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SEMI-CONVERGENCE OF THE GENERALIZED LOCAL HSS METHOD FOR SINGULAR SADDLE POINT PROBLEMS

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ABSTRACT. Recently, Zhu [*Appl. Math. Comput.*, 218 (2012), 8816–8824] considered the generalized local HSS (GLHSS) method for solving nonsingular saddle point problems and studied its convergence. In this paper, we prove the semi-convergence of the GLHSS method when it is applied to solve the singular saddle point problems.

1. Introduction

Consider the non-Hermitian saddle point problems of the form

$$\mathcal{A}u = \begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times m}$ is non-Hermitian and its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite, $B \in \mathbb{C}^{m\times n}$ is a matrix of rank r, $f \in \mathbb{C}^m$ and $g \in \mathbb{C}^n$ are given vectors, with $m \geq n$. When $r = n$, the coefficient matrix is nonsingular and the saddle point problem (1.1) has a unique solution. When $r < n$, the coefficient matrix is singular, and at the moment, we call (1.1) a singular saddle point problem. Moreover, in such case, we suppose that the singular saddle point problem (1.1) is consistent, i.e., $b \in \mathcal{R}(\mathcal{A})$, the range of $\mathcal{A}$.

The saddle point problem (1.1) appears in many engineering and scientific computing applications such as constrained optimization, the mixed finite element of elliptic PDEs, fluid dynamics and weighted linear squares problem [21, 35]. The linear system (1.1) is also termed as a Karsh-Kuhn-Tucker (KKT) system, or an augmented system, or an equilibrium system [24, 26, 27].

For its property of large and sparsity, (1.1) is suitable for being solved by the iterative method. If $r = n$, i.e., the saddle point matrix $\mathcal{A}$ is nonsingular, many efficient iterative methods have been studied in the literature, including Uzawa-type

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methods [15, 18, 22, 28], Hermitian and skew-Hermitian splitting (HSS) iterative method and its variant schemes [5, 7, 8, 9, 10, 11, 12, 13, 29], preconditioned Krylov subspace iterative methods [3, 5, 9, 13, 23], restrictively preconditioned conjugate gradient methods [14, 17] and so on. See [19] and references therein for a comprehensive survey about iterative methods and preconditioning techniques. However, in many scientific computing and engineering applications, such as fluid dynamics problem, the rank of B is less than n. For this case, (1.1) is a singular linear system and many techniques have been proposed [6, 25, 30, 31, 32, 33, 36, 37]. When (1.1) is consistent and submatrices A and B are real matrices with A being symmetric positive definite, Zheng et al. [37], Li et al. [30, 31] and Miao [33] gave semi-convergent analyses of Uzawa-type iterative methods for solving singular saddle point problems (1.1). Bai studied a class of HSS iterative methods [6] and gave necessary and sufficient condition for guaranteeing the semi-convergence of the HSS iteration.

In this paper, for singular saddle point problems (1.1), the class of iterative methods we considered is a class of generalized local HSS (GLHSS) iterative methods. The GLHSS iterative method was proposed by Zhu in [38] for non-Hermitian nonsingular saddle point problem (1.1). It is also an extension of a class of local HSS (LHSS) iterative methods proposed by Jiang and Cao in [29]. In fact, the LHSS iterative method is a mixed variant of the well-known HSS and preconditioned HSS iterative method [11, 12, 13]. It should be also noted that both LHSS and GLHSS iterative methods belong to a class of parameterized inexact Uzawa iterative methods [15, 18]. If the GLHSS iterative method is solved inexactly, then it is an extension of a class of inexact HSS iterative method [12]. In the following, we shall apply the GLHSS iterative method to solve the singular saddle point problem (1.1) and deduce the semi-convergent properties.

The remainder of the paper is organized as follows. In Section 2, the semi-convergence concepts and two useful lemmas are given. In Section 3, after a review of the GLHSS method, we demonstrate the semi-convergence of the GLHSS method. The conclusions and outlook are drawn in Section 4.

2. Basic concepts and lemmas

Throughout this paper, for $A \in \mathbb{C}^{m\times m}$, A^*, $\sigma(A)$ and $\rho(A)$ denote the conjugate transpose, the spectral set and the spectral radius of the matrix A, respectively. I_n is the identity matrix with order n.

In what follows, basic concepts and lemmas of splitting iterative methods for solving the linear system (1.1) are given for latter use. Assume that the matrix $\mathcal{A}$ in (1.1) can be split as

$$\mathcal{A} = M - N,$$

with M being nonsingular; then we can construct a splitting iterative method

$$z_{k+1} = Tz_k + c, \quad k = 0, 1, 2, \ldots \tag{2.1}$$

for solving linear systems (1.1), where $T = M^{-1}N$ is the iterative matrix and $c = M^{-1}b$.

When $\mathcal{A}$ is nonsingular, for any initial vector z_0 the iteration scheme (2.1) converges to the exact solution of (1.1) if and only if $\rho(T) < 1$. But for the singular matrix $\mathcal{A}$, we have $1 \in \sigma(T)$ and $\rho(T) \leq 1$, so that one can require only the semi-convergence of the iterative scheme (2.1), see [16, 20]. By [20], the iterative scheme (2.1) is semi-convergent if and only if the following three conditions are satisfied:

(1) The spectral radius of the iterative matrix T is equal to one, i.e., $\rho(T) = 1$;
(2) the elementary divisors of the iterative matrix T associated with $\lambda = 1 \in \sigma(T)$ are linear, i.e., $\operatorname{rank}(I - T)^2 = \operatorname{rank}(I - T)$; here $\operatorname{rank}(\cdot)$ denotes the rank of the corresponding matrix;
(3) if $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$, i.e., $\vartheta(T) < 1$, where
$$\vartheta(T) = \max\{|\lambda|, \lambda \in \sigma(T), \lambda \neq 1\}$$
is called the semi-convergence factor of the iterative scheme (2.1).

We call a matrix T semi-convergent provided that it satisfies the above three conditions, and we say the iterative method (2.1) is semi-convergent if T is a semi-convergent matrix. When the matrix $\mathcal{A}$ is singular, the semi-convergence properties about the iterative scheme (2.1) are described in the following two lemmas.

Lemma 2.1 ([20]). *Let $\mathcal{A} = M - N$ with M nonsingular, $T = M^{-1}N$. Then for any initial vector z_0 the iterative scheme (2.1) is semi-convergent to a solution $z_\star$ of (1.1) if and only if the matrix T is semi-convergent.*

Lemma 2.2 ([37]). *Let $H \in \mathbb{R}^{l \times l}$ with positive integers l. Then the partitioned matrix*
$$T = \begin{pmatrix} H & 0 \\ L & I_t \end{pmatrix}$$
is semi-convergent if and only if either of the following conditions holds true:

(1) *$L = 0$ and H is semi-convergent;*
(2) *$\rho(H) < 1$.*

3. The semi-convergence of the GLHSS method

We first review the GLHSS method presented in [38] for solving the linear system (1.1). For the coefficient matrix of the linear system (1.1), we make the following splitting:
$$\begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} = \begin{pmatrix} Q_1 + H & 0 \\ -B^* + Q_3 & Q_2 \end{pmatrix} - \begin{pmatrix} Q_1 - S & -B \\ Q_3 & Q_2 \end{pmatrix}, \tag{3.1}$$
where $H = \frac{1}{2}(A + A^*)$, $S = \frac{1}{2}(A - A^*)$ are the Hermitian and the skew-Hermitian parts of A, respectively, $Q_1 \in \mathbb{C}^{m \times m}$, $Q_2 \in \mathbb{C}^{n \times n}$ are Hermitian positive definite matrices, $Q_3 \in \mathbb{C}^{n \times m}$ is an arbitrary matrix. With the splitting (3.1), the GLHSS method for solving nonsingular saddle point problem (1.1) is studied in [38]. It is defined by the following iterative scheme:
$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = T \begin{pmatrix} x_k \\ y_k \end{pmatrix} + c, \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

where

$$T = \begin{pmatrix} Q_1 + H & 0 \\ -B^* + Q_3 & Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_1 - S & -B \\ Q_3 & Q_2 \end{pmatrix}$$
$$= \begin{pmatrix} (Q_1 + H)^{-1}(Q_1 - S) & -(Q_1 + H)^{-1}B \\ Q_2^{-1}\left[(B^* - Q_3)(Q_1 + H)^{-1}(Q_1 - S) + Q_3\right] & I_n - Q_2^{-1}(B^* - Q_3)(Q_1 + H)^{-1}B \end{pmatrix}$$

is the GLHSS iterative matrix, and

$$c = \begin{pmatrix} Q_1 + H & 0 \\ -B^* + Q_3 & Q_2 \end{pmatrix}^{-1} \begin{pmatrix} f \\ g \end{pmatrix}.$$

Method 3.1. GLHSS Method*: Given the initial vectors $x_0,\ y_0$, the Hermitian positive definite matrices Q_1 and Q_2, and the arbitrary matrix Q_3; for $i = 0,\ 1, \ldots,$ until the iteration sequence $\{(x_k^*, y_k^*)^*\}$ is convergent, compute*

$$\begin{cases} x_{k+1} = x_k + (Q_1 + H)^{-1}(f - Ax_k - By_k), \\ y_{k+1} = y_k + Q_2^{-1}\left[(B^* - Q_3)x_{k+1} + Q_3 x_k + g\right]. \end{cases}$$

Evidently, when $Q_3 = 0$, the GLHSS method reduces to the modified LHSS method considered in [29], and when $Q_1 = 0$ and $Q_3 = 0$, the GLHSS method becomes the LHSS method [29]. When $Q_3 = 0$, $Q_1 + H = \frac{1}{\omega}P$ and $Q_2 = \frac{1}{\tau}Q$, the GLHSS method gives the parameterized inexact Uzawa method in [18].

Moreover, with different choices of the matrices Q_1, Q_2 and Q_3, the GLHSS methods lead to different iterative schemes. How to choose the matrices Q_1, Q_2 and Q_3 is the key of the GLHSS method. Note that when the matrices Q_1, Q_2 and Q_3 are scalar matrices, theoretical discussions and numerical verifications can be found in [5, 7, 13, 15]. For special cases, we refer to [23].

When the coefficient matrix $\mathcal{A}$ of (1.1) is nonsingular, the convergence of GLHSS method is studied in [38]. Moreover, if λ is an eigenvalue of T and $(u^*,\ v^*)^*$ is its corresponding eigenvector, where $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$, then from Theorem 2.4 in [38], we known that $u \neq 0$.

When $r < n$, the coefficient matrix of (1.1) is singular, the following theorem describes the semi-convergence property when the GLHSS method is applied to solve the singular saddle point poblem (1.1).

Theorem 3.1. *Assume that $A \in \mathbb{C}^{m\times m}$ is non-Hermitian and its Hermitian part $H = \frac{1}{2}(A+A^*)$ is positive definite, the skew-Hermitian part of A is $S = \frac{1}{2}(A-A^*)$, and $B \in \mathbb{C}^{m\times n}$ is a matrix of rank r with $r < n$. Let $Q_1 \in \mathbb{C}^{m\times m},\ Q_2 \in \mathbb{C}^{n\times n}$ be Hermitian positive definite matrices, $Q_3 \in \mathbb{C}^{n\times m}$ be a matrix such that $B^*Q_2^{-1}Q_3$ is Hermitian. Suppose that $(u^*,\ v^*)^*$ is the eigenvector according to an eigenvalue ($\neq 1$) of the iteration matrix T, $\mathbf{i}$ is the imaginary unit, and denote by*

$$a = \frac{u^*Hu}{u^*u},\ -b = \frac{u^*\mathbf{i}Su}{u^*u},\ c = \frac{u^*Q_1u}{u^*u},\ d = \frac{u^*B^*Q_2^{-1}Bu}{u^*u},\ e = \frac{u^*B^*Q_2^{-1}Q_3u}{u^*u}.$$

Then the GLHSS method is semi-convergent to a solution of the singular saddle point problem (1.1) if and only if

$$d \leq \frac{2(a-e)\left[(a-e)(a+e+2c)-b^2\right]}{(a-e)^2+b^2}. \tag{3.3}$$

Proof. By Lemma 2.1, we only need to demonstrate the semi-convergence of the iteration matrix T of the GLHSS method.

Let $B = U(B_r,\ 0)V^*$ be the singular value decomposition of B, where $B_r = (\Sigma_r,\ 0)^T \in \mathbb{R}^{m\times n}$ with $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, U, V are unitary matrices. Then

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

is an $(m+n)$-by-$(m+n)$ unitary matrix. Let $\widehat{T} = P^*TP$ and $(\widehat{u}^*,\ \widehat{v}^*)^* = P^*(u^*,\ v^*)^*$, then the matrix T shares the same eigenvalues with the matrix $\widehat{T}$, and $(\widehat{u}^*,\ \widehat{v}^*)^*$ be the eigenvectors of $\widehat{T}$. Hence, we only need to demonstrate the semi-convergence of the matrix $\widehat{T}$.

Define matrices

$$\widehat{A} = U^*AU, \quad \widehat{H} = U^*HU, \quad \widehat{S} = U^*SU, \quad \widehat{B} = U^*BV$$

and

$$\widehat{Q}_1 = U^*Q_1U, \quad \widehat{Q}_2 = V^*Q_2V, \quad \widehat{Q}_3 = V^*Q_3U.$$

Then it holds that $\widehat{B} = (B_r,\ 0)$,

$$\widehat{Q}_2^{-1} = \begin{pmatrix} V_1^*Q_2^{-1}V_1 & V_1^*Q_2^{-1}V_2 \\ V_2^*Q_2^{-1}V_1 & V_2^*Q_2^{-1}V_2 \end{pmatrix},$$

and

$$\widehat{Q}_3 = \begin{pmatrix} V_1^*Q_3U \\ V_2^*Q_3U \end{pmatrix};$$

here we have partitioned the unitary matrix V into the block form $V = (V_1,\ V_2)$, with $V_1 \in \mathbb{C}^{m\times r}$ and $V_2 \in \mathbb{C}^{m\times(m-r)}$. Moreover, direct operations yield

$$\begin{aligned} U^*(Q_1+H)^{-1}(Q_1-S)U &= \left(U^*(Q_1+H)^{-1}U\right)\left(U^*(Q_1-S)U\right) \\ &= \left(U^*(Q_1+H)U\right)^{-1}\left(U^*Q_1U - U^*SU\right) \\ &= (\widehat{Q}_1+\widehat{H})^{-1}(\widehat{Q}_1-\widehat{S}), \end{aligned}$$

$$\begin{aligned} U^*(Q_1+H)^{-1}BV &= (U^*(Q_1+H)^{-1}U)(U^*BV) \\ &= (\widehat{Q}_1+\widehat{H})^{-1}(B_r,\ 0), \end{aligned}$$

$$\begin{aligned} &V^*Q_2^{-1}\left[(B^*-Q_3)(Q_1+H)^{-1}(Q_1-S)+Q_3\right]U \\ &\quad = (V^*Q_2^{-1}V)(V^*(B^*-Q_3)U)(U^*(Q_1+H)^{-1}(Q_1-S)U + (V^*Q_2^{-1}V)(V^*Q_3U) \\ &\quad = \begin{pmatrix} V_1^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}(\widehat{Q}_1-\widehat{S}) + V_1^*Q_2^{-1}Q_3U \\ V_2^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}(\widehat{Q}_1-\widehat{S}) + V_2^*Q_2^{-1}Q_3U \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned}
&V^*Q_2^{-1}(B^*-Q_3)(Q_1+H)^{-1}BV\\
&\qquad=(V^*Q_2^{-1}V)\left(V^*(B^*-Q_3)U\right)\left(U^*(Q_1+H)^{-1}U\right)U^*BV\\
&\qquad=\begin{pmatrix} V_1^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}B_r & 0\\ V_2^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}B_r & 0_{n-r}\end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\widehat{T}=P^*TP\\
&=\left(\begin{array}{c|c} U^*(Q_1+H)^{-1}(Q_1-S)U & -U^*(Q_1+H)^{-1}BV\\ \hline V^*Q_2^{-1}(B^*-Q_3)(Q_1+H)^{-1}(Q_1-S)U & -V^*Q_2^{-1}(B^*-Q_3)(Q_1+H)^{-1}BV\\ +V^*Q_2^{-1}Q_3U & +I_n\end{array}\right)\\
&=\begin{pmatrix}\widehat{H} & 0\\ \widehat{L} & I_{n-r}\end{pmatrix},
\end{aligned}$$

where

$$\widehat{H}=\begin{pmatrix}\widehat{H}_{11} & \widehat{H}_{12}\\ \widehat{H}_{21} & \widehat{H}_{22}\end{pmatrix}\quad\text{and}\quad \widehat{L}=\begin{pmatrix}\widehat{L}_{11} & \widehat{L}_{12}\end{pmatrix},$$

with

$$\begin{aligned}
\widehat{H}_{11}&=(\widehat{Q}_1+\widehat{H})^{-1}(\widehat{Q}_1-\widehat{S}),\\
\widehat{H}_{12}&=-(\widehat{Q}_1+\widehat{H})^{-1}B_r,\\
\widehat{H}_{21}&=V_1^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}(\widehat{Q}_1-\widehat{S})+V_1^*Q_2^{-1}Q_3U,\\
\widehat{H}_{22}&=I_r-V_1^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}B_r,\\
\widehat{L}_{11}&=V_2^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}(\widehat{Q}_1-\widehat{S})+V_2^*Q_2^{-1}Q_3U,\\
\widehat{L}_{12}&=I_r-V_2^*Q_2^{-1}V_1(B_r^*-V_1^*Q_3U)(\widehat{Q}_1+\widehat{H})^{-1}B_r.
\end{aligned}$$

As $\widehat{L}\neq 0$, from Lemma 2.2 we know that the matrix $\widehat{T}$ is semi-convergent if and only if $\rho(\widehat{H})<1$.

In fact, when we consider the GLHSS method for solving the following nonsingular saddle point problem

$$\begin{pmatrix}\widehat{A} & B_r\\ B_r^* & 0\end{pmatrix}\begin{pmatrix}\widehat{x}\\ \widehat{y}\end{pmatrix}=\begin{pmatrix}\widehat{f}\\ \widehat{g}\end{pmatrix}\tag{3.4}$$

with the matrices $\widehat{Q}_1$, $\widetilde{Q}_2=(V_1^*Q_2^{-1}V_1)^{-1}$ and $\widetilde{Q}_3=V_1^*Q_3U$, and vectors $\widehat{y}$, $\widehat{g}\in\mathbb{R}^r$, the iterative matrix is $\widehat{H}$. From Theorem 2.5 in [38], we know that $\rho(\widehat{H})<1$ if and only if

$$\widetilde{d}\leq\frac{2(\widetilde{a}-\widetilde{e})\left[(\widetilde{a}-\widetilde{e})(\widetilde{a}+\widetilde{e}+2\widetilde{c})-\widetilde{b}^2\right]}{(\widetilde{a}-\widetilde{e})^2+\widetilde{b}^2};\tag{3.5}$$

here

$$\widetilde{a}=\frac{\widehat{u}^*\widehat{H}\widehat{u}}{\widehat{u}^*\widehat{u}},\ -\widetilde{b}=\frac{\widehat{u}^*\mathbf{i}\widehat{S}\widehat{u}}{\widehat{u}^*\widehat{u}},\ \widetilde{c}=\frac{\widehat{u}^*\widehat{Q}_1\widehat{u}}{\widehat{u}^*\widehat{u}},\ \widetilde{d}=\frac{\widehat{u}^*B_r^*\widetilde{Q}_2^{-1}B_r\widehat{u}}{\widehat{u}^*\widehat{u}},\ e=\frac{\widehat{u}^*B_r^*\widetilde{Q}_2^{-1}\widetilde{Q}_3\widehat{u}}{\widehat{u}^*\widehat{u}}.$$

It is easy to show that condition (3.5) is equivalent to condition (3.3). The proof is completed. □

When $Q_3=0$, the GLHSS method becomes the modified LHSS method [29], and when $Q_1=0$ and $Q_3=0$, the GLHSS method becomes the LHSS method, thus from Theorem 3.1, we can derive the following corollaries directly.

Corollary 3.1. *Assume that $A\in\mathbb{C}^{m\times m}$ is non-Hermitian and its Hermitian part $H=\frac{1}{2}(A+A^*)$ is positive definite, the skew-Hermitian part of A is $S=\frac{1}{2}(A-A^*)$, and $B\in\mathbb{C}^{m\times n}$ is a matrix of rank r with $r<n$. Let $Q_1\in\mathbb{C}^{m\times m}$, $Q_2\in\mathbb{C}^{n\times n}$ be Hermitian positive definite matrices. Suppose that $(u^*,\ v^*)^*$ is the eigenvector according to an eigenvalue ($\neq 1$) of the MLHSS iteration matrix, $\mathbf{i}$ is the imaginary unit, and denote by*

$$a=\frac{u^*Hu}{u^*u},\ -b=\frac{u^*\mathbf{i}Su}{u^*u},\ c=\frac{u^*Q_1u}{u^*u},\ d=\frac{u^*B^*Q_2^{-1}Bu}{u^*u}.$$

Then the modified LHSS method is semi-convergent to a solution of the singular saddle point problem (1.1) if and only if

$$d\le\frac{2a^3+4a^2d-2ab^2}{a^2+b^2}.$$

Corollary 3.2. *Assume that $A\in\mathbb{C}^{m\times m}$ is non-Hermitian and its Hermitian part $H=\frac{1}{2}(A+A^*)$ is positive definite, the skew-Hermitian part of A is $S=\frac{1}{2}(A-A^*)$, and $B\in\mathbb{C}^{m\times n}$ is a matrix of rank r with $r<n$. Let $Q_2\in\mathbb{C}^{n\times n}$ be Hermitian positive definite matrix. Suppose that $(u^*,\ v^*)^*$ is the eigenvector according to an eigenvalue ($\neq 1$) of the LHSS iteration matrix, $\mathbf{i}$ is the imaginary unit, and denote by*

$$a=\frac{u^*Hu}{u^*u},\ -b=\frac{u^*\mathbf{i}Su}{u^*u},\ d=\frac{u^*B^*Q_2^{-1}Bu}{u^*u}.$$

Then the LHSS method is semi-convergent to a solution of the singular saddle point problem (1.1) if and only if

$$d\le\frac{2a(a^2-b^2)}{a^2+b^2}.$$

Note that Corollary 3.1 and 3.2 are the main results (Theorem 2.1 and Theorem 2.2) in [32]. For the real case, there is a better result, which is summarized in the following corollary.

Corollary 3.3. *Assume that $A\in\mathbb{R}^{m\times m}$ is non-symmetric and its symmetric part $H=\frac{1}{2}(A+A^T)$ is positive definite, the skew-symmetric part of A is $S=\frac{1}{2}(A-A^T)$, and $B\in\mathbb{R}^{m\times n}$ is a matrix of rank r with $r<n$. Let $Q_1\in\mathbb{R}^{m\times m}$, $Q_2\in\mathbb{R}^{n\times n}$ be symmetric positive definite matrices, $Q_3\in\mathbb{R}^{n\times m}$ be a matrix such that $B^TQ_2^{-1}Q_3$ is symmetric. Then the GLHSS method is semi-convergent to a solution of the singular saddle point problem (1.1) if and only if $2(H+2Q_1+B^*Q_2^{-1}Q_3)-B^TQ_2^{-1}B$ is positive definite.*

From Corollary 3.3, we can derive the semi-convergence conditions of the LHSS method and the modified LHSS method for the real case similarly. These semi-convergence conditions have been omitted as they are included in [32].

4. Conclusions and outlook

In this paper, we consider the GLHSS method for solving singular saddle point problems. The semi-convergence conditions are given, which generalize the result of Zhu [38] for nonsingular saddle point problems to singular saddle point problems. Meanwhile, from the proof of Theorem 3.1, we note that the semi-convergence factor of the GLHSS method for the singular saddle point problem (1.1) is the convergence factor of the GLHSS method for the nonsingular saddle point problem (3.4). For other splitting iterative methods, the same observations can be reached, see [30, 31, 32, 33, 37].

It is known that there are many efficient iterative methods for solving non-Hermitian positive definite linear systems, such as efficient splitting iteration methods [4], additive and multiplicative splitting iterations [2], multiplicative Schwarz iteration [1], skew-Hermitian triangular splitting iteration methods [34] and so on. These iterative methods are proposed and proven to be very efficient for solving nonsingular linear systems. For solving singular linear systems, especially the singular saddle point problems, we can also apply these iterative methods and deduce the corresponding semi-convergence properties. These will be studied in future work.

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