ON TAUBERIAN CONDITIONS FOR \((C,1)\) SUMMABILITY OF INTEGRALS

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Abstract. We investigate some Tauberian conditions in terms of the general control modulo of the oscillatory behavior of integer order of continuous real functions on \([0,\infty)\) for \((C,1)\) summability of integrals. Moreover, we obtain a Tauberian theorem for a real bounded function on \([0,\infty)\).

1. Introduction

Throughout this paper we assume that \(f(x)\) is a real valued continuous function on \([0,\infty)\) and \(s(x) = \int_{0}^{x} f(t) \, dt\). The symbols \(s(x) = o(1)\) and \(s(x) = O(1)\) mean that \(\lim_{x \to \infty} s(x) = 0\) and \(s(x)\) is bounded on \([0,\infty)\), respectively. The identity

\[ s(x) - \sigma(x) = v(x), \]

where \(v(x) = \frac{1}{x} \int_{0}^{x} t f(t) \, dt\) and \(\sigma(x) = \sigma(s(x)) = \frac{1}{x} \int_{0}^{x} s(t) \, dt\), is well-known and will be used in various steps of the proofs.

The classical control modulo of the oscillatory behavior of the function \(s(x)\) is denoted by \(\omega_{0}(x) = xf(x)\), and the general control modulo of the oscillatory behavior of integer order \(m \geq 1\) of the function \(s(x)\) is defined in [1] by \(\omega_{m}(x) = \omega_{m-1}(x) - \sigma(\omega_{m-1}(x))\). We note that \(\omega_{m}(x) = v(\omega_{m-1}(x))\) for any integer \(m \geq 1\). Moreover, \(\omega_{m}(x)\) can be written as \(\omega_{m}(x) = (v(v(\ldots(v(\omega_{0}(x))))\ldots))\) for any integer \(m \geq 1\).

For each integer \(m \geq 0\), \(\sigma_{m}(x)\) and \(v_{m}(x)\) are defined in [2] by

\[
\sigma_{m}(x) = \begin{cases} 
\frac{1}{x} \int_{0}^{x} \sigma_{m-1}(t) \, dt & m > 1 \\
\sigma(x) & m = 1 
\end{cases}
\]

and

\[
v_{m}(x) = \begin{cases} 
\frac{1}{x} \int_{0}^{x} v_{m-1}(t) \, dt & m \geq 1 \\
v(x) & m = 0
\end{cases}
\]

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respectively. The relationship between the functions \( \sigma_m(x) \) and \( v_m(x) \) is given by the identity \( \sigma_m(x) - \sigma_{m+1}(x) = v_m(x) \).

For a function \( s(x) \), we define
\[
\left( x \frac{d}{dx} \right)_m s(x) = \left( x \frac{d}{dx} \right)_{m-1} \left( x \frac{d}{dx} s(x) \right) = x \frac{d}{dx} \left( x \frac{d}{dx} \right)_{m-1} s(x)
\]
for any positive integer \( m \) and nonnegative \( x \), where \( \left( x \frac{d}{dx} \right)_0 s(x) = s(x) \), and \( \left( x \frac{d}{dx} \right)_1 s(x) = x \frac{d}{dx} s(x) \). A more useful identity for the general control modulo of the oscillatory behavior of integer order \( m \geq 1 \) of \( s(x) \) is
\[
\omega_m(x) = \left( x \frac{d}{dx} \right)_m v_{m-1}(x)
\]
(see [2] for the proof of this identity).

A function \( s(x) \) is said to be \((C,1)\) summable to a finite number \( \ell \) if
\[
\lim_{x \to \infty} \sigma(x) = \ell.
\]
(3)

It is well known that if \( s(x) \) is \((C,1)\) summable to \( \ell \), then \( \sigma(x) \) is \((C,1)\) summable to \( \ell \). But the converse is not true. For example, let \( s \) be defined by \( s(x) = \int_0^x (2 \sin t + t \cos t) \, dt \). Then, \( \sigma(x) \) is \((C,1)\) summable to 1, but \( s(x) \) is not \((C,1)\) summable to 1.

If the integral
\[
\int_0^\infty f(t) \, dt = \ell
\]
exists, then clearly \( s(x) \) is \((C,1)\) summable to \( \ell \). The converse is not necessarily true. For example, the function \( s \) defined by \( s(x) = \int_0^x \sin t \, dt \) is \((C,1)\) summable to 1, but it is not convergent in the ordinary sense. However, (3) may imply (4) by adding some suitable condition on \( s(x) \). Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

A real valued function \( s(x) \) is slowly oscillating \([3]\) if
\[
\lim_{\lambda \to 1^+} \lim_{x \to \infty} \sup_{x \leq t \leq \lambda x} |s(t) - s(x)| = 0.
\]
(5)

Note that \( \sigma(x) \) is slowly oscillating for every slowly oscillating function \( s(x) \).

On the other hand, \( \omega_0(x) = O(1) \) implies that \( s(x) \) is an slowly oscillating function. Indeed,
\[
|s(t) - s(x)| = \left| \int_x^t s'(u) \, du \right| \leq \int_x^t |f(u)| \, du \leq \int_x^t \frac{du}{u} = \log \frac{t}{x}
\]
By the definition of slowly oscillating function, we have
\[
\lim_{\lambda \to 1^+} \lim_{x \to \infty} \sup_{x \leq t \leq \lambda x} |s(t) - s(x)| \leq 0.
\]

The following classical Tauberian theorem is known as Littlewood type Tauberian theorem \([4]\).
Theorem 1. Let \( s(x) \) be \((C, 1)\) summable to \( \ell \). If 
\[
\omega_0(x) = O(1)
\]
then \( \int_0^\infty f(t) dt \) converges to \( \ell \).

Different proofs of Theorem 1 are given by Laforgia [5], Çanak and Totur [6].

Çanak and Totur [1] have proved the following theorems for \((C, 1)\) summability.

Theorem 2. Let \( s(x) \) be \((C, 1)\) summable to \( \ell \). If \( s(x) \) is slowly oscillating, then 
\[
\int_0^\infty f(t) dt \text{ converges to } \ell.
\]

Theorem 3. Let \( s(x) \) be \((C, 1)\) summable to \( \ell \). If \( v(x) \) is slowly oscillating, then 
\[
\int_0^\infty f(t) dt \text{ converges to } \ell.
\]

The aim of this paper is to investigate some Tauberian conditions in terms of the general control modulo of the oscillatory behavior of integer order of continuous real functions on \([0, \infty)\) for \((C, 1)\) summability of integrals. Also, we obtain a Tauberian theorem for a real bounded function \( s(x) \).

2. A Lemma

For the proofs of the main theorems in the next section we need the following Lemma in [1] showing the differences between \( s(x) \) and \( \sigma(\lambda x) \) for \( \lambda > 1 \) and \( 0 < \lambda < 1 \).

Lemma 1.
(i) For \( \lambda > 1 \),
\[
s(x) - \sigma(\lambda x) = \frac{1}{\lambda - 1} (\sigma(\lambda x) - \sigma(x)) - \frac{1}{\lambda x - x} \int_x^{\lambda x} (s(t) - s(x)) dt.
\]
(ii) For \( 0 < \lambda < 1 \),
\[
s(x) - \sigma(\lambda x) = \frac{1}{1 - \lambda} (\sigma(x) - \sigma(\lambda x)) + \frac{1}{x - \lambda x} \int_{\lambda x}^x (s(x) - s(t)) dt.
\]

3. Main results

In this section, the main theorems of the paper will be presented.

Theorem 4. Let \( s(x) \) be \((C, 1)\) summable to \( \ell \). If \( \sigma(\omega_m(x)) \) is slowly oscillating for some integer \( m \geq 0 \), then \( \int_0^\infty f(t) dt \) converges to \( \ell \).

Proof. Since \( s(x) \) is \((C, 1)\) summable to \( \ell \), then \( \sigma(x) \) is also \((C, 1)\) summable to \( \ell \). Thus, it follows from the identity (1) that \( v(x) \) is \((C, 1)\) summable to 0. By the definition of the general control modulo of oscillatory behavior of order \( m \), we obtain that
\[
\sigma(\omega_m(x)) \text{ is } (C, 1) \text{ summable to } 0 \tag{6}
\]

for some integer $m \geq 1$. On the other hand, since $\sigma(\omega_m(x))$ is slowly oscillating, we get, by Lemma 1(i),
\[
\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x)) = \frac{1}{\lambda - 1} (\sigma_2(\omega_m(\lambda x)) - \sigma_2(\omega_m(x)))
- \frac{1}{\lambda x - x} \int_x^{\lambda x} (\sigma(\omega(t)) - \sigma(\omega_m(x))) \, dt.
\] (7)

By (7) we have
\[
|\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x))| \leq \frac{1}{\lambda - 1} |\sigma_2(\omega_m(\lambda x)) - \sigma_2(\omega_m(x))| + \max_{x \leq t \leq \lambda x} |\sigma(\omega(t)) - \sigma(\omega_m(x))|.
\] (8)

Taking the lim sup of both sides of (8) as $x \to \infty$, we obtain
\[
\limsup_{x \to \infty} |\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x))| \leq \frac{1}{\lambda - 1} \limsup_{x \to \infty} |\sigma_2(\omega_m(\lambda x)) - \sigma_2(\omega_m(x))|
+ \limsup_{x \to \infty} \max_{x \leq t \leq \lambda x} |\sigma(\omega(t)) - \sigma(\omega_m(x))|.
\] (9)

Since $\sigma_2(\omega_m(x))$ converges by (6), the first term on the right-hand side of the inequality (9) vanishes. Therefore, the inequality (9) becomes
\[
\limsup_{x \to \infty} |\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x))| \leq \limsup_{x \to \infty} \max_{x \leq t \leq \lambda x} |\sigma(\omega(t)) - \sigma(\omega_m(x))|. \tag{10}
\]

Since $\sigma(\omega_m(x))$ is slowly oscillating, we have from (10) that $\lim_{x \to \infty} \sigma(\omega_m(x)) = 0$.

Analogously, since $s(x)$ is $(C,1)$ summable to $\ell$, we obtain that $\sigma_2(\omega_{m-1}(x))$ is $(C,1)$ summable to 0. It follows from the identity
\[
\sigma(\omega_m(x)) = \left( \frac{d}{dx} \right)_m v_m(x) = \frac{d}{dx} \left( \frac{d}{dx} \right)_{m-1} v_m(x) = \frac{d}{dx} \sigma_2(\omega_{m-1}(x))
\]
that $\frac{d}{dx} \sigma_2(\omega_{m-1}(x)) = O(1)$. If we apply Theorem 1 to $\sigma_2(\omega_{m-1}(x))$, we see that $\sigma_2(\omega_{m-1}(x)) = o(1)$. From the identity
\[
\sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = \sigma(\omega_m(x))
\]
we have $\lim_{x \to \infty} \sigma(\omega_{m-1}(x)) = 0$. Continuing in this vein, we get $\sigma(\omega_0(x)) = v(x) = o(1)$ as $x \to \infty$. Since $s(x)$ is $(C,1)$ summable to $\ell$ and $\lim_{x \to \infty} v(x) = 0$, $\lim_{x \to \infty} s(x) = \ell$. This completes the proof. \qed

Remark 5. Theorem 4 includes not only Theorem 3 but also Theorem 2. If $\sigma(\omega_m(x))$ is slowly oscillating for some integer $m \geq 0$, it follows from the identity $\sigma(\omega_{m+1}(x)) = v(\sigma(\omega_m(x)))$ that $\sigma(\omega_{m+1}(x))$ is slowly oscillating. However, the slow oscillation of $\sigma(\omega_m(x))$ does not imply the slow oscillation of $\sigma(\omega_{m-1}(x))$, since $\sigma(\omega_m(x)) = v(\sigma(\omega_{m-1}(x)))$. If we take $\sigma(\omega_{m-1}(x)) = \log(x+1) + \int_0^x \frac{\log(t+1)}{t+1} \, dt$, we have that $\sigma(\omega_{m-1}(x))$ is not slowly oscillating, but $\sigma(\omega_m(x)) = \log(x+1)$ is slowly oscillating. If we take $m = 0$ in Theorem 4, then we have Theorem 3.

An equivalent definition of slow oscillation of $s(x)$ given by Çanak and Totur [1] says that $s(x)$ is slowly oscillating if and only if $v(x)$ is slowly oscillating and bounded. If $v(x)$ is slowly oscillating, then $s(x)$ may not be slowly oscillating. For
example, if we take \( f(x) = \frac{1}{x+1} + \frac{\log(x+1)}{x+1} \), it is clear that \( s(x) \) is not slowly oscillating. It follows from the identity \( \sigma \) that \( v(x) = \log(x+1) \) is slowly oscillating.

**Corollary 6.** If \( s(x) \) is \((C,1)\) summable to \( \ell \) and \( \frac{d}{dx}(\sigma_1(\omega_m(x))) = O\left(\frac{p'(x)}{p(x)}\right) \) as \( x \to \infty \), where

\[
\lim_{\lambda \to 1^+} \limsup_{x \to \infty} \frac{p(\lambda x)}{p(x)} = 1, \tag{11}
\]

then \( \lim_{x \to \infty} s(x) = \ell \).

**Proof.** For any \( x \leq t \leq \lambda x \), we have

\[
|\sigma_1(\omega_m(t)) - \sigma_1(\omega_m(x))| = \left| \int_x^t \frac{d}{dx} \sigma_1(\omega_m(u)) du \right| \leq C \int_x^t \frac{p'(u)}{p(u)} du = C \log \frac{p(t)}{p(x)},
\]

whence we conclude that

\[
\limsup_{x \to \infty} \max_{x \leq t \leq \lambda x} |\sigma_1(\omega_m(t)) - \sigma_1(\omega_m(x))| \leq C \log \limsup_{x \to \infty} \frac{p(\lambda x)}{p(x)}.
\]

Taking the limit of both sides as \( \lambda \to 1^+ \), we obtain

\[
\lim_{\lambda \to 1^+} \limsup_{x \to \infty} \max_{x \leq t \leq \lambda x} |\sigma_1(\omega_m(t)) - \sigma_1(\omega_m(x))| = 0,
\]

i.e. \( \sigma_1(\omega_m(x)) \) is slowly oscillating. \( \square \)

**Corollary 7.** Let \( s(x) \) be \((C,1)\) summable to \( \ell \). If \( \omega_m(x) \) is slowly oscillating for some integer \( m \geq 0 \), then \( \int_0^\infty f(t) dt \) converges to \( \ell \).

**Proof.** Since \( \omega_m(x) \) is slowly oscillating, then \( \sigma(\omega_m(x)) \) is also slowly oscillating. Hence, the conditions in Theorem 4 are satisfied. \( \square \)

In the following theorem, we obtain convergence of \( s(x) \) out of the \((C,1)\) summability of \( \sigma(\omega_m(x)) \) instead of the \((C,1)\) summability of \( s(x) \) with the same condition as in Theorem 4.

**Theorem 8.** Let \( s(x) \) be \((C,1)\) summable to \( \ell \). If \( \sigma(\omega_m(x)) \) is slowly oscillating for some integer \( m \geq 0 \), then \( \int_0^\infty f(t) dt \) converges to \( \ell \).

**Proof.** Since \( \sigma(x) \) is \((C,1)\) summable to \( \ell \), from the identity \( \square \), we have \( v_1(x) \) is \((C,1)\) summable to 0. Thus, \( \sigma_2(\omega_m(x)) \) is \((C,1)\) summable to 0. Since \( \sigma(\omega_m(x)) \) is slowly oscillating for some nonnegative integer \( m \), then we have \( v(\sigma(\omega_m(x))) = \sigma(\omega_{m+1}(x)) = O(1) \). It follows from the identity \( \sigma(\omega_{m+1}(x)) = x \frac{d}{dx} \sigma_2(\omega_m(x)) \) that we have \( x \frac{d}{dx} \sigma_2(\omega_m(x)) = O(1) \). If we apply Theorem 4 to \( \sigma_2(\omega_m(x)) \), we see that

\[
\sigma_2(\omega_m(x)) = o(1). \tag{12}
\]

Hence, \( \sigma_1(\omega_m(x)) \) is \((C,1)\) summable to 0 and \( \sigma_1(\omega_m(x)) \) is slowly oscillating for some nonnegative integer \( m \). Analogously, continuing as in the proof of Theorem 4, we obtain \( \sigma_1(\omega_m(x)) = o(1) \) as \( x \to \infty \). It follows from the identity \( \sigma(\omega_m(x)) = x \frac{d}{dx} \sigma_2(\omega_{m-1}(x)) \) that we have \( x \frac{d}{dx} \sigma_2(\omega_{m-1}(x)) = O(1) \). If we apply Theorem 4 to \( \sigma_2(\omega_{m-1}(x)) \), we see that

\[
\sigma_2(\omega_{m-1}(x)) = o(1). \tag{13}
\]
By the assumption and (13) we obtain from the identity
\[ \sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = \sigma(\omega_m(x)) \]
that \( \sigma(\omega_{m-1}(x)) = o(1). \)

Continuing in this vein, we obtain \( \sigma_2(\omega_1(x)) = o(1). \) Since \( \sigma(x) \) is \((C,1)\) summable to \( \ell \), we have \( v_2(x) = o(1) \). Therefore, from the identity
\[ \sigma_2(\omega_1(x)) = \sigma_2(\omega_0(x)) - \sigma_3(\omega_0(x)) = v_1(x) - v_2(x) \]
we get \( v_1(x) = o(1) \). Since \( \sigma(x) \) is \((C,1)\) summable to \( \ell \), we have \( \sigma_2(x) \to \ell \) as \( x \to \infty \). From the identity \( \sigma(x) - \sigma_2(x) = v_1(x) \), we obtain \( \sigma(x) \to \ell \) as \( x \to \infty \). Thus, \( s(x) \) is \((C,1)\) summable to \( \ell \). Since the conditions in Theorem 4 are satisfied, the proof is completed. \( \square \)

**Theorem 9.** Let \( s(x) \) be bounded. If \( \frac{d}{dx} (\sigma(\omega_m(x))) \) is slowly oscillating for some integer \( m \geq 0 \), then \( f(x) \) converges to 0 as \( x \to \infty \).

**Proof.** Since \( s(x) \) is assumed to be bounded, \( \sigma(\omega_m(x)) \) is bounded for every nonnegative integer \( m \). Thus, we have
\[ \sigma \left( \frac{d}{dx} \sigma(\omega_m(x)) \right) = \frac{1}{x} \int_0^x \frac{d}{dx} (\sigma(\omega_m(x))) = \frac{\sigma(\omega_m(x))}{x} = o(1), \quad x \to \infty. \]

Thus, we obtain that \( \frac{d}{dx} (\sigma(\omega_m(x))) \) is \((C,1)\) summable to 0. By hypothesis, if we apply Theorem 4 to \( \frac{d}{dx} (\sigma(\omega_m(x))) \), we see that \( \frac{d}{dx} (\sigma(\omega_m(x))) \) converges to 0. If we write \( \frac{d}{dx} (\sigma(\omega_{m-1}(x))) \) instead of \( s(x) \) in the identity (1), we have
\[ \frac{d}{dx} (\sigma(\omega_{m-1}(x))) - \frac{\sigma(\omega_m(x))}{x} = \frac{d}{dx} (\sigma(\omega_m(x))). \]

So, \( \frac{d}{dx} (\sigma(\omega_{m-1}(x))) = o(1) \). Continuing in this manner, we obtain \( \frac{d}{dx} (\sigma(\omega_0(x))) = v'(x) = o(1) \) as \( x \to \infty \). On the other hand, boundedness of \( s(x) \) implies boundedness of \( v(x) \). Finally, applying the identity (1) to \( \frac{d}{dx} s(x) \), we have
\[ s'(x) = \frac{v(x)}{x} + v'(x) = o(1). \]

Hence \( s'(x) = f(x) = o(1) \). This completes the proof. \( \square \)

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