ABSTRACT. In this article, we consider graph painting and introduce the Borel-de Siebenthal property. We study this property on graphs of type $E$. We also study its applications in Hermitian symmetric spaces and hyperbolic Kac-Moody Lie algebras.

1. INTRODUCTION

A painting on a connected graph $\Gamma$ is a color assignment of white (unpainted) or black (painted) on its vertices. If $p$ is a painting on $\Gamma$ and $\alpha$ is a black vertex, we define another painting $F_{\alpha} p$ by

$$F_{\alpha} : \text{reverse the colors of all vertices adjacent to } \alpha.$$ (1.1)

We make the convention that the adjacent vertices do not include $\alpha$ itself, so $\alpha$ remains black. We say that two paintings $p, q$ on $\Gamma$ are equivalent if there exists a sequence of paintings on $\Gamma$,

$$p = p_0 \to p_1 \to \ldots \to p_m = q$$

such that each $p_i \to p_{i+1}$ is given by some $F_{\alpha}$. We always ignore the painting with all white vertices, because it is not equivalent to any other painting. For a given painting, we look for an equivalent painting with the least number of black vertices. The following terminology is motivated by a theorem of Borel and de Siebenthal [6, Thm. 6.96].

Definition 1.1.

(a) We say that $\Gamma$ is a Borel-de Siebenthal graph if every painting on $\Gamma$ is equivalent to a painting with a single black vertex.

(b) We say that $\Gamma$ is a strictly Borel-de Siebenthal graph if every connected subgraph of $\Gamma$ is Borel-de Siebenthal.

We say that a graph is of type $E$ if it looks like Figure 1. If it has $n$ vertices, as indicated in the figure, we call it $E_n$. We assume that $n \geq 6$. The indices in

2010 Mathematics Subject Classification. 05C25, 17B22.

Key words and phrases. Graph painting, Lie algebra.
Figure 1.

The following is the main result of this article.

**Theorem 1.2.** For all \( n \), \( E_n \) is a strictly Borel-de Siebenthal graph.

We prove the theorem in Section 2. While the theorem and its proof are purely combinatorial, they have unexpected algebraic applications in Lie theory. We discuss in Section 3 the relations of this theorem with Hermitian symmetric spaces and hyperbolic Kac-Moody Lie algebras, leading to Corollaries 3.3 and 3.4.

**Acknowledgement.** The author is grateful to the referee, who read the manuscript carefully and improved the presentation of this article.

## 2. Proof of the main theorem

In this section, we prove Theorem 1.2. Label the vertices of \( E_n \) by \( \{1, \ldots, n\} \) as in Figure 1. Let \( D_{n-1} \) be the subgraph formed by vertices \( 1, \ldots, n - 1 \), and let \( A_{n-2} \) be the subgraph formed by vertices \( 1, \ldots, n - 2 \). Hence \( A_{n-2} \subset D_{n-1} \subset E_n \).

**Proposition 2.1.** [1, Table 1] For all \( n \), \( A_n \) and \( D_n \) are Borel-de Siebenthal graphs.

To show that \( E_n \) is also a Borel-de Siebenthal graph, we introduce the following notations. Let \( (i_1, \ldots, i_r) \) denote the painting on \( E_n \) whose vertices \( i_1, \ldots, i_r \) are black. If a black vertex \( \alpha \) is vertex \( i \) of Figure 1, we write \( F_i \) to denote \( F_\alpha \) of (1.1). For example if \( p = (2,3) \), then \( F_{2p} = F_2(2,3) = (1,2) \). We also underline \( i \) to denote \( F_i \), for example \( (2,3) \rightarrow (1,2) \).

**Proposition 2.2.** On \( E_n \),

(a) \( (i, n) \sim (i + 1, n - 1) \) for \( i < n - 3 \);

(b) \( (i, n - 1) \sim (i + 1, n - 2) \) for \( i < n - 3 \);

(c) \( (i, n - 2) \sim (i + 2) \) for \( i < n - 4 \).

**Proof.** For part (a), start with \( (i, n) \rightarrow (i, n - 2, n) \rightarrow (i, n - 3, n - 2) \). Then apply \( F_{n-3}, F_{n-4}, \ldots, F_{i+1} \) to \( (i, n - 3, n - 2) \) and get \( (i + 1, n - 1) \). This proves part (a).
For part (b), apply $F_{n-1}, F_{n-3}, F_{n-4}, \ldots, F_{i+1}$ to $(i, n - 1)$ and get $(i + 1, n - 2)$. Finally for part (c),

$$(F_{i+2} \ldots F_{n-3} F_{n-2} F_n) (F_{i+1} \ldots F_{n-3} F_{n-2}) (i, n - 2)$$

$$= (F_{i+2} \ldots F_{n-3} F_{n-2} F_n) (i + 1, n - 1, n)$$

$$= (i + 2).$$

This proves the proposition. 

Proposition 2.3. For all $n$, $E_n$ is a Borel-de Siebenthal graph.

Proof. Let $p$ be a painting on $E_n$. Let $\overline{p}$ denote its restriction to $D_{n-1}$. By Proposition 2.1, $\overline{p} \sim (r)$. Hence a sequence $F_{i_1}, \ldots, F_{i_m}$ transforms $\overline{p}$ to $(r)$. The same sequence transforms $p$ to either $(r)$ or $(r, n)$, depending on the parity of occurrence of $n - 2$ in $\{i_1, \ldots, i_m\}$. If $p \sim (r)$, there is nothing to prove. Hence we may assume that $p \sim (r, n)$. (2.1)

We want to show that in (2.1), $(r, n)$ is equivalent to a painting with a single black vertex. We divide the arguments into several cases depending on $r$.

$r = n - 1$:

$(n - 1, n) \to (n - 2, n - 1, n) \to (n - 3, n - 2, n - 1) \to (n - 4, n - 3) \to \ldots \to (1)$.

$r = n - 2$:

$(n - 2, n) \to (n)$.

$r = n - 3$:

$(n - 3, n) \to (n - 3, n - 2, n) \to (n - 2)$.

For $r = n - 4$, we have

$$(n - 4, n) \sim (n - 3, n - 1) \quad \text{by Proposition 2.2(a)}$$

$$\sim (n - 1) \quad \text{by } F_{n-1}.$$

For $r = n - 5$, we have

$$(n - 5, n) \sim (n - 4, n - 1) \quad \text{by Proposition 2.2(a)}$$

$$\sim (n - 3, n - 2) \quad \text{by Proposition 2.2(b)}$$

$$\sim (n - 2, n) \sim (n) \quad \text{by } F_{n-2} \text{ and } F_n.$$

For $r = n - 6$, we start with $(n - 6, n) \sim (n - 5, n - 1) \sim (n - 4, n - 2)$ by Proposition 2.2(a,b). This is followed by

$$(n - 4, n - 2) \to (n - 4, n - 3, n - 2, n) \to (n - 3, n - 1, n)$$

$$\to (n - 3, n - 2, n - 1, n) \to (n - 2, n - 1) \to (n - 3, n - 2, n - 1).$$

Then $(n - 3, n - 2, n - 1) \sim (n - 4, n - 3) \sim \ldots \sim (1)$ by $F_{n-3}, F_{n-4}, \ldots, F_1$. 
Finally let $r \leq n - 7$. Since $r$ is sufficiently far from the right end of the graph, we can apply the general arguments of Proposition 2.2 by

\[ (r, n) \sim (r + 1, n - 1) \quad \text{by Proposition 2.2(a)} \]
\[ \sim (r + 2, n - 2) \quad \text{by Proposition 2.2(b)} \]
\[ \sim (r + 4) \quad \text{by Proposition 2.2(c)} \]

We have shown that in all cases of $r$ in (2.1), $(r, n)$ is equivalent to a painting with a single black vertex. This proves Proposition 2.3. \(\square\)

**Proof of Theorem 1.2**: Any connected subgraph of $E_n$ is $A_k$, $D_k$ or $E_k$. Propositions 2.1 and 2.3 say that $A_k$, $D_k$ and $E_k$ are all Borel-de Siebenthal graphs. Hence $E_n$ is a strictly Borel-de Siebenthal graph. \(\square\)

**Remark 2.4.** In view of Theorem 1.2, the equivalence classes of paintings on $E_n$ can be parametrized by paintings with one black vertex. However, the classification of equivalence classes is a nontrivial matter, and various $n$ have different results. We list a few cases here:

- $E_6$ has 2 equivalence classes, parametrized by (1) and (5).
- $E_7$ has 3 equivalence classes, parametrized by (1), (2) and (6).
- $E_8$ has 2 equivalence classes, parametrized by (1) and (8).
- $E_9$ has 3 equivalence classes, parametrized by (1), (2) and (9).

The above results on $E_6$, $E_7$ and $E_8$ are given in [1, p. 23-24], and the result on $E_9$ is given in [2, p. 115]. These examples show that the situation varies greatly among various $E_n$. It will be interesting to systematically solve this problem for all $E_n$.

### 3. Applications in Lie Theory

In this section, we discuss the significance of Theorem 1.2 for $n = 9$ and $n = 10$. A finite dimensional complex simple Lie algebra $\mathfrak{g}$ is said to be *simply-laced* if its root system contains only one root length. The simply-laced Lie algebras are $A_n$, $D_n$, $E_6$, $E_7$ and $E_8$ [6, Appendix C], so their Dynkin diagrams $\Gamma$ are just graphs in the usual sense. Recall the notion of Borel-de Siebenthal graph and strictly Borel-de Siebenthal graph in Definition 1.1.

**Theorem 3.1.** Every simply-laced Dynkin diagram is a strictly Borel-de Siebenthal graph.

**Proof.** By a theorem of Borel and de Siebenthal [6, Thm. 6.96], every simply-laced Dynkin diagram is a Borel-de Siebenthal graph (the theorem also works for non-simply-laced diagrams, by modifying (1.1) suitably). Every subset of a simple system is itself a simple system of a smaller root system, so every subdiagram of
a Dynkin diagram is a Dynkin diagram. It follows that all Dynkin diagrams are strictly Borel-de Siebenthal.

The theorem of Borel and de Siebenthal provides an indirect proof for our theorem, in the sense that it does not explicitly show how any given painting is simplified to a painting with one black vertex. The explicit simplification process is given in [1], as summarized in [1, Cor. 5.2]. □

The natural question is the converse of Theorem 3.1, namely whether every strictly Borel-de Siebenthal graph is a Dynkin diagram. If this were true, it would provide an equivalent definition of the simply-laced Dynkin diagrams. In [1, §5], the authors state without proof that $E_9$ is strictly Borel-de Siebenthal though it is not Dynkin. The proof is now given by Theorem 1.2 confirming that “strict Borel-de Siebenthal is weaker that simply-laced Dynkin.”

Corollary 3.2.

$$\{\text{strictly Borel-de Siebenthal graphs}\} = \{\text{simply-laced Dynkin diagrams}\} \cup \{E_9, E_{10}, \ldots\}.$$  

Proof. The $\supset$ part of Corollary 3.2 follows from Theorems 1.2 and 3.1. It remains to prove its $\subset$ part.

We first introduce some terminologies and notations. The degree of a vertex of a graph is the number of edges joint to it. The vertex is said to be a branch point if its degree is $\geq 3$. If a graph has a unique branch point $p$ of degree 3, it can be expressed as $(a, b, c)$, where $a \leq b \leq c$ are positive integers which describe the lengths of the “legs” joint to $p$. For example, $E_6$ is $(1, 2, 2)$ and $E_7$ is $(1, 2, 3)$.

We recall that in [1, §5] it is stated that $E_9$ is a Borel-de Siebenthal graph, however the other non-Dynkin diagrams in [1, (5.1)] fail to be Borel-de Siebenthal graph because the paintings given in [1, (5.2)] are not equivalent to paintings with a single black vertex. Since the first graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph is acyclic, i.e. it cannot contain any loop. Since the second graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph cannot contain any branch point with degree $\geq 4$. Since the third graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph can have at most one branch point. Hence a strictly Borel-de Siebenthal graph is either $A_n$ or contains a unique branch point with degree 3. Assume the latter and express it as $(a, b, c)$ as described above. Since the fourth graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph $(a, b, c)$ satisfies $a = 1$. Since the fifth graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal
graph \((1, b, c)\) satisfies \(b < 3\). So we get either \((1, 1, c)\) or \((1, 2, c)\). The former is \(D_n\) and the latter is \(E_n\). We conclude that a strictly Borel-de Siebenthal graph is \(A_n\), \(D_n\) or \(E_n\). This proves the \(\subset\) part of Corollary 3.2. \(\square\)

The graphs in [1] (5.1) are actually all extended simply-laced Dynkin diagrams \(\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7\) and \(\widetilde{E}_8\) [4] Ch. X-5 Table 1] [5] Ch. 4 Table Aff 1]. In other words,
\[
\begin{array}{l}
(a) \quad \widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7 \text{ are not Borel-de Siebenthal;}
(b) \quad \widetilde{E}_8 \text{ is Borel-de Siebenthal.}
\end{array}
\]

We now discuss the algebraic significance of (3.1) in terms of symmetric spaces.

Let \(G\) be a real simple Lie group with Lie algebra \(\mathfrak{g}\). Let \(K\) be a maximal compact subgroup of \(G\). Then \(G/K\) is a Riemannian symmetric space [4] Ch. IV]. We say that \(G\) is of Hermitian type if the \(G\)-invariant Riemannian structure on \(G/K\) is Hermitian.

**Corollary 3.3.** \(E_8\) is the only simply-laced Lie algebra without any real form of Hermitian type.

**Proof.** Let \(G\) be a real simple Lie group with Lie algebra \(\mathfrak{g}\). Let \(L = \mathfrak{g} \otimes \mathbb{C}\). Let \(\Gamma\) be the extended Dynkin diagram of \(L\), and suppose that it is simply-laced.

Let \(\theta\) be a Cartan involution of \(\mathfrak{g}\), extended by complex linearity to \(L\). Then \(\theta\) is represented by a painting \(p\) on \(\Gamma\). Furthermore, \(G\) is of Hermitian (resp. non-Hermitian) type if and only if \(p\) is a painting with two (resp. one) black vertices, and in that case \(L^\theta\) has a 1-dimensional center (resp. \(L^\theta\) is semisimple) [3] §2]. Here \(L^\theta\) denotes the fixed points of \(\theta\). If \(q\) is another painting on \(\Gamma\) with a single black vertex, then \(q\) represents an \(L\)-automorphism \(\sigma\) such that \(L^\sigma\) is semisimple [4] Ch. X-5 Thm. 5.15(ii)][5] Ch. 8 Prop. 8.6]. Therefore, \(\mathfrak{g}\) is of Hermitian type if and only if \(\theta\) is not conjugate to such \(\sigma\), namely \(p\) is not equivalent to such \(q\).

In [1] (5.2)], there are paintings with two black vertices on \(\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7\) which are not equivalent to paintings with a single black vertex, so each of them represents a real form of Hermitian type. Such a painting does not exist on \(E_0 = \widetilde{E}_8\), so \(E_8\) has no real form of Hermitian type. This proves the corollary. \(\square\)

The vertices of an extended Dynkin diagram are equipped with positive integers, known as their labels [4] Ch. X-5 Table 1] [5] Ch. 4 Table Aff 1]. When \(\mathfrak{g}\) is represented by a painting with two black vertices, the black vertices occur on vertices with label 1. The list of extended Dynkin diagrams shows that \(\widetilde{E}_8\) is the only simply-laced diagram without a pair of vertices with label 1. This explains why it does not accept a painting of two black vertices that is not equivalent to a single black vertex.

Next we consider another application of Theorem [1,2] by setting \(n = 10\). Here \(E_{10}\) is the Dynkin diagram of a hyperbolic Kac-Moody Lie algebra of highest rank.
We denote the Lie algebra of $E_{10}$ by $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Pi \subset \mathfrak{h}^*$ be a simple system of roots. Let $\theta$ be a $\mathfrak{g}$-involution. We say that $\theta$ preserves $\mathfrak{h}$ and $\Pi$ if $\theta(\mathfrak{h}) = \mathfrak{h}$, and $\theta$ maps a simple root space to another simple root space.

**Corollary 3.4.** Let $\mathfrak{g}$ be the hyperbolic Kac-Moody Lie algebra of $E_{10}$, and let $\theta$ be a $\mathfrak{g}$-involution which preserves $\mathfrak{h}$ and a simply system $\Pi$. Then there exists another simple system $\Pi'$ such that $\theta = -1$ on one simple root space of $\Pi'$ and $\theta = 1$ on the remaining root spaces.

**Proof.** Suppose that $\theta$ preserves $\mathfrak{h}$ and $\Pi$. Let $\mathfrak{g}_\alpha$ be the root space of $\alpha$. Then $\theta$ leads to a permutation $d$ on $\Pi$ by $\theta\mathfrak{g}_\alpha = \mathfrak{g}_{d\alpha}$ for all $\alpha \in \Pi$. The Dynkin diagram of $E_{10}$ does not have nontrivial diagram symmetry, hence $d$ is trivial. It follows that $\theta$ preserves each root space. Since $\theta$ has order 2, it acts as $\pm 1$ on each root space.

If we represent $\Pi$ by the vertices of $E_{10}$, we obtain a painting $p$ where a vertex is white (resp. black) if and only if $\theta = 1$ (resp. $\theta = -1$) on the corresponding root space. This is the motivation for Vogan diagrams [6, Ch. VI-8]. Suppose that $\alpha$ is a black vertex and $\mathfrak{g}_\alpha$ is its reflection map on the root system. Then the new painting which represents $\theta$ with respect to $r_\alpha \Pi$ is $F_\alpha p$, where $F_\alpha$ is given by (1.1). By Theorem 1.2 there exists a sequence of some $F_\alpha$ such that $q = F_{\alpha_1} \cdots F_{\alpha_m} p$ has a single black vertex. Let $\Pi' = r_{\alpha_1} \cdots r_{\alpha_m} \Pi$, so that $q$ represents $\theta$ with respect to $\Pi'$. Since $\alpha \in \Pi'$ is white (resp. black) with respect to $q$ if and only if $\theta = 1$ (resp. $\theta = -1$) on the corresponding root space, $\Pi'$ is our desired simple system. □

**References**


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*Received: May 12, 2012*

*Accepted: December 28, 2012*