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# WIENER'S LEMMA: PICTURES AT AN EXHIBITION

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ABSTRACT. In this expository paper we present various general extensions of the celebrated Wiener's Tauberian Lemma, outline several ingredients that are often used in proofs and discuss a few applications to localization of frames.

## 1. INTRODUCTION

The title of this paper is inspired by two different sources. The first one is, of course, the famous suite by M. Mussorgsky which provided a refreshing background in time of this writing. The second, less obvious motivation comes from an excellent expository paper [30] by K. Gröchenig which also bears a musical name and deals with various aspects of Wiener's lemma. It may, indeed, seem presumptuous to follow [30] with another expository paper on roughly the same subject. The purpose of this paper, however, is not to present an all-in-one account of the history and modern development of Wiener's lemma and related results. Neither is this a set of lecture notes for beginning graduate students, even though they might find it useful as an introduction and an abbreviated reference guide. Our primary purpose is to exhibit a few sketches that may show the reader the versatility of the subject and, hopefully, enrich their appreciation. We strive to achieve a delicate balance and make the presented ideas clear with the least amount of detail possible. Some of the comments that are not necessary for the exposition but, nonetheless, are too interesting to omit are included in remarks.

We also offer a slightly different perspective compared to [30] as well as mention a few recent developments that became available in the past year [7, 32, 39, 40, 55]. An attentive observer will notice the contours of the results that are yet to be proven; however, as any prudent artist would do, we will refrain from unveiling such results prematurely.

The first of our sketches emphasizes various ways in which the statement of Wiener's lemma can be looked at. We shall see that the essence of this kind of results is in the spectral invariance with respect to different Banach algebras or, more precisely, in the preservation of spectral decay by the inverse elements. The second sketch exhibits several kinds of spectral decay which are preserved (or nearly preserved) under inversion. There we also highlight the relation between the spectral decay and the smoothness of elements which is akin to the relation between the smoothness of a periodic function and the decay of its Fourier coefficients. The

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following sketch briefly outlines the ideas typically used in the proofs: holomorphic extensions, "boot-strap" estimates, Brandenburg's trick, Bochner-Phillips theorem, etc. In the final sketch, we show the importance of the spectral decay in the study of localization of frames — the subject that was largely responsible for a flurry of recent developments concerning Wiener's lemma.

At a risk of carrying the analogy too far, we suggest that the eclectic nature of this paper is somewhat similar to that of Mussorgsky's suite and, while we certainly don't claim that this paper is a world-class masterpiece, we hope that the style will not have a detrimental effect upon the exposition. We certainly tried to do our best to make the reading enjoyable.

## 2. Various guises of Wiener's lemma and its extensions

The classical Wiener's Tauberian Lemma [59] states that if a (continuous) periodic function $f$ has an absolutely convergent Fourier series and never vanishes then the function $1/f$ also has an absolutely convergent Fourier series. In this section, we guide the reader through a promenade of equivalent or more general statements that culminates with a formulation that, on one hand, closely resembles the original and, on the other hand, gives one of the most far-reaching generalizations available.

Let us begin by introducing some standard notation. By $C(\mathbb{R}^d)$ we denote the Banach algebra of bounded continuous (complex-valued) functions on $\mathbb{R}^d$, $d \in \mathbb{N}$. We let $C(\mathbb{T}^d)$ be the subalgebra of $C(\mathbb{R}^d)$ containing all $\mathbb{Z}^d$-periodic functions. We use the customary identification of the group $\mathbb{T}^d$ with $[0,1)^d$. As usually, $\ell^p(\mathbb{Z}^d)$, $1 \le p \le \infty$, is the Banach space of $p$-summable (bounded for $p = \infty$) complex sequences indexed by $\mathbb{Z}^d$, and $L^p(\mathbb{R}^d)$ and $L^p(\mathbb{T}^d)$ are the Banach space of equivalence classes of $p$-integrable functions on $\mathbb{R}^d$ or $\mathbb{T}^d$, respectively. The norms in these spaces are denoted by $\|\cdot\|_p$, $p \in [1,\infty]$. The Fourier series of a function $a \in C(\mathbb{T}^d)$ is then given by

$$a(t) \simeq \sum_{n\in\mathbb{Z}^d} a_n e^{2\pi i n\cdot t}, \tag{2.1}$$

where the Fourier coefficients are

$$a_n = \int_{\mathbb{T}^d} a(t) e^{-2\pi i n\cdot t} dt, \; n \in \mathbb{Z}^d. \tag{2.2}$$

With this notation we get the following classical statement.

**Theorem 2.1.** [59]. *Assume that a function $a \in C(\mathbb{T})$ has a Fourier series given by (2.1), the sequence $(a_n)_{n\in\mathbb{Z}}$ belongs to $\ell^1(\mathbb{Z})$, and $a(t) \neq 0$ for all $t \in \mathbb{T}$. Then $b = 1/a \in C(\mathbb{T})$ has an absolutely convergent Fourier series $b(t) = \sum_{n\in\mathbb{Z}} b_n e^{2\pi i n\cdot t}$, that is the sequence $(b_n)_{n\in\mathbb{Z}}$ also belongs to $\ell^1(\mathbb{Z})$.*

The algebra of periodic functions with summable Fourier coefficients is often called the Wiener algebra. Rephrasing Theorem 2.1 in the Banach algebra terminology, we get that the Wiener algebra is *inverse closed* in $C(\mathbb{T})$.

Let us now restate the above theorem by considering the multiplication action of the algebra $L^\infty(\mathbb{T})$ on the Banach space $L^2(\mathbb{T})$. For $a \in L^\infty(\mathbb{T})$ we define the operator $A_a : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by $(A_a f)(t) = a(t)f(t)$, $f \in L^2(\mathbb{T})$. Clearly, $A_a$ belongs to the Banach algebra $B(L^2(\mathbb{T}))$ of all bounded linear operators on $L^2(\mathbb{T})$ and $\|A_a\| = \|a\|_\infty$. Observe that the operator $A_a$ is invertible in $B(L^2(\mathbb{T}))$ if and only if $b = 1/a \in L^\infty(\mathbb{T})$, i.e., $a$ is invertible in $L^\infty(\mathbb{T})$, in which case, $A_b = A_a^{-1}$. Let us now consider the matrix $\mathcal{A}_a$ of the operator $A_a$ with respect to the Fourier basis of $L^2(\mathbb{T})$. Using (2.1) we see that

$$\mathcal{A}_a = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_1 & a_2 & \ddots \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ \ddots & a_{-2} & a_{-1} & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

is the bi-infinite Laurent matrix comprised of the Fourier coefficients of the function $a$. We say that a bi-infinite Laurent matrix has summable diagonals if its rows (equivalently, columns) belong to $\ell^1(\mathbb{Z})$. With the observations just made it is obvious that the following statement is merely a reformulation of Theorem 2.1.

**Theorem 2.2.** *Assume that a bi-infinite Laurent matrix has summable diagonals and defines a (bounded) invertible operator on $\ell^2(\mathbb{Z})$. Then the inverse (bi-infinite Laurent) matrix also has summable diagonals.*

*Remark* 2.1. A detailed proof of the equivalence can be found, for example, in [30]. It is also explained there how this reformulation represents the simplest form of symbolic calculus – a mapping from a function space to an operator space that preserves certain useful properties.

At this point a person interested in more general statements would naturally ask if the condition for the matrix to be Laurent may somehow be eliminated. Somewhat surprisingly, despite many formidable developments in the area that we shall not discuss here, a positive answer to this question was not given in the first 55 years since Theorem 2.1 was proven. Shortly afterwards, however, the result appeared in [9, 26, 42] which were all written independently and at about the same time. To state the theorem we need to expand the definition of "summable diagonals" to general bi-infinite matrices $\mathcal{A} = (a_{jk})$, $j, k \in \mathbb{Z}$. In fact, since the dimension of the index set plays only a minor (if any) role in the proofs, we shall assume that $j, k \in \mathbb{Z}^d$.

**Definition 2.1.** We say that a matrix $\mathcal{A} = (a_{jk})$, $j, k \in \mathbb{Z}^d$, has *summable diagonals* if

$$\|\mathcal{A}\|_{\mathcal{W}} := \sum_{n \in \mathbb{Z}^d} \sup_{j-k=n} |a_{jk}| < \infty. \tag{2.3}$$

It is an easy exercise to show that the set of matrices satisfying (2.3) forms a Banach algebra $\mathcal{W}$ with respect to this norm. Identifying the matrices with the operators on $\ell^p$, as we shall do from now on, it is nearly obvious that $\mathcal{W} \subset$

$B(\ell^p(\mathbb{Z}^d))$. This generalization of the Wiener algebra is sometimes called the Baskakov-Gohberg-Sjöstrand class (see [30] and references therein).

The following non-commutative extension of the Wiener's Lemma follows from the corresponding theorems in [9, 26, 42].

**Theorem 2.3.** *Assume that $\mathcal{A} \in \mathcal{W}$ is invertible as an operator on $B(\ell^p(\mathbb{Z}^d))$ for some $p \in [1, \infty)$. Then $\mathcal{A}^{-1} \in \mathcal{W}$.*

*Remark* 2.2. The case $p = \infty$ has to be treated separately since a non-zero bounded operator on $\ell^\infty(\mathbb{Z}^d)$ may have a zero matrix.

It turns out that many of the known proofs of the above theorem extend easily to the case of block matrices. In special cases, the Wiener norm in (2.3) becomes, essentially, a summability method for the Fourier series. More generally, instead of considering a scalar matrix, we let each entry $a_{jk}$, $j, k \in \mathbb{Z}^d$, be an operator from $L(X_j, X_k)$ – the Banach space of bounded linear operators between (complex) Banach spaces $X_j$ and $X_k$. In this case $|a_{jk}|$ means $\|a_{jk}\|_{L(X_j,X_k)}$ and $\mathcal{A} \in \mathcal{W}$ defines a bounded linear operator on the Banach spaces $\mathcal{X}_p = \ell^p(\mathbb{Z}^d, (X_j))$, $p \in [1, \infty)$, which consist of the sequences $x = (x(j))_{j\in\mathbb{Z}^d}$, $x(j) \in X_j$, such that $\|x\|_p = \left(\sum_{j\in\mathbb{Z}^d} \|x(j)\|^p_{X_j}\right)^{1/p} < \infty$. Theorem 2.3 remains valid [11] if $B(\ell^p(\mathbb{Z}^d))$ is replaced with $B(\mathcal{X}_p)$.

The formulation of Theorem 2.3 has no mention of Fourier series. It is natural to ask if one can explain the Wiener norm in (2.3) in terms of some kind of a Fourier series. In [11, 12], Baskakov introduced his notion of Fourier series of linear operators precisely for this purpose. To present the concept, we use the following notation.

From now on $\mathcal{X}$ and $\mathcal{Y}$ will be complex Banach spaces and $\mathfrak{B}$ – a Banach algebra with a unit denoted by $e = e_{\mathfrak{B}}$. As usually, $L(\mathcal{X}, \mathcal{Y})$ will be the Banach space of all bounded linear operators between $\mathcal{X}$ and $\mathcal{Y}$ and $B(\mathcal{X})$ will be the Banach algebra $L(\mathcal{X}, \mathcal{X})$ with the unit being the identity operator $I$. This algebra and some of its closed subalgebras provide the most common examples of $\mathfrak{B}$. To simplify the exposition, in this paper we restrict ourselves to the case of operators acting on one Banach space (Banach algebras) and refer to [7, 12] for the results involving operators between different Banach spaces.

The Fourier series for elements of $\mathfrak{B}$ will be defined using a bounded periodic $d$-parameter group of automorphisms $\mathcal{T}(t) = \mathcal{T}_{\mathfrak{B}}(t) \in B(\mathfrak{B})$, $t \in \mathbb{R}^d$. We remind the reader that an automorphism means that $\mathcal{T}(t)e = e$ and $\mathcal{T}(t)(ab) = (\mathcal{T}(t)a)(\mathcal{T}(t)b)$, $a, b, e \in \mathfrak{B}$, $t \in \mathbb{R}^d$. We shall assume that the group of automorphisms is strongly continuous, that is, the function $a_{\mathcal{T}} : \mathbb{R}^d \to \mathfrak{B}$ given by $a_{\mathcal{T}}(t) = \mathcal{T}(t)a$ is continuous for every $a \in \mathfrak{B}$. In general, elements of $\mathfrak{B}$ with the above property are called $\mathcal{T}$-continuous. For our purposes strong continuity can be assumed without loss of generality even though it is typically not the case if $\mathfrak{B} = B(\mathcal{X})$. No loss of generality occurs due to the fact that the Wiener algebra $\mathcal{W}$ defined below is always a subset of the subalgebra $\mathfrak{B}_{\mathcal{T}} \subset \mathfrak{B}$ of $\mathcal{T}$-continuous elements of $\mathfrak{B}$. It is also worth mentioning that the subalgebra $\mathfrak{B}_{\mathcal{T}}$ is always inverse closed in $\mathfrak{B}$.

**Definition 2.2.** The Fourier series of an element $a \in \mathfrak{B}$ with respect to a strongly continuous $\mathbb{Z}^d$-periodic group of automorphisms $\mathcal{T}$ is defined as the abstract Fourier series of the (continuous) periodic function $a_{\mathcal{T}}$. In particular, it is given by the standard formula (2.1)

$$a_{\mathcal{T}}(t) \simeq \sum_{n\in\mathbb{Z}^d} e^{2\pi i n\cdot t} a_n,\ t \in \mathbb{R}^d,$$

where the Fourier coefficients $a_n \in \mathfrak{B}$ are computed via

$$a_n = \int_{\mathbb{T}^d} e^{-2\pi i n\cdot t} \mathcal{T}(t) a dt.$$

Not surprisingly, the Wiener algebra $\mathcal{W} = \mathcal{W}(\mathfrak{B}, \mathcal{T})$ of elements of $\mathfrak{B}$ with respect to the group $\mathcal{T}$ consists of $a \in \mathfrak{B}$ such that

$$\|a\|_{\mathcal{W}} = \sum_{n\in\mathbb{Z}^d} \|a_n\|_{\mathfrak{B}} < \infty. \tag{2.4}$$

To see that (2.3) is a special case of (2.4) one needs to make specific choices of $\mathcal{T}$ and $\mathfrak{B}$ as in [11, 12]. In the matrix case, given $\mathcal{X} = \mathcal{X}_p = \ell^p(\mathbb{Z}^d, (X_j))$, $p \in [1, \infty)$, the group $\mathcal{T}$ is defined via

$$((\mathcal{T}(t)a))x(j) = \sum_{k\in\mathbb{Z}^d} e^{2\pi i (j-k)\cdot t} a_{jk} x(k),\ x \in \mathcal{X}_p,\ t \in \mathbb{R}^d,\ j \in \mathbb{Z}^d,$$

where $a = (a_{jk})_{j,k\in\mathbb{Z}^d} \in B(\ell^p(\mathbb{Z}^d))$. A straightforward computation shows that the Fourier coefficients $a_n$, $n \in \mathbb{Z}^d$, of the element $a$ with respect to $\mathcal{T}$ are then precisely the "one-diagonal" matrices with entries $a_{jk}\delta_{n,j-k}$, $j, k \in \mathbb{Z}^d$, where $\delta$ is the usual Kronecker delta. To provide a better understanding of the above group $\mathcal{T}$ we observe that it arises by conjugation of $a$ with a modulation representation $M$ on $\mathcal{X}_p$:

$$M\colon\ \mathbb{R}^d \to B(\mathcal{X}_p),\quad M(t)x(k) = e^{2\pi i k\cdot t},\ x \in \mathcal{X}_p,\ t \in \mathbb{R}^d.$$

More precisely, we have $\mathcal{T}(t)a = M(t)aM(-t)$ which clearly implies that $\mathcal{T}$ is indeed a group of automorphisms. This group is not strongly continuous on all of $B(\mathcal{X})$ but, as mentioned above, the Wiener algebra $\mathcal{W}$ is contained in $(B(\mathcal{X}))_{\mathcal{T}}$ which will be our algebra $\mathfrak{B}$ in this case. By the way, this restriction is sufficient to include the case of $\mathcal{X}_\infty$.

In a more general case, one can define the group of automorphisms in the following way. One starts with a Banach space $\mathcal{X}$ equipped with a disjunctive resolution of the identity that has an additional property that guarantees unconditional convergence.

**Definition 2.3.** A sequence $\mathcal{E} = \{E_n,\ n \in \mathbb{Z}^d\}$ of idempotents (projections) in $B(\mathcal{X})$ is called a *resolution of the identity* if $x = \sum\limits_{n\in\mathbb{Z}^d} E_n x$ for all $x \in \mathcal{X}$, where the series converges unconditionally. A resolution of the identity $\mathcal{E}$ is *disjunctive* if $E_m E_n = 0$ whenever $m \neq n$.

If a disjunctive resolution of the identity satisfies

$$\left\| \sum_{n\in\mathbb{Z}^d} \alpha_n E_n x \right\| \le C\|x\|,\ C>0,\ x\in\mathcal{X},\ \alpha_n=\pm1,\ n\in\mathbb{Z}^d,$$

then the group $\mathcal{T}$ and its action on $B(\mathcal{X})$ is well defined by the following unconditionally convergent series

$$(\mathcal{T}(t)a)x = \sum_{m,n\in\mathbb{Z}^d} e^{2\pi i(m-n)\cdot t} E_m a E_n x,\ x\in\mathcal{X},\ a\in B(\mathcal{X}),\ t\in\mathbb{R}^d.$$

This group is also constructed using the conjugation by a representation which, this time, is given by

$$U:\mathbb{R}^d\to B(\mathcal{X}),\ U(t)x = \sum_{n\in\mathbb{Z}^d} e^{2\pi i n\cdot t} E_n x,\ x\in\mathcal{X},\ t\in\mathbb{R}^d.$$

We refer to [7, Section 10.1] for more details and an alternative way of defining $\mathcal{T}$.

The above observations suggest that different representations of $\mathbb{R}^d$ play an important role in generalizing Wiener's lemma. This is, indeed, the case when specific examples are considered. The following observation shows, however, that for the purposes of the theory one only needs to consider the algebras $C(\mathbb{R}^d,\mathfrak{B})$ which are the generalizations of $C(\mathbb{R}^d)$ that contain $\mathfrak{B}$-valued functions. We shall also consider the $\mathfrak{B}$-valued $\ell^p$ and $L^p$ spaces defined in the obvious way. Clearly, $C(\mathbb{R}^d,\mathfrak{B})$ is a Banach algebra with respect to pointwise multiplication and the norm $\|\cdot\|_\infty$ given by

$$\|a\|_\infty = \sup_{t\in\mathbb{R}^d}\|a(t)\|_{\mathfrak{B}},\quad a\in C(\mathbb{R}^d,\mathfrak{B}).$$

As before, the subalgebra of periodic functions is denoted by $C(\mathbb{T}^d,\mathfrak{B})$. The natural group of automorphisms of $C(\mathbb{R}^d,\mathfrak{B})$ is given by translations

$$T:\mathbb{R}^d\to B(C(\mathbb{R}^d,\mathfrak{B})),\ T(t)a(s)=a(s+t),\ t\in\mathbb{R}^d,\ a\in C(\mathbb{R}^d,\mathfrak{B}).$$

Clearly, the group $T$ becomes $\mathbb{Z}^d$-periodic and strongly continuous when restricted to $C(\mathbb{T}^d,\mathfrak{B})$. Observe now that, given a strongly continuous periodic group $\mathcal{T}$ of automorphisms of a Banach algebra $\mathfrak{B}$, invertibility of $a$ in $\mathfrak{B}$ and in $\mathcal{W}(\mathfrak{B},\mathcal{T})$ is equivalent to that of $a_\mathcal{T}$ in $C(\mathbb{T}^d,\mathfrak{B})$ and in $\mathcal{W}(C(\mathbb{T}^d,\mathfrak{B}),T)$, respectively. Moreover, it is easily seen that the Fourier coefficients satisfy $(a_n)_\mathcal{T}=(a_\mathcal{T})_n$. Hence, we get the following generalization of Theorem 2.3 that appears, for example, in [7, Section 10.2].

**Theorem 2.4.** *Assume that a $\mathfrak{B}$-valued function $a\in\mathcal{W}(C(\mathbb{T}^d,\mathfrak{B}),T)$ is invertible in $C(\mathbb{T}^d,\mathfrak{B})$. Then $a^{-1}\in\mathcal{W}(C(\mathbb{T}^d,\mathfrak{B}),T)$.*

Clearly, a restatement of the above result gives Theorem 2.1 for $\mathfrak{B}$-valued continuous periodic functions with Fourier coefficients in $\mathfrak{B}$. We provide the exact formulation for future reference.

**Theorem 2.5.** *Assume that a function $a \in C(\mathbb{T}^d, \mathfrak{B})$ has a Fourier series given by (2.1), the sequence $(a_n)_{n\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d, \mathfrak{B})$, and $a(t)$ is invertible in $\mathfrak{B}$ for all $t \in \mathbb{T}^d$. Then $b = a^{-1} \in C(\mathbb{T}^d, \mathfrak{B})$ has an absolutely convergent Fourier series $b(t) = \sum_{n\in\mathbb{Z}^d} b_n e^{2\pi i n\cdot t}$, that is the sequence $(b_n)_{n\in\mathbb{Z}^d}$ also belongs to $\ell^1(\mathbb{Z}, \mathfrak{B})$.*

The last avenue of generalizing Wiener's lemma that we consider in this section is aimed at loosening the periodicity requirement for functions in $C(\mathbb{R}^d, \mathfrak{B})$. We will discuss two different ways of doing it. The first of them deals with almost periodic functions. In the classical case it follows easily from Gelfand's abstract proof [24]. The most general result of this ilk that is known to us appears in [6]. Before stating an equivalent version of the result we remind the reader the definition of almost periodic functions in $C(\mathbb{R}^d, \mathfrak{B})$.

**Definition 2.4.** A function $a \in C(\mathbb{R}^d, \mathfrak{B})$ is (Bohr) *almost periodic* if for every $\varepsilon > 0$, the set $\Omega(\varepsilon) = \{\omega \in \mathbb{R}^d\colon \sup\limits_{t\in\mathbb{R}^d} \|a(t+\omega) - a(t)\| < \varepsilon\}$ of its $\varepsilon$-periods is relatively dense in $\mathbb{R}^d$, *i.e.*, there exists a compact set $K = K_\varepsilon \subset \mathbb{R}^d$ such that $(t + K) \cap \Omega(\varepsilon) \neq \emptyset$ for all $t \in \mathbb{R}^d$.

The set of all almost periodic functions in $C(\mathbb{R}^d, \mathfrak{B})$ will be denoted by $AP = AP(\mathbb{R}^d, \mathfrak{B})$. We refer to [8, 13, 43, 44] for various facts about $AP$. For us, the crucial property is the existence of almost periodic Fourier series for each $a \in AP$:

$$a(t) \simeq \sum_{n\in\mathbb{Z}^d} a_n e^{2\pi i \lambda_n\cdot t}, \ \lambda_n \in \mathbb{R}^d, \ a_n \in \mathfrak{B}. \tag{2.5}$$

**Theorem 2.6.** [6]. *Assume that a function $a \in AP(\mathbb{R}^d, \mathfrak{B})$ has a Fourier series given by (2.5), the sequence $(a_n)_{n\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d, \mathfrak{B})$, $a(t)$ is invertible in $\mathfrak{B}$ for all $t \in \mathbb{R}^d$ and $\sup_{t\in\mathbb{R}^d} \|(a(t))^{-1}\|_{\mathfrak{B}} < \infty$. Then $b = a^{-1} \in AP(\mathbb{R}^d, \mathfrak{B})$ has an absolutely convergent Fourier series $b(t) = \sum_{n\in\mathbb{Z}^d} b_n e^{2\pi i \mu_n\cdot t}$, $\mu_n \in \mathbb{R}^d$, that is the sequence $(b_n)_{n\in\mathbb{Z}^d}$ also belongs to $\ell^1(\mathbb{Z}^d, \mathfrak{B})$.*

*Remark* 2.3. In [6] almost periodic functions on locally compact Abelian groups are considered. Here we restricted ourselves to the case of $\mathbb{R}^d$ merely to simplify the exposition.

The second way of going beyond periodic functions is to consider an analog of the Sjöstrand's class [29, 48, 49] which we call the $\mathfrak{B}$-*valued Wiener algebra* $\mathcal{W} = \mathcal{W}(\mathbb{R}^d, \mathfrak{B})$. The definition of $\mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ is based on a partition of unity in $C(\mathbb{R})$. We let $\phi \in L^1(\mathbb{R})$ be defined by its Fourier transform $\hat\phi(\lambda) = \max\{0, 1-|\lambda|\}$. Then the translates $\hat\phi_n = \hat\phi(\cdot - n)$, $n \in \mathbb{Z}$, satisfy $\sum_{n\in\mathbb{Z}} \hat\phi_n \equiv 1$. We also define $\phi^d \in L^1(\mathbb{R}^d)$ via $\hat\phi^d(\lambda) = \prod_{n=1}^d \hat\phi(\lambda_n)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^d$, and let $\phi^d_s$ be given by $\hat\phi^d_s = \hat\phi^d(\cdot - s)$, $s \in \mathbb{R}^d$.

**Definition 2.5.** A function $a \in C(\mathbb{R}^d, \mathfrak{B})$ belongs to $\mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ if

$$\|a\|_{\mathcal{W}} = \int_{\mathbb{R}^d} \|\phi^d_s * a\|_\infty ds = \int_{\mathbb{R}^d} \sup_{t\in\mathbb{R}^d} \|(\phi^d_s * a)(t)\|_{\mathfrak{B}} ds < \infty,$$

where $*$ denotes the usual convolution of two functions. The subalgebra of periodic functions in $\mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ is denoted by $\mathcal{W}(\mathbb{T}^d, \mathfrak{B})$.

*Remark* 2.4. The function $\phi^d$ in the above definition is often called a *window*. It is not hard to show that the definition of $\mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ is independent of the choice of the window $\phi$ as long as it is integrable and generates a partition of unity in the Fourier domain. We also observe that $\mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ is a generalization of the *modulation space* $M^{\infty,1} = M^{\infty,1}(\mathbb{R}^d)$. We refer to [27] for the theory of this and other modulation spaces.

A straightforward computation shows that $\mathcal{W}(C(\mathbb{T}^d, \mathfrak{B}), T)$ is isometrically isomorphic to $\mathcal{W}(\mathbb{T}^d, \mathfrak{B})$ and, hence, can be considered a subalgebra of $\mathcal{W}(\mathbb{R}^d, \mathfrak{B})$. The algebra of almost periodic functions, however, is not a subset of the $\mathfrak{B}$-valued Wiener algebra.

**Theorem 2.7.** [7]. *Assume that a $\mathfrak{B}$-valued function $a \in \mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ is invertible in $C(\mathbb{R}^d, \mathfrak{B})$. Then $a^{-1} \in \mathcal{W}(\mathbb{R}^d, \mathfrak{B})$.*

The above result concludes our promenade through successive generalizations of Wiener's lemma. Hopefully, by now we made it clear that the essence of the result is the preservation of a specific kind of spectral decay by inverse elements in Banach algebras. In the next section we discuss different kinds of spectral decay which are also preserved (or nearly preserved) by the inverse elements.

## 3. Other types of spectral decay

Summability of Fourier coefficients (diagonals of a matrix) is a natural way to describe spectral decay of functions (matrices). It is equally natural to discuss other types of decay that appear in the literature and determine which of them are preserved by the inverse elements. In this section we present several kinds of spectral decay and relate some of them to the smoothness properties of elements in $C(\mathbb{R}^d, \mathfrak{B})$.

The most common way of generalizing Wiener's summability condition is to consider weighted Wiener norms.

**Definition 3.1.** An even function $\nu : \mathbb{R}^d \to [1, \infty)$ is called a *weight*. A weight is *submultiplicative* if

$$\nu(t + s) \le \nu(t)\nu(s), \quad \text{for all} \quad t, s \in \mathbb{R}^d.$$

A weight is *admissible* if it satisfies the *GRS-condition* [25, 30]

$$\lim_{n\to\infty} n^{-1} \ln \nu(nt) = 0, \qquad \text{for all} \qquad t \in \mathbb{R}^d.$$

A typical example of weights is given by

$$\nu(t) = e^{a|t|^b}(1 + |t|)^s, \;\; a, b, s \ge 0,$$

where $|t| = |t|_1$ and $|t|_p$, $p \in [1, \infty]$, are the standard $p$-norms on $\mathbb{R}^d$. This weight is submultiplicative when $b \in [0, 1]$ and admissible when $b \in [0, 1)$. If $b = 0$ the weight is called *polynomial*. If $s = 0$ and $b = 1$, the weight is called *exponential*.

The weighted ($\mathfrak{B}$-valued) Wiener algebra $\mathcal{W}_\nu(\mathbb{R}^d,\mathfrak{B})$ contains all functions $a \in C(\mathbb{R}^d,\mathfrak{B})$ such that

$$\|a\|_{\mathcal{W}_\nu} = \int_{\mathbb{R}^d} \nu(s)\|\phi_s^d * a\|_\infty ds < \infty.$$

We shall use similar notation for weighted periodic and almost periodic $\mathfrak{B}$-valued functions.

The abstract proof of Gelfand [24] applies to scalar Wiener algebras with admissible weights. More generally we have the following result.

**Theorem 3.1.** [6, 7, 11, 12]. *Consider $a \in \mathcal{W}_\nu(\mathbb{R}^d,\mathfrak{B})$ such that $a^{-1} \in C(\mathbb{R}^d,\mathfrak{B})$.*

- *If $\nu$ is exponential then there exists another exponential weight $\mu$ such that $a^{-1} \in \mathcal{W}_\mu(\mathbb{R}^d,\mathfrak{B})$;*
- *If $\nu$ is polynomial with $s \in \mathbb{N}$ then $a^{-1} \in \mathcal{W}_\nu(\mathbb{R}^d,\mathfrak{B})$;*
- *If $\nu$ is admissible and $a$ is almost periodic then $a^{-1} \in AP_\nu(\mathbb{R}^d,\mathfrak{B})$.*

*Remark* 3.1. For these and more exotic kinds of weighted Wiener-type spectral decay appearing in various settings see, e.g., [1, 2, 6, 7, 11, 12, 15, 48, 49, 51, 53].

A stronger type of spectral decay is provided by generalizations of the Beurling algebra [7, 14, 55]. The weighted ($\mathfrak{B}$-valued) Beurling algebra $\mathcal{B}_\nu(\mathbb{R}^d,\mathfrak{B})$ contains the functions $a \in C(\mathbb{R}^d,\mathfrak{B})$ such that

$$\|a\|_{\mathcal{B}_\nu} = \sum_{k\in\mathbb{Z}} \nu(k) \max_{|n|_\infty \geq k} \|\phi_n^d * a\|_\infty < \infty.$$

Again, we shall use similar notation for weighted periodic $\mathfrak{B}$-valued functions and omit the index if the weight is trivial.

**Theorem 3.2.** [7, 55]. *Consider $a \in \mathcal{B}_\nu(\mathbb{R}^d,\mathfrak{B})$ such that $a^{-1} \in C(\mathbb{R}^d,\mathfrak{B})$.*

- *If $\nu$ is polynomial with $s \in \mathbb{N}$ then $a^{-1} \in \mathcal{B}_\nu(\mathbb{R}^d,\mathfrak{B})$;*
- *If $\nu$ is admissible and $a$ is periodic then $a^{-1} \in \mathcal{B}_\nu(\mathbb{T}^d,\mathfrak{B})$.*

A weaker type of spectral decay is provided by Jaffard and Schur algebras. The Jaffard algebra $\mathcal{J}_r$, $r > d$, is defined by the norm

$$\|a\|_{\mathcal{J}_r} = \sup_{n\in\mathbb{Z}^d} (1+|n|)^d \|\phi_n^d * a\|_\infty, n \in \mathbb{Z}^d, a \in C(\mathbb{R}^d,\mathfrak{B}).$$

In [38] Jaffard proved inverse closedness of the algebra $\mathcal{J}_r$ for matrices (equivalently, for periodic functions in $C(\mathbb{R}^d,\mathfrak{B})$).

The Schur algebra is defined for matrices $a = (a_{jk})_{j,k\in\mathbb{Z}^d}$ by the norm

$$\|a\|_{\mathcal{S}_r} = \max\left\{\sum_{j\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} (1+|j-k|)^r \|a_{jk}\|, \sum_{k\in\mathbb{Z}^d} \sup_{j\in\mathbb{Z}^d} (1+|j-k|)^r \|a_{jk}\|\right\}.$$

For $r > 0$ this algebra is inverse closed in $B(\ell^2)$, however, this is not the case for $r = 0$ [30, 34, 58].

The spectral decay of a function $a \in C(\mathbb{R}^d,\mathfrak{B})$ is tightly connected with its smoothness properties. Most recently this was thoroughly illustrated in [32] for matrix algebras (periodic functions) and in [7] for functions in $\mathcal{W}(\mathbb{R}^d,\mathfrak{B})$. The connection can be shown using the (infinitesimal) generators $\delta_k$, $k = 1, \dots, d$, of the 1-parameter automorphism groups $T_k : \mathbb{R} \to C(\mathbb{R}^d, \mathfrak{B})$, given by

$$T_k(t)a(s) = T(te_k)a(s) = a(s + te_k),\ t \in \mathbb{R}, a \in C(\mathbb{R}^d, \mathfrak{B}).$$

In the above formula $\{e_k, k = 1, \dots, d\}$, is the standard basis in $\mathbb{R}^d$. Recall [21] that $\delta_k : D(\delta_k) \subset C(\mathbb{R}^d, \mathfrak{B}) \to C(\mathbb{R}^d, \mathfrak{B})$ are defined by

$$\delta_k(a) = \lim_{t \to 0} \frac{1}{t}(T_k(t)a - a),$$

that is each $\delta_k$ is nothing but the partial derivative $\frac{\partial}{\partial t_k}$. It is easily checked that the partial derivatives form a family of *commuting derivations* of the Banach algebra $C(\mathbb{R}^d, \mathfrak{B})$, i.e., $\delta_k \delta_j = \delta_j \delta_k$, $k = 1, \dots, d$, and

$$\delta_k(ab) = \delta_k(a)b + a\delta_k(b),\ a, b \in C(\mathbb{R}^d, \mathfrak{B}).$$

The above all but trivial observation makes it possible to use the powerful theory of derivations developed, for example, in [18, 32]. In particular, the crucial fact is that the domains of derivations are typically inverse closed subalgebras (see the following Section for more details).

For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we shall use the standard multi-index notation $\delta_\alpha = \delta_1^{\alpha_1} \delta_2^{\alpha_2} \dots \delta_d^{\alpha_d}$. The Banach algebras $C^m(\mathbb{R}^d, \mathfrak{B})$, $m \in \mathbb{N}$, of $m$ times continuously differentiable $\mathfrak{B}$-valued functions are defined in the usual way by the norm

$$\|a\|_{C^m} = \|a\|_\infty + \|\delta_{\alpha^m}(a)\|_\infty,\ \alpha^m = (m, m, \dots, m).$$

The Wiener-Sobolev algebras $\mathcal{W}^m(\mathbb{R}^d, \mathfrak{B})$ are then defined to contain all $a \in C^m(\mathbb{R}^d, \mathfrak{B})$ such that $\delta_\alpha(a) \in \mathcal{W}(\mathbb{R}^d, \mathfrak{B})$ for all $|\alpha|_\infty \le m$. The connection between smoothness and spectral decay is then revealed [7, 32] by the equality

$$\mathcal{W}^m(\mathbb{R}^d, \mathfrak{B}) = \mathcal{W}_{\nu_m}(\mathbb{R}^d, \mathfrak{B}),$$

where $\nu_m(t) = (1 + |t|)^m$, $t \in \mathbb{R}^d$. A similar equality holds for Beurling-Sobolev algebras defined in a natural way [7].

More subtle relations between smoothness and spectral decay can be described using fractional powers of the generators $\delta_k$ and the associated Hölder-Zygmund continuity [32, 40]. We mention, for example, that $D(\delta^{1+\varepsilon}) \subset \mathcal{W}(\mathbb{R}, \mathfrak{B})$, $\varepsilon > 0$, $\delta = \frac{d}{dx}$.

## 4. Some ideas useful for the proofs

In this section we discuss several ideas that are often used to prove that various subalgebras characterized by the spectral decay are inverse closed. Our goal is not to present the complete proofs or even their outlines. Instead, we strive to illustrate richness and diversity of the material that can be used in those proofs. With this in mind, however, it is still not feasible to mention all the different proofs that appeared over the years. We refer to [30] for references to the proofs by Levy, Zygmund, Newman, etc.

Wiener's original proof [59] from 1932 uses a localization property and a partition of unity argument. These ideas can be viewed as a precursor to the "boot-strap" approach which we shall discuss below in greater detail. We begin, however, with the proof that is most commonly encountered in textbooks. It is based on the conceptual approach developed by Gelfand [24, 25].

The main idea of Gelfand's theory is the characterization of (unital) commutative Banach algebras as algebras of continuous functions on a compact Hausdorff space of all of their maximal ideals. For functions in $\mathcal{W}_\nu(\mathbb{T}^d) \subset C(\mathbb{T}^d)$ each set

$$\mathcal{I}_x = \{a \in \mathcal{W}_\nu : \ a(x) = 0\}$$

is a maximal ideal. The key observation is that no other set is a maximal ideal in $\mathcal{W}_\nu$ only if $1/a \in \mathcal{W}_\nu$ for every non-vanishing $a \in \mathcal{W}_\nu$. In [24] Gelfand found that the space of maximal ideals of $\mathcal{W}_\nu(\mathbb{T})$ is isomorphic to the strip $\{t \in \mathbb{C} : \ |\Im m\ t| \le a_\nu\}$, where $a_\nu = \lim\limits_{t\to\infty} \frac{\ln \nu(t)}{t}$. This immediately lead to the conclusion that the weighted commutative Wiener's lemma holds if and only if the weight is admissible. Almost immediately, Bochner and Phillips [16] generalized this result to non-commutative Banach algebras in the following way.

Let $\mathfrak{B}$ be a unital Banach algebra with the following properties

- There exist a closed subalgebra $\mathfrak{F} \subset \mathfrak{B}$ and a closed commutative subalgebra $\mathfrak{A}$ from the center of $\mathfrak{B}$ such that the elements
$$(a, f) = \sum_{k=1}^{n} a_k f_k \quad a = (a_1, \dots, a_n) \in \mathfrak{A}^n, \ f = (f_1, \dots, f_n) \in \mathfrak{F}^n,$$
are dense in $\mathfrak{B}$.
- $\|a_0 f_0\| = \|a_0\|\,\|f_0\|$ for all $a_0 \in \mathfrak{A}$, $f_0 \in \mathfrak{F}$.
- $\left\| \sum\limits_{k=1}^{n} \chi(a_i) f_i \right\| \le \|(a, f)\|$ for all $a = (a_1, \dots, a_n) \in \mathfrak{A}^n$, $f = (f_1, \dots, f_n) \in \mathfrak{F}^n$ and any character (complex algebra homomorphism) $\chi$ from the spectrum $Sp\mathfrak{A}$ of the algebra $\mathfrak{A}$.

An algebra homomorphism $\bar\chi : \mathfrak{B} \to \mathfrak{F}$ is called a *generalized character* if there exists a complex character $\chi \in Sp\mathfrak{A}$ such that $\bar\chi(af) = \chi(a)f$ for all $a \in \mathfrak{A}$, $f \in \mathfrak{F}$. The set of all generalized characters is denoted by $Sp(\mathfrak{B}, \mathfrak{F})$.

**Theorem 4.1.** [16]. *An element $b \in \mathfrak{B}$ has a left (right) inverse if and only if for every generalized character $\bar\chi \in Sp(\mathfrak{B}, \mathfrak{F})$ the element $\bar\chi(b) \in \mathfrak{F}$ has a a left (right) inverse in $\mathfrak{F}$.*

Several non-commutative extensions of the Wiener's lemma, such as, for example, in [6, 11], are proved, essentially, as special cases of the above result. For example, the almost periodic extension in [6] is based on the following corollary that arises when a specific choice of algebras in Theorem 4.1 is made. There $\mathbb{R}^d_c$ denotes the Bohr compactification of $\mathbb{R}^d$.

**Corollary 4.2.** *An element $f \in AP_\nu(\mathbb{R}^d, \mathfrak{B})$ is invertible in $AP_\nu(\mathbb{R}^d_d, \mathfrak{B})$ if and only if all elements in $\mathfrak{B}$ of the form*

$$\hat f(\omega) = \sum_{t\in\mathbb{R}^d} f(t)\omega(-t), \quad \omega \in \mathbb{R}^d_c,$$

*are invertible in* $\mathfrak{B}$*.*

An interesting approach to proving extensions of Wiener's lemma was developed by Hulanicki [37]. It works in the context of $*$-algebras and does not use any Fourier series based techniques. Its centerpiece is the following lemma.

**Lemma 4.3.** [30, 37]. *Assume that* $\mathfrak{F} \subseteq \mathfrak{B}$ *are two Banach* $*$*-algebras with common unit element and involution. Assume that* $\mathfrak{B}$ *is symmetric, i.e., the spectrum of its positive elements is positive. Then the following are equivalent:*

(1) $\mathfrak{F}$ *is inverse-closed in* $\mathfrak{B}$*;*
(2) $r_{\mathfrak{F}}(a) \le r_{\mathfrak{B}}(a)$ *for all* $a = a^* \in \mathfrak{F}$*;*
(3) $r_{\mathfrak{F}}(a^*a) = r_{\mathfrak{B}}(a^*a)$ *for all* $a \in \mathfrak{F}$*.*

*If one of the above conditions is satisfied, then* $\mathfrak{F}$ *is also symmetric.*

We refer to [30] for the standard definitions and notation of $*$-algebras, involution, spectra, and spectral radii in Banach algebras used above.

Hulanicki's lemma is often used in conjunction with the Brandenburg's trick [17]. It applies when for every $a \in \mathfrak{F}$ there exists a sequence $c_n = c_n(a) > 0$, such that $\lim_{n\to\infty} c_n^{1/n} = 1$ and

$$\|a^{2n}\|_{\mathfrak{F}} \le c_n \|a^n\|_{\mathfrak{F}} \|a^n\|_{\mathfrak{B}}.$$

The above property holds, for example, for a Beurling algebra in place of $\mathfrak{F}$ and the corresponding Wiener algebra in place of $\mathfrak{B}$ [7, 55].

Both of the approaches outlined above typically provide short and elegant proofs. Their main drawback is that they give no quantitative information about the norm of the inverse element. One special kind of inverse closed subalgebras where such information can be obtained rather easily is given by the domains of the derivations discussed in the previous section. Recall that an unbounded linear operator $\delta : D(\delta) \subseteq \mathfrak{B} \to \mathfrak{B}$ is a derivation on a Banach algebra $\mathfrak{B}$ if

$$\delta(ab) = \delta(a)b + a\delta(b),\ \ a, b \in D(\delta) \subseteq \mathfrak{B}.$$

We cite [32] for useful examples of derivations. As shown in [18, 32] derivations from a large class including the generators of bounded automorphism groups satisfy

$$\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1},\ \ a \in \mathfrak{B}.$$

Hence, not only $D(\delta)$ is an inverse closed subalgebra in $\mathfrak{B}$ but we also have an estimate on the graph norm:

$$\|a^{-1}\|_{D(\delta)} \equiv \|a^{-1}\| + \|\delta(a^{-1})\| \le \|a^{-1}\| + \|a^{-1}\|^2 \|\delta(a)\|.$$

The above estimate is unusual since it requires only $\|a^{-1}\|$ and $\|\delta(a)\|$. For more general subalgebras with spectral decay such estimates are impossible [46].

The most common way to obtain quantitative results is often called the "bootstrap" approach. Its different variations have been used by many authors including [3, 7, 12, 10, 48, 51, 57]. This approach is rather technical but can be used in most general settings. We outline the ideas in case of $\mathcal{W}(\mathbb{R}^d, \mathfrak{B}) \subset C(\mathbb{R}^d, \mathfrak{B})$ and refer to [7] for the details.

The first step of the approach is to consider invertible functions $a \in C(\mathbb{R}^d, \mathfrak{B})$ with compact support of the Fourier transform (in the distributional sense). Such functions extend to entire functions in the complex plane. Since invertibility is preserved under small perturbations, these entire functions are invertible in a horizontal strip containing the real line. Hence, the function $a^{-1}$ admits a holomorphic extension into this strip which is equivalent to exponential spectral decay, that is $a^{-1} \in \mathcal{W}_\nu$ with an exponential weight $\nu$. The crucial outcome of this step is an estimate on the norms of the "Fourier coefficients" of $a^{-1}$, that is, the elements $\phi_s^d * a^{-1}$, where the functions $\phi_s^d$ are the same as in the definition of $\mathcal{W}$. The argument is then extended to include $a$ with exponential decay rather than just $a$ with compact support of $\hat{a}$.

The second step of the proof is to represent $a \in \mathcal{W}$ as a sum $a = c + d$, where $\operatorname{supp} \hat{c}$ is compact and $\|d\|_{\mathcal{W}}$ is as small as needed. The main difficulty is that the estimates obtained in the previous step depend on the measure of $\operatorname{supp} \hat{c}$, and making $\|d\|_{\mathcal{W}}$ small is only possible by enlarging $\operatorname{supp} \hat{c}$. The way to overcome this hurdle is based on the fact that similar estimates remain valid for dilations of the functions $\phi_s^d$. Using such dilations provides the necessary degree of freedom that allows one to compensate for the growth of $\operatorname{supp} \hat{c}$. The idea is best illustrated in the language of matrices. Compact support of $\hat{c}$ is translated into the matrix being band-limited and the rate of the exponential decay of $c^{-1}$ depends on the band-width. Using the dilations of $\phi_s^d$ is equivalent to considering block matrices, in particular, the banded matrix can be assumed to be 3-diagonal [12]. Thus, dependence on the band-width is shifted to dependence on the size of the block which can be controlled.

The final step of the approach is to use the Neumann series

$$a^{-1} = c^{-1}(1 + dc^{-1})^{-1} = c^{-1} \sum_{n=0}^{\infty} (-dc^{-1})^n$$

to bound $\|\phi_s^d * a^{-1}\|_{\mathcal{W}}$ and $\|a^{-1}\|_{\mathcal{W}}$. The bound is obtained in terms of $\|a\|_{\mathcal{W}}$, $\|a^{-1}\|_\infty$, and the size of the dilation $N \in \mathbb{N}$ of $\phi_s^d$ needed to make $\|d\|_{\mathcal{W}}$ sufficiently small [7].

## 5. An application: localization of frames

The primary motivation for the recent activity in the study of inverse closed subalgebras is provided by the idea of localization in frame theory [1, 4, 5, 23, 28, 31, 33, 41, 47] and its application to pseudo-differential operators [29, 35, 36, 50] and sampling theory [1, 45, 47, 52, 54]. Here, we limit ourselves to presenting only two of the great multitude of the ideas in the papers cited above. The first, illustrates how Wiener's lemma is used to prove that certain types of frame localization is preserved by dual frames. Conversely, the second idea illustrates how frame theory itself leads to a few Wiener-type results. We begin by defining the basic notions of frame theory.

**Definition 5.1.** Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\Phi = \{\varphi_n \in \mathcal{H},\ n \in \mathbb{Z}^d\}$, is a *frame* for $\mathcal{H}$ if for some $0 < a \le b < \infty$

$$a\,\|f\|^2 \le \sum_{n\in\mathbb{G}} |\langle f, \varphi_n\rangle|^2 \le b\,\|f\|^2 \tag{5.1}$$

for all $f \in \mathcal{H}$. A frame $\Phi$ is *tight* if $a = b$ and *Parseval* if $a = b = 1$. If only the right hand side inequality holds in (5.1) then $\Phi$ is called a *Bessel* sequence.

For an orthonormal basis $\{e_n,\ n \in \mathbb{N}\}$ of $\mathcal{H}$, an example of a frame that is not a basis is given by the system

$$\{e_1, e_1, e_2, e_2, \ldots\}. \tag{5.2}$$

**Definition 5.2.** For a Bessel sequence $\Phi = \{\varphi_n \in \mathcal{H},\ n \in \mathbb{Z}^d\}$, the *analysis operator* $T_\Phi : \mathcal{H} \to \ell^2(\mathbb{Z}^d)$ is defined by $(T_\Phi f)(n) = \langle f, \varphi_n\rangle$. Its adjoint $T_\Phi^* : \ell^2(\mathbb{Z}^d) \to \mathcal{H}$ is called the *synthesis operator*, and $S_\Phi : \mathcal{H} \to \mathcal{H}$, $S_\Phi = T_\Phi^* T_\Phi$, is the *frame operator*.

It is easily seen that for a sequence $c = \{c_n \in \mathbb{C},\ n \in \mathbb{Z}^d\}$ we have $T_\Phi^* c = \sum\limits_{n\in\mathbb{Z}^d} c_n\varphi_n$ and $S_\Phi f = \sum\limits_{n\in\mathbb{Z}^d} \langle f, \varphi_n\rangle\varphi_n$, $f \in \mathcal{H}$, with convergence in $\mathcal{H}$. Also, $\Phi$ is a frame for $\mathcal{H}$ if and only if $aI \le S_\Phi \le bI$, and the frame is Parseval if and only if $S_\Phi = I$. The sequence $\tilde\Phi = \{\tilde\varphi_n = S_\Phi^{-1}\varphi_n,\ n \in \mathbb{Z}^d\}$ is called the *canonical dual* frame for $\Phi$. Clearly, for any $f \in \mathcal{H}$,

$$f = \sum_{n\in\mathbb{Z}^d} \langle f, \tilde\varphi_n\rangle\varphi_n = \sum_{n\in\mathbb{Z}^d} \langle f, \varphi_n\rangle\tilde\varphi_n. \tag{5.3}$$

Even though localization of a frame is understood differently by different authors, in many cases it is equivalent to some kind of spectral decay of the frame operator $S$. Hence, once a connection between the localization and the spectral decay of $S$ is established one only needs to check if this kind of spectral decay is preserved by inverses according to an extension of Wiener's lemma. If the answer is positive, this allows one to conclude that the canonical dual frame has the same kind of localization. This is particularly important for special kinds of frames such as wavelet or Gabor frames. In these cases, the spectral decay of $S$ means that it is invariant with respect to many important subsets of $L^2(\mathbb{R}^d)$. If the generators of the frame are then chosen to belong to some of those invariant subsets, one gets better convergence rates of the corresponding frame expansions. We illustrate the concept using Gabor frames.

**Definition 5.3.** For $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R}^d)$, let

$$M_{\beta m} g(x) = e^{-2\pi i\beta m\cdot x} g(x), \quad T_{\alpha n} g(x) = g(x - \alpha n).$$

The collection $\mathcal{G}(g, \alpha, \beta) = \{g_{m,n} = M_{\beta m} T_{\alpha n} g,\ m, n \in \mathbb{Z}^d\} \subset L^2(\mathbb{R}^d)$ is called a Gabor system. Naturally, a Gabor system that is a frame for $L^2(\mathbb{R}^d)$ is called a *Gabor frame*.

A central question in Gabor analysis is to find conditions on $\alpha, \beta > 0$ and $g \in L^2$ such that $\mathcal{G}(g, \alpha, \beta)$ is a Gabor frame for $L^2$. Moreover, in many applications it is desirable to find a Gabor frame such that the generator $g$ and its canonical dual $\tilde{g}$ have the same properties, e.g., same type of decay and/or smoothness. For example, let $\varphi(x) = e^{-x^2}$ and define the short time Fourier transform (STFT) of $f \in L^2(\mathbb{R}^d)$ by

$$V_\varphi f(x,\omega) = \langle f, M_\omega T_x \varphi\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{\varphi(t-x)}\,e^{-2\pi i x\cdot\omega}\,dt.$$

The space $M^1(\mathbb{R}^d)$ of all $f \in L^2(\mathbb{R}^d)$ such that

$$\|f\|_{M^1} = \iint_{\mathbb{R}^{2d}} |V_\varphi f(x,\omega)|\,dx\,d\omega < \infty$$

is known as the *Feichtinger algebra* [27]. In this context, Gröchenig and Leinert proved [33] that if $g \in M^1(\mathbb{R}^d)$ and $\alpha, \beta > 0$ are such that $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame, then also $\tilde{g} \in M^1(\mathbb{R}^d)$. This was done using Janssen's representation of the Gabor frame operator

$$S_{\mathcal{G}} f = \frac{1}{\alpha\beta} \sum_{m,n\in\mathbb{Z}^d} \langle g, M_{m/\beta} T_{n/\alpha} g\rangle M_{m/\beta} T_{n/\alpha} f,$$

which converges absolutely in $B(L^2(\mathbb{R}^d))$ whenever $g \in M^1$. Gröchenig and Leinert proved the non-commutative extension of the Wiener's lemma that handles this kind of spectral decay. Similarly, in [41], we used Walnut's representation of $S_{\mathcal{G}}$ to show that it leaves invariant certain Wiener amalgam spaces. A slightly more general version of Theorem 2.3 appearing in [11, 12] was sufficient for this purpose.

Extensions of Wiener's lemma that are used for localized frames typically deal with matrix algebras. It is not surprising, therefore, that the extensions obtained using frame theory also deal with matrix algebras. We shall formulate the most general extension of this kind that is known to us. It is based on the notion of g-frames introduced by W. Sun [56]. We follow [40] in the exposition below.

**Definition 5.4.** A family of operators $\Lambda = \{\Lambda_j,\ j \in \mathbb{Z}^d\} \subset B(\mathcal{H})$ is a *g-frame* if there exist $a, b > 0$ such that

$$a\,\|x\|^2 \le \sum_{j\in\mathbb{G}} \|\Lambda_j x\|^2 \le b\,\|x\|^2,$$

for all $x \in \mathcal{H}$. The *g-analysis operator* $T_\Lambda : \mathcal{H} \to \ell^2(\mathbb{Z}^d, \mathcal{H})$ is defined by $(T_\Lambda x)(j) = \Lambda_j x$. Its adjoint $T_\Lambda^* : \ell^2(\mathbb{Z}^d,\mathcal{H}) \to \mathcal{H}$ is called the *g-synthesis operator*, and $S_\Lambda : \mathcal{H} \to \mathcal{H}$, $S_\Lambda = T_\Lambda^* T_\Lambda$, is the *g-frame operator*. The canonical dual g-frame $\widetilde{\Lambda} = \{\widetilde{\Lambda}_j,\ j \in \mathbb{Z}^d\}$ is defined via $\widetilde{\Lambda}_j = \Lambda_j S_\Lambda^{-1}$. W. Sun showed [56] that those are well-defined and satisfy

$$S_\Lambda x = \sum_{j\in\mathbb{Z}^d} \Lambda_j^* \Lambda_j x, \quad x = \sum_{j\in\mathbb{G}} \widetilde{\Lambda}_j^* \Lambda_j x = \sum_{j\in\mathbb{G}} \Lambda_j^* \widetilde{\Lambda}_j x,$$

for all $x \in \mathcal{H}$.

The most well-known example of g-frames (other than the classical frames) is furnished by fusion frames [19, 20].

The matrix $\mathcal{A}_\Lambda = (a^\Lambda_{mn})$ of an operator $A \in B(\mathcal{H})$ with respect to a g-frame $\Lambda$ is defined by $(a^\Lambda_{mn}) = \Lambda_m A \Lambda^*_n$, $m, n \in \mathbb{Z}^d$. An operator $A \in B(\mathcal{H})$ is *localized* with respect to a g-frame $\Lambda$ and a weight $\nu$ (or, simply, $(\Lambda, \nu)$-localized), if its matrix $\mathcal{A}_\Lambda$ is $\nu$*-localized*, i.e.

$$\|\mathcal{A}\|_{\mathcal{W}_\nu} := \sum_{k\in\mathbb{Z}^d} \nu(k) \sup_{m-n=k} \|a_{mn}\| < \infty.$$

We conclude our paper with a generalization of Theorem 2.3 for $(\Lambda, \nu)$-localized operators.

**Theorem 5.1.** *Let $\Lambda$ be a g-frame and $A \in B(\mathcal{H})$ be an invertible $(\Lambda, \nu)$-localized operator. If $\nu$ is an admissible weight, then the inverse operator $B = A^{-1}$ is localized with respect to the same weight $\nu$ and the canonical dual g-frame $\widetilde{\Lambda}$. If $\nu$ is an exponential weight, then $B$ is $(\widetilde{\Lambda}, \mu)$-localized for some other exponential weight $\mu$.*

The distinctive feature of the above result is that the matrix $\mathcal{A}_\Lambda$ typically does not define an invertible operator on $\ell^2(\mathbb{Z}^d, \mathcal{H})$. This operator, however, turns out to be pseudo-invertible and the standard trick using Riesz-Dunford calculus [22, 29, 40] provides an easy reduction to the case of invertible operators.

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