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METALLIC STRUCTURES ON RIEMANNIAN MANIFOLDS

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Abstract. Our aim in this paper is to focus on some applications in differential geometry of the metallic means family (a generalization of the golden mean) and generalized Fibonacci sequences, using a class of polynomial structures defined on Riemannian manifolds. We search for properties of the induced structure on a submanifold by metallic Riemannian structures and we find a necessary and sufficient condition for a submanifold to be also a metallic Riemannian manifold in terms of invariance. Also, the totally geodesic, minimal and respectively totally umbilical hypersurfaces in metallic Riemannian manifolds are analyzed and the Euclidean space and its hypersphere is treated as example.

1. Introduction

In this paper we find some applications of the metallic means family and generalized Fibonacci sequences on Riemannian manifolds. Recall that a very interesting generalization of the golden mean was introduced in 1997 by Vera W. de Spinadel in [26]-[30] and called *metallic means family* or *metallic proportions*. The members of the metallic means family have the property of carrying the name of a metal, like the golden mean and its relatives: the silver mean, the bronze mean, the copper mean and many others.

The outline of this paper is as follows: in section 2 we provide a detailed survey regarding the metallic means family, secondary Fibonacci numbers, and we define the (p, q)-Fibonacci hyperbolic sine and cosine functions useful in our approach.

In section 3 we introduce the notion of metallic structure on a Riemmannian manifold and we establish several properties of the metallic structure as a generalization of the golden structure defined and studied by the present authors in [3], [14] and [15]. More precisely, a metallic structure is a polynomial structure as defined by Goldberg, Yano and Petridis in [11] and [12], with the structural polynomial $Q(J) = J^2 - pJ - qI$.

In section 4 we focus on the geometry of submanifolds endowed with structures induced by metallic Riemannian structures, due to an analogy with the theory of submanifolds in almost product manifolds ([1]-[2], [24], [25]). We find conditions

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for this kind of submanifold to be also a metallic Riemannian manifold in terms of invariance.

In the last section we search for properties of induced structures on hypersurfaces in metallic Riemannian manifolds with a special view towards totally geodesic, minimal and respectively totally umbilical hypersurfaces. An example of metallic Riemannian structure is given on the Euclidean space, and its hypersphere is analyzed with the tools of the previous section.

2. Preliminaries

The members of the metallic means family share important mathematical properties that constitute a bridge between mathematics and design; e.g., the silver mean has been used in describing fractal geometry, [4]. Some members of the metallic means family (golden mean and silver mean) appeared already in the sacred art of Egypt, Turkey, India, China and other ancient civilizations, [31]. The members of the metallic means family are closely related to quasiperiodic dynamics, [26]. The relationships between the Hausdorff dimension of higher order Cantor sets and the golden mean or silver mean was investigated by El Naschie ([18]-[21]), who proved: Bijection Theorem, Golden Mean Theorem, Modified Fibonacci Theorem, Silver Mean Theorem, Arithmetic Mean Theorem, and showed that the n-dimensional triadic Cantor set has the same Hausdorff dimension as the dimension of a random inverse golden mean Sierpinski space to the power $n-1$ ([18, 17]).

Fix two positive integers p and q. The positive solution of the equation

$$x^2 - px - q = 0$$

is named member of the metallic means family ([26]-[30]). These numbers, denoted by:

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}, \tag{1}$$

are also called (p, q)-*metallic numbers*.

Some properties studied in this paper are related with the generalized secondary Fibonacci sequence (GSFS) ([30], [36], [16]), given by relations of the type:

$$G(n+1) = pG(n) + qG(n-1), \quad n \geq 1 \tag{2}$$

where p, q, $G(0) = a$ and $G(1) = b$ are real numbers. The members of generalized secondary Fibonacci sequences are: a, b, $pb + qa$, $p(pb + qa) + qb$, $\ldots$.

The ratio $G(n+1)/G(n)$ of two consecutive generalized secondary Fibonacci numbers converges to:

- the golden mean $\phi = \frac{1+\sqrt{5}}{2}$ if $p = q = 1$, determined by the ratio of two consecutive classical Fibonacci numbers;
- the silver mean $\sigma_{2,1} = 1 + \sqrt{2}$ for $p = 2$ and $q = 1$, determined by the ratio of two consecutive Pell numbers ([7]);
- the bronze mean $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ for $p = 3$ and $q = 1$, which plays an important role in studying topics such as dynamical systems and quasicrystals;

- the subtle mean $\sigma_{4,1} = 2 + \sqrt{5} = \phi^3$ ([22]), which plays a significant role in the theory of Cantorian fractal-like micro-space-time E^{∞} and also is involved in a fundamental way in noncommutative geometry and four manifold theory ([23]);
- the copper mean $\sigma_{1,2} = 2$ for $p = 1$ and $q = 2$,
- the nickel mean $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$ for $p = 1$ and $q = 3$ and so on.

If $q = 1$ and $p = k$ in (2) we obtain the k-Fibonacci sequence ([5]) given recurrently by:

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad F_{k,0} = 0, \quad F_{k,1} = 1,$$

for $n \geq 1$. If $k := x$ is a real variable then $F_{x,n}$ correspond to the Fibonacci polynomials defined and studied in different contexts in [6]-[10].

Based on the analogy given by Binet's formulas between extended Fibonacci numbers and the classical hyperbolic function, the *symmetrical Fibonacci sine* and the *symmetrical Fibonacci cosine* given respectively as follows:

$$sFs(x) = \frac{\phi^x - \phi^{-x}}{\sqrt{5}}, \quad cFs(x) = \frac{\phi^x + \phi^{-x}}{\sqrt{5}} \tag{3}$$

are studied in [31]-[36]. Also, in [10], k-Fibonacci hyperbolic sine and cosine functions are studied, given respectively by:

$$sF_kh(x) = \frac{\sigma_k^x - \sigma_k^{-x}}{\sigma_k + \sigma_k^{-1}}, \quad cF_kh(x) = \frac{\sigma_k^x + \sigma_k^{-x}}{\sigma_k + \sigma_k^{-1}}, \tag{4}$$

under the condition that $\sigma_k = \frac{k+\sqrt{k^2+4}}{2}$ is the positive root of the characteristic equation associated to the k-Fibonacci sequence.

In analogous way as in (3) and (4), using generalized secondary Fibonacci sequences and the metallic number $\sigma_{p,q}$ given in (1), we can define the (p, q)-Fibonacci hyperbolic sine and cosine functions given respectively as follows:

$$sF_{(p,q)}h(x) = \frac{(\sigma_{p,q})^x - (q^{-1}\sigma_{p,q})^{-x}}{\sigma_{p,q} + (q^{-1}\sigma_{p,q})^{-1}}, \quad cF_{(p,q)}h(x) = \frac{(\sigma_{p,q})^x + (q^{-1}\sigma_{p,q})^{-x}}{\sigma_{p,q} + (q^{-1}\sigma_{p,q})^{-1}}. \tag{5}$$

We pointed out that the (p, q)-Fibonacci hyperbolic sine and cosine functions verify the following properties:

$$sF_{(p,q)}h(2) = p, \quad cF_{(p,q)}h(1) = 1,$$

$$sF_{(p,q)}h(1) + cF_{(p,q)}h(1) = \frac{2\sigma_{p,q}}{2\sigma_{p,q} - p}, \quad sF_{(p,q)}h(1) - cF_{(p,q)}h(1) = -2\frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p}.$$

3. Metallic Riemannian structures

In this section a class of polynomial structures, named by us metallic structures, is introduced in Riemannian manifolds.

Definition 3.1. A polynomial structure on a manifold M is called a *metallic structure* if it is determined by an $(1, 1)$ tensor field J which satisfies the equation

$$J^2 = pJ + qI$$

where p, q are positive integers and I is the identity operator on the Lie algebra $\mathfrak{X}(M)$ of the vector fields on M.

Since the Riemannian geometry is the most used framework of the differential geometry, let us add a metric to our study. We say that a Riemannian metric g is *J-compatible* if:

$$g(JX, Y) = g(X, JY)$$

for every $X, Y \in \mathfrak{X}(M)$, which means that J is a self-adjoint operator with respect to g. This condition is equivalent in our framework with:

$$g(JX, JY) = p \cdot g(X, JY) + q \cdot g(X, Y).$$

Definition 3.2. A Riemannian manifold (M, g) endowed with a metallic structure J so that the Riemannian metric g is J-compatible is named a *metallic Riemannian manifold* and (g, J) is called a *metallic Riemannian structure* on M.

Remark 3.1. If we consider $p = q = 1$ in Definition 3.2 then we obtain that (g, J) is a golden Riemannian structure of M ([3], [15]).

The following proposition gives the main properties of a general metallic structure:

Proposition 3.1. *A metallic structure J has the following properties:*

(1) For every integer number $n \geq 1$:

$$J^n = G(n)J + qG(n-1)I,$$

where $(G(n))_{n\geq 0}$ is the generalized secondary Fibonacci sequence with $G(0) = 0$ and $G(1) = 1$.

(2) J is an isomorphism on the tangent space T_xM for every $x \in M$. It follows that J is invertible and its inverse $\overline{J} = J^{-1} = \frac{1}{q}J - \frac{p}{q}I$ is not a metallic structure but is still polynomial, more precisely a quadratic one:

$$q\overline{J}^2 + p\overline{J} - I = 0.$$

(3) The eigenvalues of J are the metallic numbers $\sigma_{p,q}$ and $p - \sigma_{p,q}$.

It is known, from [12], that a polynomial structure on a manifold M defined by a smooth tensor field of $(1, 1)$-type induces an generalized almost product structure F, i.e. $F^2 = I$, on M with the number of distributions of F equal to the number of distinct irreducible factors of the structure polynomial over the real field while the projectors are expressed as polynomials in F.

Proposition 3.2. *Every almost product structure F induces two metallic structures on M given as follows:*

$$J_1 = \frac{p}{2}I + \left(\frac{2\sigma_{p,q} - p}{2}\right)F, \qquad J_2 = \frac{p}{2}I - \left(\frac{2\sigma_{p,q} - p}{2}\right)F.$$

Conversely, every metallic structure J on M induces two almost product structures on this manifold:

$$F = \pm\left(\frac{2}{2\sigma_{p,q} - p}J - \frac{p}{2\sigma_{p,q} - p}I\right). \tag{6}$$

In particular, if the almost product structure F is a Riemannian one then J_1, J_2 are also metallic Riemannian structures.

Proposition 3.3. *On a metallic manifold (M, J) there are two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators*

$$l = \frac{\sigma_{p,q}}{2\sigma_{p,q} - p} \cdot I - \frac{1}{2\sigma_{p,q} - p} \cdot J, \qquad m = \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p} \cdot I + \frac{1}{2\sigma_{p,q} - p} \cdot J.$$

Remark 3.2. The above operators l and m verify:

$$l+m = I, \quad l^2 = l, \quad m^2 = m, \quad J \circ l = l \circ J = (p-\sigma_{p,q})l, \quad J \circ m = m \circ J = \sigma_{p,q} m.$$

Thus l and m define complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to these projections.

Using (5), the projection operators have the following form:

$$l = \frac{sF_{(p,q)}h(1) + cF_{(p,q)}h(1)}{2}(I - \frac{1}{\sigma_{p,q}}J),$$
$$m = -\frac{sF_{(p,q)}h(1) - cF_{(p,q)}h(1)}{2}(I + \frac{\sigma_{p,q}}{q}J).$$

Also, the almost product structure F from (6) induced by a metallic structure J is:

$$F = \frac{sF_{(p,q)}h(1) + cF_{(p,q)}h(1)}{\sigma_{p,q}} \cdot (J - \frac{p}{2} \cdot I). \tag{7}$$

Remark 3.3. The complementary distributions $\mathcal{D}_l, \mathcal{D}_m$ are orthogonal with respect to the J-compatible metric g, i.e. $g(l\cdot, m\cdot) = 0$.

4. Induced structures on submanifolds by metallic Riemannian structures

Let us consider that M is an n-dimensional submanifold of codimension r, isometrically immersed in an $(n + r)$-dimensional metallic Riemannian manifold $(\overline{M}, g, J)$ with $n, r \in \mathbb{N}^*$.

We denote by $T_x M$ the tangent space of M in a point $x \in M$ and by $T_x^{\perp} M$ the normal space of M in x. Let i_* be the differential of the immersion $i : M \to \overline{M}$. The induced Riemannian metric g on M is given by $g(X, Y) = g(i_* X, i_* Y)$ for every $X, Y \in \mathfrak{X}(M)$. We consider a local orthonormal basis $\{N_1, \ldots, N_r\}$ of the normal space $T_x^{\perp} M$. Hereafter we assume that the indices α, β, γ run over the range $\{1, \ldots, r\}$.

For every $X \in T_x M$ the vector fields $J(i_* X)$ and $J(N_\alpha)$ can be decomposed in tangential and normal components as follows:

$$\begin{cases} J(i_* X) = i_*(P(X)) + \sum_{\alpha=1}^{r} u_\alpha(X) N_\alpha \\ J(N_\alpha) = i_*(\xi_\alpha) + \sum_{\beta=1}^{r} a_{\alpha\beta} N_\beta \end{cases}$$

where P is an $(1,1)$ tensor field on M, $\xi_\alpha \in \mathfrak{X}(M)$, u_α are 1-forms on M and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on M.

In a similar manner as in [1]-[2] and [14]-[15], it is easy to verify:

Proposition 4.1. *The structure $\Sigma = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on the submanifold M by the metallic Riemannian structure (g, J) on $\overline{M}$ satisfies the following equalities:*

$$\begin{cases} \text{(i)} & P^2(X) = pP(X) + qX - \sum_\alpha u_\alpha(X)\xi_\alpha, \\ \text{(ii)} & u_\alpha(P(X)) = pu_\alpha(X) - \sum_\beta a_{\alpha\beta}u_\beta(X), \\ \text{(iii)} & a_{\alpha\beta} = a_{\beta\alpha}, \\ \text{(iv)} & u_\beta(\xi_\alpha) = q\delta_{\alpha\beta} + pa_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma}a_{\gamma\beta}, \\ \text{(v)} & P(\xi_\alpha) = p\xi_\alpha - \sum_\beta a_{\alpha\beta}\xi_\beta \\ \text{(vi)} & u_\alpha(X) = g(X, \xi_\alpha), \\ \text{(vii)} & g(PX, Y) = g(X, PY), \\ \text{(viii)} & g(PX, PY) = pg(X, PY) + qg(X, Y) + \sum_\alpha u_\alpha(X)u_\alpha(Y), \end{cases} \tag{8}$$

for every $X, Y \in \mathfrak{X}(M)$, where $\delta_{\alpha\beta}$ is the Kronecker delta.

Definition 4.1. ([2]) A submanifold M in a manifold $\overline{M}$ endowed with a structural tensor field J (i.e. J is a tensor field on $\overline{M}$) is called *invariant* with respect to J if $J(T_xM) \subset T_xM$ for every $x \in M$.

Remark 4.1. The induced structure $\Sigma = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ on a submanifold M by the metallic Riemannian structure (g, J) is invariant if and only if $u_\alpha = 0$ (equivalently $\xi_\alpha = 0$) for every $\alpha \in \{1, \dots, r\}$.

Proposition 4.2. *The matrix $\mathcal{A} := (a_{\alpha\beta})_r$ of the structure Σ induced on an invariant submanifold M by the metallic Riemannian structure (g, J) from the Riemannian manifold $(\overline{M}, g)$ is a metallic matrix, that is a matrix which verifies*

$$\mathcal{A}^2 = p \cdot \mathcal{A} + q \cdot I_r,$$

where I_r is the identically matrix of order r.

Proof. Using that M is an invariant submanifold in $\overline{M}$, from (8) (iv) we obtain $\sum_\gamma a_{\alpha\gamma}a_{\gamma\beta} = p \cdot a_{\alpha\beta} + q \cdot \delta_{\alpha\beta}$ for every $\alpha, \beta \in \{1, \dots, r\}$. □

In a similar manner as in [1] we obtain the following property:

Proposition 4.3. *Let $\Sigma = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on a submanifold M by the metallic Riemannian structure (g, J) on $\overline{M}$. Then M is an invariant submanifold with respect to J if and only if the induced structure (P, g) on M is a metallic Riemannian structure whenever P is non-trivial.*

Proof. From (8) (i), if (g, P) is a metallic Riemannian structure then $\sum_\alpha u_\alpha(X)\xi_\alpha = 0$. Thus, we get (by taking the g-product with X) that

$$\sum_\alpha u_\alpha(X)g(X, \xi_\alpha) = \sum_\alpha (u_\alpha(X))^2 = 0$$

which is equivalent with $u_\alpha(X) = 0$ for every $\alpha \in \{1, 2, \ldots, r\}$ and this fact implies that M is invariant.

Conversely, if M is an invariant submanifold then $P^2 = pP + qI$ and we obtain, using (7), (8) (i) and (8) (vii), that (P, g) is a metallic Riemannian structure. □

5. Hypersurfaces in metallic Riemannian manifolds

In this section we find some properties of an n-dimensional hypersurface M (i.e. submanifold of codimension $r = 1$) in a metallic Riemannian manifold $(\overline{M}_{n+1}, g, J)$. The totally geodesic, minimal and respectively umbilical hypersurfaces in metallic Riemannian manifolds are studied.

We denote the covariant differential in $\overline{M}$ by $\overline{\nabla}$ and the covariant differential in M determined by the induced metric g on M by ∇. We denote by A the Weingarten operator on TM with respect to the local unit normal vector field N of M in $\overline{M}$.

Proposition 5.1. *The structure $\Sigma = (P, g, u, \xi, a)$ verifies the equalities:*

$$\begin{cases} \text{(i)} & P^2(X) = pP(X) + qX - u(X)\xi, \\ \text{(ii)} & u(P(X)) = (p-a)u(X), \\ \text{(iii)} & u(\xi) = q + pa - a^2, \\ \text{(iv)} & P(\xi) = (p-a)\xi, \\ \text{(v)} & u(X) = g(X, \xi), \\ \text{(vi)} & g(PX, Y) = g(X, PY), \\ \text{(vii)} & g(PX, PY) = pg(X, PY) + qg(X, Y) + u(X)u(Y), \end{cases} \tag{9}$$

for every $X, Y \in \mathfrak{X}(M)$.

Remark 5.1. (i) If a is the constant function on M equal with p then $u \circ P = 0$, $P(\xi) = 0$ and $\|\xi\|^2 = q$. More generally, if M is a non-invariant hypersurface then $\operatorname{Im}(a) \in (p - \sigma_{p,q}; \sigma_{p,q})$ and

$$\|\xi\| = \sqrt{-a^2 + p \cdot a + q}. \tag{10}$$

(ii) Let us remark that for $\xi = 0$ (which is equivalent with $u = 0$) we have $J|_M = P$ and $J(N) = aN$, thus (M, g, P) is a metallic Riemannian manifold. In other words, M is an invariant hypersurface of a metallic Riemannian manifold $(\overline{M}, g, J)$ if and only if the normal vector N on M is an eigenvector of the structure J (i.e. $J(N) = a \cdot N$) with the eigenvalue the function a from the structure $\Sigma = (P, g, u, \xi, a)$ on M.

Proposition 5.2. *M is an invariant and orientable hypersurface in a metallic Riemannian manifold $(\overline{M}, g, J)$ if and only if the structure $\Sigma = (P, g, u, \xi, a)$ induced on M by the metallic Riemannian structure (J,g) has the function a equal with the metallic number $a = \sigma_{p,q}$ or $a = p - \sigma_{p,q}$.*

Proposition 5.3. *If $\Sigma = (P, g, u, \xi, a)$ is the induced structure on a hypersurface M isometrically immersed in a metallic Riemannian manifold $(\overline{M}, g, J)$ then for every $X \in \mathfrak{X}(M)$:*

$$(P(X) - (p-a)X) \perp \xi.$$

In particular, for $a = p$:

$$P(X) \in \xi^{\perp} := \{X \in \mathfrak{X}(\overline{M}) / X \perp \xi\}$$

for every $X \in \mathfrak{X}(M)$ and $TM = \operatorname{Ker} P \oplus \xi^{\perp}$.

The Gauss and Weingarten formulae for a hypersurface M isometrically immersed in the Riemannian manifold $(\overline{M}, g)$ are respectively given as follows:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)N, \qquad \bar{\nabla}_X N = -A(X),$$

where h is the second fundamental tensor corresponding to N and $h(X,Y) = g(AX, Y)$.

In a similar manner as in [14] we remark that:

Proposition 5.4. *If M is a hypersurface in a metallic Riemannian manifold $(\overline{M}, g, J)$ and J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ (i.e. $\overline{\nabla} J = 0$) then the elements of the structure $\Sigma = (P, g, u, \xi, a)$ have the following properties:*

$$\begin{cases} \text{(i)} & (\nabla_X P)(Y) = g(AX, Y)\xi + u(Y)AX, \\ \text{(ii)} & (\nabla_X u)(Y) = -g(AX, PY) + ag(AX, Y), \\ \text{(iii)} & \nabla_X \xi = -P(AX) + aAX, \\ \text{(iv)} & X(a) = -2u(AX). \end{cases} \tag{11}$$

Remark 5.2. (i) Using (11) (iii) and (8) (vii) in (11) (ii), we obtain

$$(\nabla_X u)(Y) = g(\nabla_X \xi, Y). \tag{12}$$

(ii) Let M be an invariant hypersurface in a metallic Riemannian manifold $(\overline{M}, g, J)$ with J parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ and let $\Sigma = (P, g, \xi, u, a)$ be the structure induced on M. Then P is parallel with respect to ∇.

Recall the following well-known definition: M is said to be *totally geodesic* if its second fundamental form vanishes identically, that is $h = 0$ or equivalently the Weingarten operator $A = 0$.

In a similar manner as in [1] and [2] where T. Adati studied the properties of submanifolds in an almost product manifold, we obtain some properties for submanifolds in metallic Riemannian manifolds, as follows:

Theorem 5.1. *Let M be a non-invariant hypersurface of a metallic Riemannian manifold $(\overline{M}, g, J)$ with J parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ (i.e. $\overline{\nabla} J = 0$) and let $\Sigma = (P, g, \xi, u, a)$ be the induced structure on M by (g, J). Then the following properties are equivalent: (i) M is totally geodesic; (ii) $\nabla P = 0$; (iii) $\nabla \xi = 0$; (iv) $\nabla u = 0$.*

Proof. If we suppose that M is totally geodesic (i.e. $AX = 0$), from (11) (i)-(iii) we obtain $\nabla P = 0$, $\nabla u = 0$ and $\nabla \xi = 0$.

If we suppose that $\nabla P = 0$ then, from (11) (i) we obtain that $g(AX,Y)\xi + g(Y,\xi)AX = 0$ for every $X, Y \in \mathfrak{X}(M)$. Applying $g(Y,\cdot)$ in the last equality it follows that $2 \cdot g(AX,Y)g(Y,\xi) = 0$. If we consider that $Y = \xi$ in the last equality, we obtain $g(AX,\xi)\|\xi\|^2 = 0$ and using $\|\xi\| \neq 0$ (because M is a non-invariant hypersurface in the metallic Riemannian manifold $(\overline{M}, g, J)$) we obtain $g(AX,\xi) = 0$ for every $X \in \mathfrak{X}(M)$.

On the other hand, using $Y = AX$ in $g(AX,Y)\xi + g(Y,\xi)AX = 0$, we obtain $g(AX,AX)\xi = 0$. Since $\xi \neq 0$, we have that $\|AX\| = 0$, therefore M is totally geodesic. Also, using $AX = 0$ in (11) (ii)-(iii) we obtain $\nabla\xi = 0$, $\nabla u = 0$.

If we suppose that $\nabla \xi = 0$, from (11) (iii) we have $P(AX) = a \cdot AX$ and $P^2(AX) = a \cdot P(AX) = a^2 \cdot AX$.

From (9) (i) we have $P^2(AX) = p \cdot P(AX) + q \cdot AX - u(AX)\xi$ and using (11) (iv), $u(AX) = -\frac{1}{2}X(a)$ and (10), $\|\xi\|^2 = q + pa - a^2$ where $\|\xi\| \neq 0$ (M is a non-invariant hypersurface in $\overline{M}$) we obtain:

$$AX = -\frac{X(a)}{2\|\xi\|^2} \cdot \xi, \tag{13}$$

for every $X \in \mathfrak{X}(M)$.

Using $P(AX) = a \cdot AX$ and (9) (iv) we obtain $(p - 2a) \cdot \frac{X(a)}{2\|\xi\|^2}\xi = 0$. From $\|\xi\| \neq 0$ we obtain $X(a) = 0$ or the function a is a constant $a = \frac{p}{2}$ on M, which imply $X(a) = 0$. Using (13) we obtain $AX = 0$ for every $X \in \mathfrak{X}(M)$, therefore M is totally geodesic. Also, from $AX = 0$ in (11) (i)-(ii) we obtain $\nabla P = 0$ and $\nabla u = 0$.

If we suppose that $\nabla u = 0$, using (12) we obtain $\nabla \xi = 0$, which implies that M is totally geodesic and $\nabla P = 0$. ☐

Recall that a hypersurface M in the Riemannian manifold $(\overline{M}, g)$ is said to be *minimal* if:

$$\mathrm{trace}(A_N) = \sum_{j=1}^{n} g(A_N e_j, e_j)$$

vanishes identically, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of the tangent space $T_x M$ in every point $x \in M$.

Theorem 5.2. *Let M be a non-invariant hypersurface of a metallic Riemannian manifold $(\overline{M}, g, J)$ with J parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$, and let $\Sigma = (P, g, \xi, u, a)$ be the induced structure on M by (g, J). If $\sum_{j=1}^{n}(\nabla_{e_j} P)e_j = \sum_{j=1}^{n} u(e_j)A(e_j)$ then M is minimal.*

Remark 5.3. If $\Sigma = (P, g, \xi, u, a)$ is the induced structure on an umbilical ($A = \lambda I$) hypersurface M in a metallic Riemannian manifold $(\overline{M}, g, J)$ with $\overline{\nabla} J = 0$ we have for any $X, Y \in \mathfrak{X}(M)$:

$$\begin{cases} \text{(i)} & (\nabla_X P)(Y) = \lambda[g(X,Y)\xi + g(Y,\xi)X], \\ \text{(ii)} & (\nabla_X u)(Y) = \lambda[ag(X,Y) - g(X,PY)], \\ \text{(iii)} & \nabla_X \xi = \lambda(aX - P(X)), \quad \nabla_\xi \xi = \lambda(2a-p)\xi \\ \text{(iv)} & X(a) = -2\lambda g(X,\xi). \end{cases}$$

As consequence, from (11) (iii) and (10) we get:

Theorem 5.3. *If M is an invariant umbilical ($\lambda \neq 0$) hypersurface in a metallic Riemannian manifold $(\overline{M}, g, J)$ with $\overline{\nabla} J = 0$ and $\Sigma = (P, g, \xi, u, a)$ is the induced structure on M by (g, J), then $P = aI$, where a is a constant function on M equal with the metallic number $a = \sigma_{p,q}$ or $a = p - \sigma_{p,q}$.*

Remark 5.4. Conversely, if M is a hypersurface in a metallic Riemannian manifold $(\overline{M}, g, J)$ with $\overline{\nabla} J = 0$ and $\Sigma = (P, g, \xi, u, a)$ is the induced structure on M by (g, J) with $P = aI$, then $\nabla \xi = 0$. Thus, we obtain:

Theorem 5.4. *If M is a hypersurface in a metallic Riemannian manifold $(\overline{M}, g, J)$ with $\overline{\nabla} J = 0$ and $\Sigma = (P, g, \xi, u, a)$ is the induced structure on M by (g, J) with $P = aI$, then we have only one of the following conclusions:*

(i) M is an invariant hypersurface and a is a metallic number;
(ii) M is a non-invariant totally geodesic hypersurface in the metallic Riemannian manifold $(\overline{M}, g, J)$.

An example of induced structure on a hypersphere in a metallic Riemannian manifold. We consider the $(a+b)$-dimensional Euclidean space E^{a+b} as ambient space ($a, b \in \mathbb{N}^*$). Let $J : E^{a+b} \to E^{a+b}$ be the $(1,1)$ tensor field defined for every point $(x^1, \ldots, x^a, y^1, \ldots, y^b) \in E^{a+b}$ by:

$$J(x^1, \ldots, x^a, y^1, \ldots, y^b) = (\sigma_{p,q} x^1, \ldots, \sigma_{p,q} x^a, (p - \sigma_{p,q}) y^1, \ldots, (p - \sigma_{p,q}) y^b).$$

It follows that $J^2 = pJ + qI$ and the scalar product $\langle , \rangle$ on E^{a+b} is J-compatible. Thus $(E^{a+b}, \langle\rangle, J)$ is a metallic Riemannian manifold.

In E^{a+b} we can get the hypersphere

$$S^{a+b-1}(r) = \Big\{(x^1, \ldots, x^a, y^1, \ldots, y^b), \underbrace{\sum_{i=1}^{a} (x^i)^2}_{r_1^2} + \underbrace{\sum_{j=1}^{b} (y^j)^2}_{r_2^2} = r^2\Big\},$$

which is a submanifold of codimension 1 in E^{a+b}.

In every point $(x^1, \ldots, x^a, y^1, \ldots, y^b) := (x^i, y^j) \in S^{a+b-1}(r)$ ($i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b\}$) we consider the normal vector field to $S^{a+b-1}(r)$ given by $N = \frac{1}{r}(x^i, y^j)$.

In every point $(x^i, y^j) \in E^{a+b}$ we have a tangent vector on $S^{a+b-1}(r)$

$$(X^1, \ldots, X^a, Y^1, \ldots, Y^b) := (X^i, Y^j) \in T_{(x^1, \ldots, x^a, y^1, \ldots, y^b)}(S^{a+b-1}(r))$$

if and only if $\sum_{i=1}^{a} x^i X^i + \sum_{j=1}^{b} y^j Y^j = 0$.

From the decompositions of $J(N)$ and $J(X^i, Y^j)$ respectively, in tangential and normal components on $T_{(x^i,y^j)}S^{a+b-1}(r)$ we obtain:

$$J(N) = \xi + \mathcal{A} \cdot N; \quad J(X^i, Y^j) = P(X^i, Y^j) + u(X^i, Y^j) \cdot N,$$

where $(X^1, \ldots, X^a, Y^1, \ldots, Y^b) := (X^i, Y^j)$ is a tangent vector field on $S^{a+b-1}(r)$, P is an $(1,1)$ tensor field on $S^{a+b-1}(r)$, $\xi \in \mathfrak{X}(S^{a+b-1}(r))$, u is an 1-form on $S^{a+b-1}(r)$, and $\mathcal{A}$ is a smooth real function on $S^{a+b-1}(r)$.

Using $\mathcal{A} = \langle J(N), N\rangle$, $\xi = J(N) - \mathcal{A} \cdot N$, $u(X^i, Y^j) = \langle (X^i, Y^j), \xi\rangle$ and $P(X^i, Y^j) = J(X^i, Y^j) - u(X^i, Y^j) \cdot N$, the elements of the induced structure $\Sigma = (P, \langle\rangle, \xi, u, \mathcal{A})$ on $S^{a+b-1}(r)$ by the metallic Riemannian structure $(J, \langle\rangle)$ on E^{a+b} are given as follows:

$$\begin{cases} \mathcal{A} = \dfrac{\sigma_{p,q} r_1^2 + (p - \sigma_{p,q}) r_2^2}{r^2} \\ \xi = \dfrac{2\sigma_{p,q} - p}{r^3} (r_2^2 \cdot x^i, -r_1^2 \cdot y^j) \\ u(X) = \dfrac{2\sigma_{p,q} - p}{r} \mu \\ P(X) = (\sigma_{p,q} X^i - \dfrac{2\sigma_{p,q} - p}{r^2} \mu x^i, (p - \sigma_{p,q}) Y^j - \dfrac{2\sigma_{p,q} - p}{r^2} \mu y^j) \end{cases}$$

where $X := (X^i, Y^j) = (X^1, \ldots, X^a, Y^1, \ldots, Y^b)$ is a tangent vector field on $S^{a+b-1}(r)$ and $\mu := \sum_{i=1}^a x^i X^i = -\sum_{j=1}^b y^j Y^j$. In conclusion, $S^{a+b-1}(r)$ is a non-invariant hypersurface.

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