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GRAPH PAINTING AND THE BOREL-DE SIEBENTHAL PROPERTY

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Abstract. In this article, we consider graph painting and introduce the Borel-de Siebenthal property. We study this property on graphs of type E. We also study its applications in Hermitian symmetric spaces and hyperbolic Kac-Moody Lie algebras.

1. Introduction

A *painting* on a connected graph Γ is a color assignment of white (unpainted) or black (painted) on its vertices. If p is a painting on Γ and α is a black vertex, we define another painting $F_\alpha p$ by

$$F_\alpha : \text{ reverse the colors of all vertices adjacent to } \alpha. \tag{1.1}$$

We make the convention that the adjacent vertices do not include α itself, so α remains black. We say that two paintings p, q on Γ are equivalent if there exists a sequence of paintings on Γ,

$$p = p_0 \to p_1 \to \ldots \to p_m = q$$

such that each $p_i \to p_{i+1}$ is given by some F_α. We always ignore the painting with all white vertices, because it is not equivalent to any other painting. For a given painting, we look for an equivalent painting with the least number of black vertices. The following terminology is motivated by a theorem of Borel and de Siebenthal [6, Thm. 6.96].

Definition 1.1.
(a) We say that Γ is a *Borel-de Siebenthal graph* if every painting on Γ is equivalent to a painting with a single black vertex.
(b) We say that Γ is a *strictly Borel-de Siebenthal graph* if every connected subgraph of Γ is Borel-de Siebenthal.

We say that a graph is of type E if it looks like Figure 1. If it has n vertices, as indicated in the figure, we call it E_n. We assume that $n \geq 6$. The indices in

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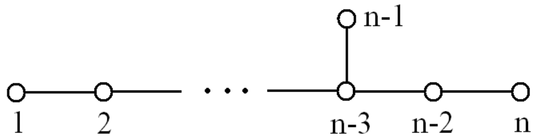

FIGURE 1.

the figure are assigned for future reference. The following is the main result of this article.

Theorem 1.2. *For all n, E_n is a strictly Borel-de Siebenthal graph.*

We prove the theorem in Section 2. While the theorem and its proof are purely combinatorial, they have unexpected algebraic applications in Lie theory. We discuss in Section 3 the relations of this theorem with Hermitian symmetric spaces and hyperbolic Kac-Moody Lie algebras, leading to Corollaries 3.3 and 3.4.

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2. Proof of the main theorem

In this section, we prove Theorem 1.2. Label the vertices of E_n by $\{1, \ldots, n\}$ as in Figure 1. Let D_{n-1} be the subgraph formed by vertices $1, \ldots, n-1$, and let A_{n-2} be the subgraph formed by vertices $1, \ldots, n-2$. Hence $A_{n-2} \subset D_{n-1} \subset E_n$.

Proposition 2.1. [1, Table 1] *For all n, A_n and D_n are Borel-de Siebenthal graphs.*

To show that E_n is also a Borel-de Siebenthal graph, we introduce the following notations. Let $(i_1, \ldots, i_r)$ denote the painting on E_n whose vertices $i_1, \ldots, i_r$ are black. If a black vertex α is vertex i of Figure 1, we write F_i to denote F_α of (1.1). For example if $p = (2, 3)$, then $F_2 p = F_2(2, 3) = (1, 2)$. We also underline i to denote F_i, for example $(\underline{2}, 3) \to (1, 2)$.

Proposition 2.2. *On E_n,*

(a) $(i, n) \sim (i+1, n-1)$ *for* $i < n-3$;

(b) $(i, n-1) \sim (i+1, n-2)$ *for* $i < n-3$;

(c) $(i, n-2) \sim (i+2)$ *for* $i < n-4$.

Proof. For part (a), start with $(i, \underline{n}) \to (i, \underline{n-2}, n) \to (i, n-3, n-2)$. Then apply $F_{n-3}, F_{n-4}, \ldots, F_{i+1}$ to $(i, n-3, n-2)$ and get $(i+1, n-1)$. This proves part (a).

For part (b), apply $F_{n-1}, F_{n-3}, F_{n-4}, \ldots, F_{i+1}$ to $(i, n-1)$ and get $(i+1, n-2)$. Finally for part (c),

$$\begin{aligned}(F_{i+2} \ldots F_{n-3}F_{n-2}F_n)&(F_{i+1} \ldots F_{n-3}F_{n-2})(i, n-2) \\ &= (F_{i+2} \ldots F_{n-3}F_{n-2}F_n)(i+1, n-1, n) \\ &= (i+2).\end{aligned}$$

This proves the proposition. □

Proposition 2.3. *For all n, E_n is a Borel-de Siebenthal graph.*

Proof. Let p be a painting on E_n. Let $\overline{p}$ denote its restriction to D_{n-1}. By Proposition 2.1, $\overline{p} \sim (r)$. Hence a sequence $F_{i_1}, \ldots, F_{i_m}$ transforms $\overline{p}$ to (r). The same sequence transforms p to either (r) or (r, n), depending on the parity of occurrence of $n-2$ in $\{i_1, \ldots, i_m\}$. If $p \sim (r)$, there is nothing to prove. Hence we may assume that

$$p \sim (r, n). \tag{2.1}$$

We want to show that in (2.1), (r, n) is equivalent to a painting with a single black vertex. We divide the arguments into several cases depending on r.

$r = n-1$:
$(n-1, \underline{n}) \to (\underline{n-2}, n-1, n) \to (\underline{n-3}, n-2, n-1) \to (\underline{n-4}, n-3) \to \ldots \to (1)$.
$r = n-2 : (n-2, \underline{n}) \to (n)$.
$r = n-3 : (n-3, \underline{n}) \to (n-3, \underline{n-2}, n) \to (n-2)$.

For $r = n-4$, we have

$$\begin{aligned}(n-4, n) &\sim (n-3, n-1) && \text{by Proposition 2.2(a)} \\ &\sim (n-1) && \text{by } F_{n-1}.\end{aligned}$$

For $r = n-5$, we have

$$\begin{aligned}(n-5, n) &\sim (n-4, n-1) && \text{by Proposition 2.2(a)} \\ &\sim (n-3, n-2) && \text{by Proposition 2.2(b)} \\ &\sim (n-2, n) \sim (n) && \text{by } F_{n-2} \text{ and } F_n.\end{aligned}$$

For $r = n-6$, we start with $(n-6, n) \sim (n-5, n-1) \sim (n-4, n-2)$ by Proposition 2.2(a,b). This is followed by

$$\begin{aligned}(n-4, \underline{n-2}) &\to (n-4, \underline{n-3}, n-2, n) \to (n-3, n-1, \underline{n}) \\ &\to (n-3, \underline{n-2}, n-1, n) \to (n-2, \underline{n-1}) \to (n-3, n-2, n-1).\end{aligned}$$

Then $(n-3, n-2, n-1) \sim (n-4, n-3) \sim \cdots \sim (1)$ by $F_{n-3}, F_{n-4}, \ldots, F_1$.

Finally let $r \leq n-7$. Since r is sufficiently far from the right end of the graph, we can apply the general arguments of Proposition 2.2 by

$$\begin{aligned} (r,n) &\sim (r+1,n-1) && \text{by Proposition 2.2(a)} \\ &\sim (r+2,n-2) && \text{by Proposition 2.2(b)} \\ &\sim (r+4). && \text{by Proposition 2.2(c)} \end{aligned}$$

We have shown that in all cases of r in (2.1), (r,n) is equivalent to a painting with a single black vertex. This proves Proposition 2.3. □

Proof of Theorem 1.2: Any connected subgraph of E_n is A_k, D_k or E_k. Propositions 2.1 and 2.3 say that A_k, D_k and E_k are all Borel-de Siebenthal graphs. Hence E_n is a strictly Borel-de Siebenthal graph. □

Remark 2.4. In view of Theorem 1.2, the equivalence classes of paintings on E_n can be parametrized by paintings with one black vertex. However, the classification of equivalence classes is a nontrivial matter, and various n have different results. We list a few cases here:

E_6 has 2 equivalence classes, parametrized by (1) and (5).
E_7 has 3 equivalence classes, parametrized by (1), (2) and (6).
E_8 has 2 equivalence classes, parametrized by (1) and (8).
E_9 has 3 equivalence classes, parametrized by (1), (2) and (9).

The above results on E_6, E_7 and E_8 are given in [1, p. 23-24], and the result on E_9 is given in [2, p. 115]. These examples show that the situation varies greatly among various E_n. It will be interesting to systematically solve this problem for all E_n.

3. Applications in Lie Theory

In this section, we discuss the significance of Theorem 1.2 for $n = 9$ and $n = 10$. A finite dimensional complex simple Lie algebra $\mathfrak{g}$ is said to be *simply-laced* if its root system contains only one root length. The simply-laced Lie algebras are A_n, D_n, E_6, E_7 and E_8 [6, Appendix C], so their Dynkin diagrams Γ are just graphs in the usual sense. Recall the notion of Borel-de Siebenthal graph and strictly Borel-de Siebenthal graph in Definition 1.1.

Theorem 3.1. *Every simply-laced Dynkin diagram is a strictly Borel-de Siebenthal graph.*

Proof. By a theorem of Borel and de Siebenthal [6, Thm. 6.96], every simply-laced Dynkin diagram is a Borel-de Siebenthal graph (the theorem also works for non-simply-laced diagrams, by modifying (1.1) suitably). Every subset of a simple system is itself a simple system of a smaller root system, so every subdiagram of a Dynkin diagram is a Dynkin diagram. It follows that all Dynkin diagrams are strictly Borel-de Siebenthal.

The theorem of Borel and de Siebenthal provides an indirect proof for our theorem, in the sense that it does not explicitly show how any given painting is simplified to a painting with one black vertex. The explicit simplification process is given in [1], as summarized in [1, Cor. 5.2]. □

The natural question is the converse of Theorem 3.1, namely whether every strictly Borel-de Siebenthal graph is a Dynkin diagram. If this were true, it would provide an equivalent definition of the simply-laced Dynkin diagrams. In [1, §5], the authors state without proof that E_9 is strictly Borel-de Siebenthal though it is not Dynkin. The proof is now given by Theorem 1.2, confirming that "strict Borel-de Siebenthal is weaker that simply-laced Dynkin."

Corollary 3.2.

$$\{\textit{strictly Borel-de Siebenthal graphs}\} = \{\textit{simply-laced Dynkin diagrams}\} \cup \{E_9, E_{10}, \dots\}.$$

Proof. The $\supset$ part of Corollary 3.2 follows from Theorems 1.2 and 3.1. It remains to prove its $\subset$ part.

We first introduce some terminologies and notations. The *degree* of a vertex of a graph is the number of edges joint to it. The vertex is said to be a *branch point* if its degree is ≥ 3. If a graph has a unique branch point p of degree 3, it can be expressed as (a, b, c), where $a \leq b \leq c$ are positive integers which describe the lengths of the "legs" joint to p. For example, E_6 is $(1, 2, 2)$ and E_7 is $(1, 2, 3)$.

We recall that in [1, §5] it is stated that E_9 is a Borel-de Siebenthal graph, however the other non-Dynkin diagrams in [1, (5.1)] fail to be Borel-de Siebenthal graph because the paintings given in [1, (5.2)] are not equivalent to paintings with a single black vertex. Since the first graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph is acyclic, i.e. it cannot contain any loop. Since the second graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph cannot contain any branch point with degree ≥ 4. Since the third graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph can have at most one branch point. Hence a strictly Borel-de Siebenthal graph is either A_n or contains a unique branch point with degree 3. Assume the latter and express it as (a, b, c) as described above. Since the fourth graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph (a, b, c) satisfies $a = 1$. Since the fifth graph of [1, (5.1)] is not Borel-de Siebenthal, it implies that a strictly Borel-de Siebenthal graph $(1, b, c)$ satisfies $b < 3$. So we get either $(1, 1, c)$ or $(1, 2, c)$. The former is D_n and the latter is E_n. We conclude that a strictly Borel-de Siebenthal graph is A_n, D_n or E_n. This proves the $\subset$ part of Corollary 3.2. ☐

The graphs in [1, (5.1)] are actually all extended simply-laced Dynkin diagrams $\widetilde{A_n}$, $\widetilde{D_n}$, $\widetilde{E_6}$, $\widetilde{E_7}$ and $\widetilde{E_8}$ [4, Ch. X-5 Table 1] [5, Ch. 4 Table Aff 1]. In other words,

$$\begin{aligned} &\text{(a)} \quad \widetilde{A_n}, \widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7} \text{ are not Borel-de Siebenthal;} \\ &\text{(b)} \quad \widetilde{E_8} \text{ is Borel-de Siebenthal.} \end{aligned} \tag{3.1}$$

We now discuss the algebraic significance of (3.1) in terms of symmetric spaces.

Let G be a real simple Lie group with Lie algebra $\mathfrak{g}$. Let K be a maximal compact subgroup of G. Then G/K is a Riemannian symmetric space [4, Ch. IV]. We say that G is of Hermitian type if the G-invariant Riemannian structure on G/K is Hermitian.

Corollary 3.3. *E_8 is the only simply-laced Lie algebra without any real form of Hermitian type.*

Proof. Let G be a real simple Lie group with Lie algebra $\mathfrak{g}$. Let $L = \mathfrak{g} \otimes \mathbb{C}$. Let Γ be the extended Dynkin diagram of L, and suppose that it is simply-laced.

Let θ be a Cartan involution of $\mathfrak{g}$, extended by complex linearity to L. Then θ is represented by a painting p on Γ. Furthermore, G is of Hermitian (resp. non-Hermitian) type if and only if p is a painting with two (resp. one) black vertices, and in that case L^θ has a 1-dimensional center (resp. L^θ is semisimple) [3, §2]. Here L^θ denotes the fixed points of θ. If q is another painting on Γ with a single black vertex, then q represents an L-automorphism σ such that L^σ is semisimple [4, Ch. X-5 Thm. 5.15(ii)][5, Ch. 8 Prop. 8.6]. Therefore, $\mathfrak{g}$ is of Hermitian type if and only if θ is not conjugate to such σ, namely p is not equivalent to such q.

In [1, (5.2)], there are paintings with two black vertices on $\widetilde{A_n}, \widetilde{D_n}, \widetilde{E_6}, \widetilde{E_7}$ which are not equivalent to paintings with a single black vertex, so each of them represents a real form of Hermitian type. Such a painting does not exist on $E_9 = \widetilde{E_8}$, so E_8 has no real form of Hermitian type. This proves the corollary. ☐

The vertices of an extended Dynkin diagram are equipped with positive integers, known as their labels [4, Ch. X-5 Table 1] [5, Ch. 4 Table Aff 1]. When $\mathfrak{g}$ is represented by a painting with two black vertices, the black vertices occur on vertices with label 1. The list of extended Dynkin diagrams shows that $\widetilde{E_8}$ is the only simply-laced diagram without a pair of vertices with label 1. This explains why it does not accept a painting of two black vertices that is not equivalent to a single black vertex.

Next we consider another application of Theorem 1.2 by setting $n = 10$. Here E_{10} is the Dynkin diagram of a hyperbolic Kac-Moody Lie algebra of highest rank

[7, p. 3767]. We denote the Lie algebra of E_{10} by $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\Pi \subset \mathfrak{h}^*$ be a simple system of roots. Let θ be a $\mathfrak{g}$-involution. We say that θ preserves $\mathfrak{h}$ and Π if $\theta(\mathfrak{h}) = \mathfrak{h}$, and θ maps a simple root space to another simple root space.

Corollary 3.4. *Let $\mathfrak{g}$ be the hyperbolic Kac-Moody Lie algebra of E_{10}, and let θ be a $\mathfrak{g}$-involution which preserves $\mathfrak{h}$ and a simply system Π. Then there exists another simple system Π' such that $\theta = -1$ on one simple root space of Π' and $\theta = 1$ on the remaining root spaces.*

Proof. Suppose that θ preserves $\mathfrak{h}$ and Π. Let $\mathfrak{g}_\alpha$ be the root space of α. Then θ leads to a permutation d on Π by $\theta\mathfrak{g}_\alpha = \mathfrak{g}_{d\alpha}$ for all $\alpha \in \Pi$. The Dynkin diagram of E_{10} does not have nontrivial diagram symmetry, hence d is trivial. It follows that θ preserves each root space. Since θ has order 2, it acts as ± 1 on each root space.

If we represent Π by the vertices of E_{10}, we obtain a painting p where a vertex is white (resp. black) if and only if $\theta = 1$ (resp. $\theta = -1$) on the corresponding root space. This is the motivation for Vogan diagrams [6, Ch. VI-8]. Suppose that α is a black vertex and r_α is its reflection map on the root system. Then the new painting which represents θ with respect to $r_\alpha \Pi$ is $F_\alpha p$, where F_α is given by (1.1). By Theorem 1.2, there exists a sequence of some F_α such that $q = F_{\alpha_1} \dots F_{\alpha_m} p$ has a single black vertex. Let $\Pi' = r_{\alpha_1} \dots r_{\alpha_m} \Pi$, so that q represents θ with respect to Π'. Since $\alpha \in \Pi'$ is white (resp. black) with respect to q if and only if $\theta = 1$ (resp. $\theta = -1$) on the corresponding root space, Π' is our desired simple system. □

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