2(a) = c$ and then $\phi(a_1, c, a)$ for some $a_1 \in I_1$. By **join** we have $a_1 \vee c = a \vee c$. Now considering $\mathbf{p_1}$ and **dist** we obtain:

$$a_1 = (a \vee a_1) \wedge (c \vee a_1) = (a \vee a_1) \wedge (c \vee a) = a,$$

therefore $a \in I_1$, proving $I_{\ker \pi_2} \subseteq I_1$. We can obtain $I_{\ker \pi_1} = I_2$ similarly. □

These results can be easily extended for the case of semilattices $\langle A, f_1, f_2, \ldots \rangle$ with *operators*, i.e., operations on A preserving $\vee$: $f_i(x \vee y) = f_i x \vee f_i y$. We only have to include in the definition of $I_1 \oplus_c I_2$ some preservation axioms analogous to **Mod1** and **Mod2**. In the case of one unary operator i, these are:

(1) I_1, I_2 are f-subalgebras,
(2) For all $x \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \vee c \geq x_1 \vee x_2$ then
$$f\big((x \vee x_1) \wedge (x \vee x_2)\big) \geq f(x \vee x_1) \wedge f(x \vee x_2).$$
(3) For all $x \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \vee c \leq x_1 \vee x_2$ then
$$f\big((x \vee x_i) \wedge (c \vee x_i)\big) \geq f(x \vee x_i) \wedge f(c \vee x_i)$$
for $i = 1, 2$.

If f is just a monotone operation, we may also recover Theorem 4 by adding:

For all $x \in A$, $x_1 \in I_1$ and $x_2 \in I_2$, if $x \vee c = x_1 \vee x_2$ then $fx \vee c = fx_1 \vee fx_2$.

Unique Factorization and Refinement. It is well-known that semilattices have the property of unique factorizability. In general, connected posets* have the *refinement property*: every two finite direct decompositions admit a common refinement. This was proved by Hashimoto [3].

Furthermore, Chen [1] showed that semilattices have a "strong form" of unique factorizability, namely the *strict refinement property (SRP)* (see, for instance, [4, 5]). This property is equivalent to the fact that the factor congruences form a Boolean algebra. By using our representation of direct decompositions it can be proved that semilattices have the SRP. We now translate the lattice operations on congruences to the set of "direct summands" of A, $\{I \leq A : \exists J \leq A, I \oplus J = A\}$. Set intersection of ideals is the meet. To describe the join, consider two pairs of ideals $I_1 \oplus I_2 = A = J_1 \oplus J_2$. Then, there exists a subsemilattice $I_2 \circledvee J_2$ of A such that $(I_1 \cap J_1) \oplus (I_2 \circledvee J_2) = A$. Namely, if we assume (by using Theorem 4) that $A = A_1 \times A_2$, $I_1 = \{\langle x', c'' \rangle : x' \in A_1\}$ and $I_2 = \{\langle c', x'' \rangle : x'' \in A_2\}$, then

$$I_2 \circledvee J_2 := \pi_1(J_2) \times A_2 = \pi_1(J_2) \times \pi_2(I_2).$$

3. Some Particular Cases

In the case our semilattice A has a 0 or a 1 —minimum or maximum element, resp.— the characterization of factor congruences by generalized ideals takes a much simpler form. By taking c to be the minimum (maximum), it turns out that the subsemilattices I_1 and I_2 are order (dual) ideals.

*I.e., such that the order relation gives rise to a connected graph.

Theorem 5. *Let A be a semilattice with 0. Then $A = I_1 \oplus_0 I_2$ if and only if $I_1, I_2 \leq A$ satisfy:*

Abs *For all $x_1, y_1 \in I_1$ and $z_2 \in I_2$, we have: $x_1 \wedge (y_1 \vee z_2) = x_1 \wedge y_1$ (and interchanging I_1 and I_2).*

onto $I_1 \vee I_2 = A$.

Moreover, I_1 and I_2 are ideals of A.

Proof. Since $y_1 \vee 0 = y_1$, **Abs** is equivalent to its new form as stated; and **Mod1**, **Mod2** are trivially true. The **join** part of ϕ_0 reduces to "$x_1 \vee x_2 = x$" and from here we conclude that **dist**, $\mathbf{p_1}$ and $\mathbf{p_2}$ hold trivially. Hence **exi** simplifies to

$$\forall x_1 \in I_1, x_2 \in I_2\ \exists x \in A: \ x_1 \vee x_2 = x,$$

which holds trivially, and **onto** reduces to:

$$\forall x \in A \exists x_1 \in I_1, x_2 \in I_2 \ : \ x_1 \vee x_2 = x.$$

and this is equivalent to $I_1 \vee I_2 = A$.

To see that both of I_1 and I_2 are ideals, we observe that by Theorem 4 each of them is the θ-class of 0 for some congruence θ, hence they are downward closed. □

Theorem 6. *Let A be a semilattice with 1. Then $A = I_1 \oplus_1 I_2$ if and only if $I_1, I_2 \leq A$ satisfy:*

exi' $I_1 \vee I_2 = \{1\}$

onto' $I_1 \wedge I_2 = A$.

Mod1' $\forall x_1 \in I_1, x_2 \in I_2 : \ \forall y \ (x_1 \wedge x_2) \vee y = (y \vee x_1) \wedge (y \vee x_2)$,

Moreover, I_1 and I_2 are dual ideals (viz. upward closed sets) of A.

Proof. We may eliminate **Abs** and **Mod2** since they are trivial when $c = 1$ (this can be seen by considering **ori** after Definition 2). Formulas $\mathbf{p_1}$ and $\mathbf{p_2}$ turn into

$$x_1 = x \vee x_1 \qquad x_2 = x \vee x_2$$

which are equivalent to say $x_1, x_2 \geq x$. Using this, we reduce **dist** to:

$$x = x_1 \wedge x_2,$$

and then we may eliminate $\mathbf{p_1}$ and $\mathbf{p_2}$ in favor of this last formula. **join** is equivalent to $x_1 \vee x_2 = 1$, and hence **exi** reduces to the statement "for all $x_1 \in I_1, x_2 \in I_2$, $x_1 \vee x_2 = 1$ and $x_1 \wedge x_2$ exists". We may state the first part as $I_1 \vee I_2 = \{1\}$, and may condense the second part with **onto** writing $I_1 \wedge I_2 = A$.

It remains to take care of **Mod1**. The hypothesis $x \vee c \geq x_1 \vee x_2$ trivializes when $c = 1$. It is obvious that **Mod1'** implies **Mod1**, by taking $y := x \vee y$. And we may obtain the new version under the hypotheses of the theorem by taking $x := x_1 \wedge x_2$ in **Mod1**. □

4. Independence and necessity of the axioms

We check independence and necessity by providing semilattices with a distinguished element c and a pair of ideals I_1 and I_2 such that they satisfy every axiom except one and it is not the case that $A = I_1 \oplus_c I_2$.

It is immediate that the minimal non-modular lattice N_5 provide two such counterexamples showing that **Abs** and **Mod1** are necessary and independent (see Figure 2 (a) and (b), respectively). In Figure 2 (a) we have taken c to be the minimum element, hence Theorem 5 (and its proof) applies and we only should check **onto**, which is obvious. Nevertheless, **Abs** fails for the labeled elements x_1, y_1 and z_2. Note that **Abs** and **onto** imply $I_1 \cap I_2 = \{c\}$, but this sole assumption does not suffices to prove uniqueness of representations: in this example we have $\phi_c(x_1, z_2, 1)$ and $\phi_c(y_1, z_2, 1)$.

In Figure 2 (b) c is the maximum of the semilattice, hence by Theorem 6 (and its proof) we only have to check **exi'** and **onto'**, again obvious, and **Mod1** fails for this model.

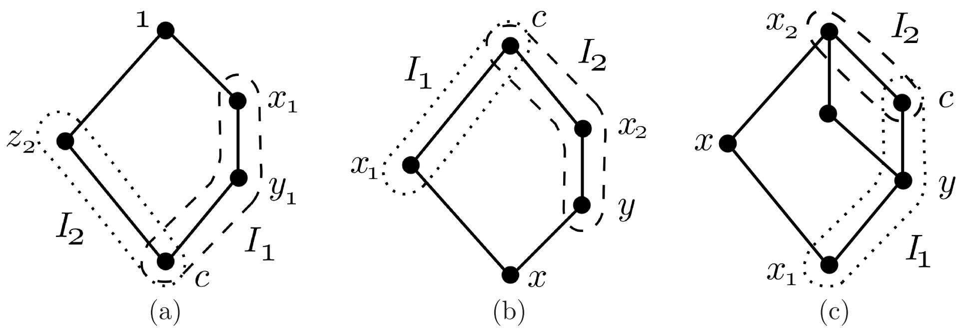

Figure 2. Counterexamples.

To see that **Mod2** is necessary and independent is almost as easy as with the other axioms. Consider the semilattice pictured in Figure 2 (c). First note that every element in I_2 is greater than or equal to every element of I_1. From this observation it is immediate to see that **Mod1** and **Abs** hold and that $\phi_c(z_1, z_2, z)$ (where $z_1 \in I_1$ and $z_2 \in I_2$) may be simplified as follows:

$z = (z \vee z_1) \wedge (z \vee z_2) = z \vee z_1$	since $z_1 \leq z_2$
$z_1 = (z \vee z_1) \wedge (c \vee z_1) = z \wedge c$	by the previous line and $c \geq z_1$
$z_2 = (z \vee z_2) \wedge (c \vee z_2) = (z \vee z_2) \wedge z_2$	(holds trivially)
$z_1 \vee z_2 = z \vee c$	and hence $z_2 = z \vee c$

Therefore, $\phi_c(z_1, z_2, z)$ is equivalent to $(z_1 = z \wedge c)$ & $(z_2 = z \vee c)$ and it is now evident that **exi** and **onto** also hold. Finally, consider the labeled elements x, x_1,

x_2 and y. We have $x \leq x_1 \vee x_2$, and if **Mod2** were true we should have:

$$\begin{aligned}\big((x \vee x_1) \wedge (c \vee x_1)\big) \vee y &= (x \vee y \vee x_1) \wedge (c \vee y \vee x_1)\\ (x \wedge c) \vee y &= x_2 \wedge c\\ x_1 \vee y &= c\\ y &= c,\end{aligned}$$

which is an absurdity.

The last two counterexamples show that **onto** and **exi** are both independent and necessary. In Figure 3 (a) there exist no pair $\langle x_1, x_2\rangle \in I_1 \times I_2$ such that $\phi_c(x_1, x_2, x)$, and in Figure 3 (b) there exists no x such that $\phi_c(x_1, x_2, x)$.

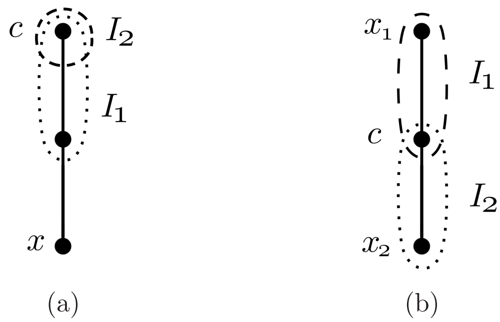

FIGURE 3. Counterexamples.

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