ON SLANT CURVES IN TRANS-SASAKIAN MANIFOLDS

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Abstract. We find the characterizations of the curvatures of slant curves in trans-Sasakian manifolds with $C$-parallel and $C$-proper mean curvature vector field in the tangent and normal bundles.

1. Introduction

Let $\gamma$ be a curve in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$. In [14], Lee, Suh and Lee introduced the notions of $C$-parallel and $C$-proper curves in the tangent and normal bundles. A curve $\gamma$ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is defined to be $C$-parallel if $\nabla T H = \lambda \xi$, $C$-proper if $\Delta H = \lambda \xi$, $C$-parallel in the normal bundle if $\nabla^\perp T H = \lambda \xi$, $C$-proper in the normal bundle if $\Delta^\perp H = \lambda \xi$, where $T$ is the unit tangent vector field of $\gamma$, $H$ is the mean curvature vector field, $\Delta$ is the Laplacian, $\lambda$ is a non-zero differentiable function along the curve $\gamma$, $\nabla^\perp$ and $\Delta^\perp$ denote the normal connection and Laplacian in the normal bundle, respectively [14]. For a submanifold $M$ of an arbitrary Riemannian manifold $\tilde{M}$, if $\Delta H = \lambda H$, then $M$ is a submanifold with proper mean curvature vector field $H$ [7]. If $\Delta^\perp H = \lambda H$, then $M$ is a submanifold with proper mean curvature vector field $H$ in the normal bundle [1].

Let $M$ be an almost contact metric manifold and $\gamma(s)$ a Frenet curve in $M$ parametrized by the arc-length parameter $s$. The contact angle $\alpha(s)$ is a function defined by $\cos[\alpha(s)] = g(T(s), \xi)$. A curve $\gamma$ is called a slant curve [8] if its contact angle is a constant. Slant curves with contact angle $\frac{\pi}{2}$ are traditionally called Legendre curves [9].

In [16], Srivastava studied Legendre curves in trans-Sasakian 3-manifolds. In [14], Inoguchi and Lee studied almost contact curves in normal almost contact 3-manifolds. In [12], the same authors studied slant curves in normal almost contact metric 3-manifolds. In [13], Lee, Suh and Lee studied slant curves in Sasakian 3-manifolds. They find the curvature characterizations of $C$-parallel and $C$-proper curves in the tangent and normal bundles. In the present study, our aim is to generalize results of [14] to a curve in a trans-Sasakian manifold.

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2. Preliminaries

A \((2n + 1)\)-dimensional Riemannian manifold \(M\) is said to be an almost contact metric manifold \([4]\), if there exist on \(M\) a \((1, 1)\) tensor field \(\varphi\), a vector field \(\xi\), a 1-form \(\eta\) and a Riemannian metric \(g\) satisfying

\[
\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi \xi = 0, \quad \eta \circ \varphi = 0
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi),
\]

for any vector fields \(X, Y\) on \(M\). Such a manifold is said to be a contact metric manifold if \(d\eta = \Phi\), where \(\Phi(X, Y) = g(X, \varphi Y)\) is called the fundamental 2-form of \(M\) \([4]\).

The almost contact metric structure of \(M\) is said to be normal if

\[
[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi,
\]

for any vector fields \(X, Y\) on \(M\), where \([\varphi, \varphi]\) denotes the Nijenhuis torsion of \(\varphi\). A normal contact metric manifold is called a Sasakian manifold \([4]\). It is easy to see that an almost contact metric manifold is Sasakian if and only if

\[
(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X.
\]

An almost contact metric manifold \(M\) is called a trans-Sasakian manifold \([17]\) if there exist two functions \(\alpha\) and \(\beta\) on \(M\) such that

\[
(\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[\varphi(X, Y)\xi - \eta(Y)\varphi(X)],
\]

for any vector fields \(X, Y\) on \(M\). From (2.1), it is easily obtained that

\[
\nabla_X \xi = -\alpha \varphi X + \beta[X - \eta(X)\xi].
\]

If \(\beta = 0\) (resp. \(\alpha = 0\)), then \(M\) is said to be an \(\alpha\)-Sasakian manifold (resp. \(\beta\)-Kenmotsu manifold). Sasakian manifolds (resp. Kenmotsu manifolds \([13]\)) appear as examples of \(\alpha\)-Sasakian manifolds (resp. \(\beta\)-Kenmotsu manifolds), with \(\alpha = 1\) (resp. \(\beta = 1\)). For \(\alpha = \beta = 0\), we get cosymplectic manifolds \([15]\). From (2.2), for a cosymplectic manifold we obtain

\[
\nabla_X \xi = 0.
\]

Hence \(\xi\) is a Killing vector field for a cosymplectic manifold \([3]\).

**Proposition 2.1.** \([16]\) A trans-Sasakian manifold of dimension greater than or equal to 5 is either \(\alpha\)-Sasakian, \(\beta\)-Kenmotsu or cosymplectic.

From now on, we state “(\(\alpha, \beta\))-trans-Sasakian manifold”, when the dimension of the manifold is 3 and \(\alpha \neq 0, \beta \neq 0\).

The contact distribution of an almost contact metric manifold \(M\) with an almost contact metric structure \((\varphi, \xi, \eta, g)\) is defined by

\[
\{X \in TM : \eta(X) = 0\}
\]

and an integral curve of the contact distribution is called a Legendre curve \([4]\).
3. Slant curves with $C$-parallel mean curvature vector field

Let $(M, g)$ be an $m$-dimensional Riemannian manifold and $\gamma : I \to M$ a curve parametrized by arc length. Then $\gamma$ is called a Frenet curve of osculating order $r$, $1 \leq r \leq m$, if there exists orthonormal vector fields $E_1, E_2, \ldots, E_r$ along $\gamma$ such that

$$E_1 = \gamma' = T,$$
$$\nabla_T E_1 = \kappa_1 E_2,$$
$$\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3,$$
$$\ldots$$
$$\nabla_T E_r = -\kappa_{r-1} E_{r-1},$$

where $\kappa_1, \ldots, \kappa_{r-1}$ are positive functions on $I$.

A geodesic is a Frenet curve of osculating order 1; a circle is a Frenet curve of osculating order 2 such that $\kappa_1$ is a non-zero positive constant; a helix of order $r$, $r \geq 3$, is a Frenet curve of osculating order $r$ such that $\kappa_1, \ldots, \kappa_{r-1}$ are non-zero positive constants; a helix of order 3 is called simply a helix.

Now let $(M, g)$ be a Riemannian manifold and $\gamma : I \to M$ a Frenet curve of osculating order $r$. By the use of (3.1), it can be easily seen that

$$\nabla_T \nabla_T T = -\kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$
$$\nabla_T \nabla_T T = -3\kappa_1 \kappa_1' E_1 + (\kappa_1'' - \kappa_1^2 - \kappa_1 \kappa_2^2) E_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 + \kappa_1 \kappa_2 \kappa_3 E_4,$$
$$\nabla_T \nabla_T T = \kappa_1' E_2 + \kappa_1 \kappa_2 E_3,$$
$$\nabla_T \nabla_T T = (\kappa_1'' - \kappa_1 \kappa_2^2) E_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') E_3 + \kappa_1 \kappa_2 \kappa_3 E_4.$$
\( \gamma \) has \( C \)-proper mean curvature vector field if and only if
\[
3\kappa_1\kappa'_1 E_1 + (\kappa^3_1 + \kappa_1\kappa^2_2 - \kappa''_1) E_2 - (2\kappa'_1\kappa_2 + \kappa_1\kappa'_2) E_3 - \kappa_1\kappa_2\kappa_3 E_4 = \lambda \xi; \quad \text{or (3.7)}
\]

\( \gamma \) has \( C \)-parallel mean curvature vector field in the normal bundle if and only if
\[
\kappa'_1 E_2 + \kappa_1\kappa_2 E_3 = \lambda \xi; \quad \text{or (3.8)}
\]

\( \gamma \) has \( C \)-proper mean curvature vector field in the normal bundle if and only if
\[
(\kappa^3_1 - \kappa''_1) E_2 - (2\kappa'_1\kappa_2 + \kappa_1\kappa'_2) E_3 - \kappa_1\kappa_2\kappa_3 E_4 = \lambda \xi, \quad \text{(3.9)}
\]

where \( \lambda \) is a non-zero differentiable function along the curve \( \gamma \).

Now, let \( \gamma : I \subseteq \mathbb{R} \rightarrow M \) be a non-geodesic slant curve of order \( r \) with contact angle \( \alpha_0 \) in an \( n \)-dimensional trans-Sasakian manifold. By the use of (2.1), (2.2) and (3.1), we obtain
\[
\eta(T) = \cos \alpha_0, \quad \text{(3.10)}
\]
\[
\kappa_1 \eta(E_2) = -\beta \sin^2 \alpha_0, \quad \text{(3.11)}
\]
\[
\nabla_T \xi = -\alpha \varphi T + \beta [T - \cos \alpha_0 \xi], \quad \text{(3.12)}
\]
\[
\nabla_T \varphi T = \alpha [\xi - \cos \alpha_0 T] - \beta \cos \alpha_0 \varphi T + \kappa_1 \varphi E_2. \quad \text{(3.13)}
\]

So we have the following theorem:

**Theorem 3.1.** Let \( \gamma : I \subseteq \mathbb{R} \rightarrow M \) be a non-geodesic slant curve of order \( r \) in a trans-Sasakian manifold. If \( \gamma \) has \( C \)-parallel or \( C \)-proper mean curvature vector field in the normal bundle, then it is a Legendre curve.

**Proof.** By the use of (3.8), (3.9) and (3.10), the proof is clear. \( \Box \)

We consider the following cases:

**Case I.** The osculating order \( r = 2 \).

For this case, we have the following results:

**Theorem 3.2.** Let \( \gamma : I \subseteq \mathbb{R} \rightarrow M \) be a non-geodesic slant curve of order 2 with contact angle \( \alpha_0 \) in a trans-Sasakian manifold. Then \( \gamma \) has \( C \)-parallel mean curvature vector field if and only if it satisfies
\[
\kappa_1 = \pm \frac{\cot \alpha_0}{c - s}, \quad \text{(3.14)}
\]
\[
\lambda = \frac{-\cot \alpha_0 \csc \alpha_0}{(c - s)^2}, \quad \text{(3.15)}
\]

where \( c \) is an arbitrary constant and \( s \) is the arc-length parameter of \( \gamma \). In this case, \( M \) becomes an \((\alpha, \beta)\)-trans-Sasakian or a \( \beta \)-Kenmotsu manifold with
\[
\beta = \frac{\cot \alpha_0 \csc \alpha_0}{c - s}.
\]
Proof. Let $\gamma$ have $C$-parallel mean curvature vector field. From (3.6), we have

$$-\kappa_1^2 E_1 + \kappa_1' E_2 = \lambda \xi.$$  \hspace{1cm} (3.16)

If $\alpha_0 = \frac{\pi}{2}$, we find $\kappa_1 = 0$, which is a contradiction. Thus, $\alpha_0 \neq \frac{\pi}{2}$.

Let $\beta \neq 0$. Hence $M$ is an $(\alpha, \beta)$-trans-Sasakian or a $\beta$-Kenmotsu manifold. Since $\eta(E_2) = \pm \sin \alpha_0$, (3.11) gives us

$$\kappa_1 = \mp \beta \sin \alpha_0.$$  \hspace{1cm} (3.17)

By the use of (3.10), (3.11) and (3.16), we get

$$\lambda = \frac{-\kappa_1^2}{\cos \alpha_0},$$  \hspace{1cm} (3.18)

$$\kappa_1' = \kappa_1 \beta \sin \alpha_0 \tan \alpha_0.$$  \hspace{1cm} (3.19)

Differentiating (3.17) and using (3.19), we have

$$\beta' = \beta^2 \sin \alpha_0 \tan \alpha_0,$$

which gives us

$$\beta = \frac{\cot \alpha_0 \csc \alpha_0}{c - s},$$  \hspace{1cm} (3.20)

where $c$ is an arbitrary constant. Using (3.20) in (3.18) and (3.19), we obtain (3.14) and (3.15).

Now, let $\beta = 0$. Hence $M$ is an $\alpha$-Sasakian or cosymplectic manifold. In this case, we have $\eta(E_2) = 0$. Thus (3.16) gives us $\kappa_1 = \text{constant}$. So we get

$$-\kappa_1^2 E_1 = \lambda \xi.$$  \hspace{1cm} (3.21)

Thus $\xi = \pm E_1$. From (3.1) and (3.12), we have

$$\nabla_T \xi = -\alpha \varphi T = 0 = \pm \kappa_1 E_2.$$  \hspace{1cm} (3.21)

Since $\gamma$ is non-geodesic, (3.21) causes a contradiction.

Conversely, if the above conditions are satisfied, one can easily show that $\gamma$ has $C$-parallel mean curvature vector field.

Using the proof of Theorem 3.2, we have the following corollary:

**Corollary 3.1.** There does not exist any non-geodesic slant curve of order 2 with $C$-parallel mean curvature vector field in an $\alpha$-Sasakian or a cosymplectic manifold.

In the normal bundle, we can state the following theorem:

**Theorem 3.3.** Let $\gamma : I \subset \mathbb{R} \to M$ be a non-geodesic slant curve of order 2 with contact angle $\alpha_0$ in a trans-Sasakian manifold. Then $\gamma$ has $C$-parallel mean curvature vector field in the normal bundle if and only if it is a Legendre curve with

$$\kappa_1 = \mp \beta, \quad \xi = \pm E_2, \quad \lambda = \pm \beta'.$$  \hspace{1cm} (3.22)

In this case, $\alpha = 0$ and $\beta$ is not a constant along the curve $\gamma$. 

Proof. Let $\gamma$ have $C$-parallel mean curvature vector field in the normal bundle. From (3.8) and Theorem 3.1, we have

$$\kappa_1' E_2 = \lambda \xi.$$  

(3.23)

So we have

$$\lambda = \pm \kappa_1',$$

$$\xi = \pm E_2.$$  

(3.24)

Differentiating (3.24), we find

$$- \alpha \varphi E_1 + \beta E_1 = \mp \kappa_1 E_1.$$  

(3.25)

(3.25) gives us (3.22) and $\alpha = 0$ along the curve. □

Case II. The osculating order $r = 3$.

For slant curves of order 3, we have the following theorem:

**Theorem 3.4.** Let $\gamma : I \subseteq \mathbb{R} \rightarrow M$ be a non-geodesic slant curve of order 3 with contact angle $\alpha_0$ in a trans-Sasakian manifold. Then $\gamma$ has $C$-parallel mean curvature vector field if and only if

i) it is a curve with

$$\kappa_1 = c e^{\sin \alpha_0 \tan \alpha_0 \int \beta(s) ds},$$

$$\kappa_2 = |\tan \alpha_0| \sqrt{\kappa_1^2 - \beta^2 \sin^2 \alpha_0},$$

$$\xi = \cos \alpha_0 E_1 - \frac{\beta \sin^2 \alpha_0}{\kappa_1} E_2 - \frac{\kappa_2 \cos \alpha_0}{\kappa_1} E_3$$

and

$$\lambda = -\frac{\kappa_1^2}{\cos \alpha_0}.$$  

(3.26)

(3.27)

(3.28)

(3.29)

where $\kappa_1^2 > \beta^2 \sin^2 \alpha_0$, $\alpha_0 \neq \frac{\pi}{2}$, $c$ is an arbitrary constant, $s$ is the arc-length parameter of $\gamma$, (in this case, $M$ becomes an $(\alpha, \beta)$-trans-Sasakian or a $\beta$-Kenmotsu manifold); or

ii) it is a helix with

$$\kappa_1 = \frac{-\kappa_1^2}{\cos \alpha_0}, \quad \alpha_0 \neq \frac{\pi}{2},$$

$$\kappa_2 = -\kappa_1 \tan \alpha_0$$

and

$$\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_3.$$  

(In this case, $\alpha \neq 0$ and $\beta = 0$ along the curve.)

Proof. Let $\gamma$ have $C$-parallel mean curvature vector field. From (3.6), we have

$$- \kappa_1^2 E_1 + \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \xi.$$  

(3.30)

If $\alpha_0 = \frac{\pi}{2}$, we find $\kappa_1 = 0$, which is a contradiction. Thus, $\alpha_0 \neq \frac{\pi}{2}$. 

Let $\beta \neq 0$. So $M$ is an $(\alpha, \beta)$-trans-Sasakian or a $\beta$-Kenmotsu manifold. (3.30) gives us $\xi \in \text{span}\{E_1, E_2, E_3\}$. Thus, we can write
\[ \xi = \cos \alpha_0 E_1 + \sin \alpha_0 (\cos \theta E_2 + \sin \theta E_3), \] (3.31)
where $\theta$ is the angle function between $E_2$ and the orthogonal projection of $\xi$ onto $\text{span}\{E_2, E_3\}$. From (3.30) and (3.31), we find
\[ \cos \theta = -\frac{\beta \sin \alpha_0}{\kappa_1}, \quad \sin \theta = -\frac{\kappa_2 \cot \alpha_0}{\kappa_1}. \]
So we obtain (3.28). We also have (3.29) using (3.30). Since $\lambda \eta(E_2) = \kappa'_1$, we can calculate
\[ \kappa'_1 = \kappa_1 \beta \sin \alpha_0 \tan \alpha_0, \] (3.32)
which gives us (3.26). Using (3.32) in (3.30), we find (3.27).

Now, let $\alpha \neq 0, \beta = 0$ along the curve. Since $\eta(E_2) = 0$, (3.30) and (3.31) give us $\kappa_1 > 0$ is a constant, $\theta = \frac{\pi}{2}$ and
\[ -\kappa^2_1 E_1 + \kappa_1 \kappa_2 E_3 = \lambda (\cos \alpha_0 E_1 + \sin \alpha_0 E_3). \] (3.33)
From (3.33), we find $\kappa_2 = -\kappa_1 \tan \alpha_0$. So $\kappa_2$ is also a constant. Hence $\gamma$ is a helix.

Finally, let $\alpha = \beta = 0$ along the curve. In this case, (3.30) and (3.31) give us
\[ -\kappa^2_1 E_1 + \kappa_1 \kappa_2 E_3 = \lambda \xi, \] (3.34)
\[ \xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_3. \] (3.35)
Differentiating (3.35) along $\gamma$, we have
\[ \frac{\kappa_2}{\kappa_1} = \cot \alpha_0. \] (3.36)
From (3.34), we get
\[ \frac{\kappa_2}{\kappa_1} = -\tan \alpha_0. \] (3.37)
By the use of (3.36) and (3.37), we obtain $\cot \alpha_0 = -\tan \alpha_0$, which has no solution. The converse statement is clear.

Using Theorem 3.4, we give the following corollary:

**Corollary 3.2.** There does not exist any non-geodesic slant curve of order 3 with $C$-parallel mean curvature vector field in a cosymplectic manifold.

In the normal bundle, we can state the following theorem:

**Theorem 3.5.** Let $\gamma : I \subseteq \mathbb{R} \rightarrow M$ be a non-geodesic slant curve of order 3 with contact angle $\alpha_0$ in a trans-Sasakian manifold. Then $\gamma$ has $C$-parallel mean curvature vector field in the normal bundle if and only if
i) it is a Legendre curve with
\[ \kappa_1 \neq \text{constant}, \]
\[ \kappa_2 = \frac{\kappa'_1 \sqrt{\kappa^2_1 - \beta^2}}{\kappa_1 \beta}, \]
\[ \xi = -\frac{\beta}{\kappa_1} E_2 - \frac{\sqrt{\kappa_1^2 - \beta^2}}{\kappa_1} E_3 \]  

\[ \lambda = -\frac{\kappa_1'}{\beta}, \]

\[ \text{(in this case, } M \text{ becomes an } (\alpha, \beta)-\text{trans-Sasakian or a } \beta-\text{Kenmotsu manifold}); \text{ or} \]

\[ \text{ii) it is a Legendre helix with} \]

\[ \xi = E_3, \quad \kappa_2 = \alpha > 0, \quad \lambda = \kappa_1 \kappa_2, \]

\[ \text{(in this case, } M \text{ becomes an } \alpha-\text{Sasakian or an } (\alpha, \beta)-\text{trans-Sasakian manifold}). \]

**Proof.** From (3.8), we have

\[ \kappa_1' E_2 + \kappa_1 \kappa_2 E_3 = \lambda \xi. \]  

(3.39)

Then we get

\[ \eta(E_1) = 0, \]

\[ \kappa_1 \eta(E_2) = -\beta. \]  

(3.40)

Firstly, let \( \beta \neq 0 \). Then \( M \) is an \((\alpha, \beta)\)-trans-Sasakian or a \( \beta \)-Kenmotsu manifold. From (3.39) and (3.40), we have

\[ \lambda = -\frac{\kappa_1'}{\beta}, \]

which gives us \( \kappa_1 \neq \text{constant} \). We also have

\[ \eta(E_3) = -\frac{\beta \kappa_2}{\kappa_1'}. \]  

(3.41)

By the use of (3.40) and (3.41), we can write

\[ \xi = -\frac{\beta}{\kappa_1} E_2 - \frac{\beta \kappa_2}{\kappa_1'} E_3. \]  

(3.42)

Since \( \xi \) is a unit vector field, we obtain

\[ \kappa_2 = \frac{\kappa_1' \sqrt{\kappa_1^2 - \beta^2}}{\kappa_1 \beta}. \]  

(3.43)

Finally, let \( \beta = 0 \) along the curve. Then (3.40) gives us \( \eta(E_2) = 0 \). From (3.39), we find \( \kappa_1 = \text{constant} \), \( \xi = E_3 \) and \( \lambda = \kappa_1 \kappa_2 \). Differentiating \( \xi = E_3 \) along the curve \( \gamma \), we get \( \kappa_2 = \alpha \). Thus \( \gamma \) is a Legendre helix. Since \( \kappa_2 = \alpha > 0 \), \( M \) cannot be cosymplectic.

The converse statement is trivial. \( \square \)

**Case III.** The osculating order \( r \geq 4 \).

For non-geodesic slant curves of osculating order \( r \geq 4 \), we give the following theorem:
Theorem 3.6. Let \( \gamma : I \subseteq \mathbb{R} \to M \) be a non-geodesic slant curve of order \( r \geq 4 \) with contact angle \( \alpha_0 \) in a trans-Sasakian manifold with \( \dim M \geq 5 \). Then \( \gamma \) has \( C \)-parallel mean curvature vector field if and only if it satisfies

\[
\kappa_1 = \text{constant},
\]
\[
\kappa_2 = -\kappa_1 \tan \alpha_0 = \text{constant},
\]
\[
\kappa_3 = \frac{-\alpha g(\varphi E_1, E_4)}{\sin \alpha_0} = \sqrt{\alpha^2 - \frac{4\kappa_1^2}{\sin^2(2\alpha_0)}} = \text{constant},
\]
\[
\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_3,
\]
\[
\varphi E_1 \in \text{span} \{ E_2, E_4 \}, \quad g(\varphi E_1, E_4) \neq 0
\]
and
\[
\lambda = \frac{-\kappa_1^2}{\cos \alpha_0} = \text{constant}.
\]
In this case, \( M \) becomes an \( \alpha \)-Sasakian manifold.

Proof. Let \( \gamma \) be a curve with \( C \)-parallel mean curvature vector field. From (3.6), we have

\[
-\kappa_1^2 E_1 + \kappa'_1 E_2 + \kappa_1 \kappa_2 E_3 = \lambda \xi. \tag{3.44}
\]
Moreover, from Proposition 2.1, \( M \) is either \( \alpha \)-Sasakian, \( \beta \)-Kenmotsu or cosymplectic. Firstly, let us consider \( \alpha \)-Sasakian case. We have

\[
\eta(E_2) = 0, \tag{3.45}
\]
\[
\nabla_T \xi = -\alpha \varphi E_1. \tag{3.46}
\]
(3.44) and (3.45) give us \( \kappa_1 \) is a constant. The Legendre case causes a contradiction with \( \gamma \) being non-geodesic; so, \( \alpha_0 \neq \frac{\pi}{2} \). From (3.44), we obtain

\[
\lambda = \frac{-\kappa_1^2}{\cos \alpha_0} = \text{constant}, \tag{3.47}
\]
\[
\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_3. \tag{3.48}
\]
Differentiating (3.48) and using (3.46), we get

\[
-\alpha \varphi E_1 = (\kappa_1 \cos \alpha_0 - \kappa_2 \sin \alpha_0) E_2 + \kappa_3 \sin \alpha_0 E_4, \tag{3.49}
\]
which gives us

\[
\varphi E_1 \in \text{span} \{ E_2, E_4 \}, \tag{3.50}
\]
\[
\kappa_3 = \frac{-\alpha g(\varphi E_1, E_4)}{\sin \alpha_0}. \tag{3.51}
\]
Since \( \kappa_3 > 0 \), we have \( g(\varphi E_1, E_4) \neq 0 \). Using (3.44), (3.47) and (3.48), we find

\[
\kappa_2 = -\kappa_1 \tan \alpha_0 = \text{constant}. \tag{3.52}
\]
Thus, from (3.49) and (3.52), we get

\[
\kappa_1 \cos \alpha_0 - \kappa_2 \sin \alpha_0 = \frac{\kappa_1}{\cos \alpha_0}
\]
and
\[
-\alpha \varphi E_1 = \frac{\kappa_1}{\cos \alpha_0} E_2 + \kappa_3 \sin \alpha_0 E_4. \tag{3.53}
\]
Since \( g(\varphi E_1, \varphi E_1) = \sin^2 \alpha_0 \), using equation (3.53), we have
\[
\kappa_3 = \sqrt{\alpha^2 - \frac{4\kappa_1^2}{\sin^2(2\alpha_0)}} = \text{constant}.
\]
So the necessity condition is proved. Conversely, if \( \gamma \) is the above curve, (3.44) is satisfied.

Now, let us consider the \( \beta \)-Kenmotsu case. The proof is done as in the proof of Theorem 3.4 and same results are found with some extra conditions which cause contradiction. Firstly, we have
\[
\kappa_1 \eta(E_2) = -\beta \sin^2 \alpha_0 , \quad (3.54)
\]
and
\[
\nabla_T \xi = \beta [T - \cos \alpha_0 \xi] . \quad (3.55)
\]
Since \( \xi \in \text{span}\{E_1, E_2, E_3\} \), we can write
\[
\xi = \cos \alpha_0 E_1 + \sin \alpha_0 \{ \cos \theta E_2 + \sin \theta E_3 \} , \quad (3.56)
\]
where \( \theta = \theta(s) \) is the angle function between \( E_2 \) and the orthogonal projection of \( \xi \) onto \( \text{span}\{E_2, E_3\} \). Since \( \kappa_3 > 0 \) and \( \sin \alpha_0 \neq 0 \); differentiating (3.56) and using (3.55), one can easily find that \( \sin \theta = 0 \). So we have
\[
\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_2 , \quad (3.57)
\]
From (3.44) and (3.57), we have \( \kappa_2 = 0 \), a contradiction.

Finally, let us consider the cosymplectic case. In this case, we have
\[
\eta(E_2) = 0 , \quad (3.58)
\]
\[
\nabla_T \xi = 0 . \quad (3.59)
\]
(3.44) and (3.58) give us
\[
\xi = \cos \alpha_0 E_1 + \sin \alpha_0 E_2 , \quad (3.60)
\]
\[
\kappa_1 = \text{constant}.
\]
Differentiating (3.60) and using (3.59), we obtain \( \kappa_3 = 0 \), which is also a contradiction. \( \square \)

The following corollaries are direct consequences of Theorem 3.6:

**Corollary 3.3.** If the osculating order \( r = 4 \) in Theorem 3.6, then \( \gamma \) is a helix.

**Corollary 3.4.** There does not exist a non-geodesic slant curve of osculating order \( r \geq 4 \) with \( C \)-parallel mean curvature vector field in a \( \beta \)-Kenmotsu or a cosymplectic manifold.

In the normal bundle, we can state the following theorem:
\textbf{Theorem 3.7.} Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle $\alpha_0$ in a trans-Sasakian manifold with $\dim M \geq 5$. Then $\gamma$ has $C$-parallel mean curvature vector field in the normal bundle if and only if it is a Legendre curve with

\begin{align*}
\kappa_1 &= \text{constant}, \\
\kappa_2 &= \alpha g(\varphi E_1, E_2), \\
\kappa_3 &= -\alpha g(\varphi E_1, E_4), \\
\kappa_2^2 + \kappa_3^2 &= \alpha, \\
\lambda &= \kappa_1 \kappa_2, \\
\xi &= E_3, \quad \alpha \neq 0
\end{align*}

and

\begin{align*}
\varphi E_1 &= \frac{\kappa_2}{\alpha} E_2 - \frac{\kappa_3}{\alpha} E_4. \tag{3.64}
\end{align*}

In this case, $M$ becomes an $\alpha$-Sasakian manifold.

\textit{Proof.} From (3.8), we have

\begin{align*}
\kappa_1' E_2 + \kappa_1 \kappa_2 E_3 &= \lambda \xi. \tag{3.65}
\end{align*}

Then we get

\begin{align*}
\eta(E_1) &= 0, \\
\kappa_1 \eta(E_2) &= -\beta. \tag{3.66}
\end{align*}

Firstly, let $\beta = 0$. Then, from (3.65) and (3.66),

\begin{align*}
\eta(E_2) &= 0, \\
\lambda &= \kappa_1 \kappa_2, \\
\xi &= E_3. \tag{3.67}
\end{align*}

Differentiating (3.67), we find

\begin{align*}
-\alpha \varphi E_1 &= -\kappa_2 E_2 + \kappa_3 E_4,
\end{align*}

which gives us (3.61), (3.62), (3.63) and (3.64), where $\alpha \neq 0$, that is, $M$ is an $\alpha$-Sasakian manifold.

Now, let us assume that $\beta \neq 0$. We have same results in Theorem 3.5 but some extra calculations lead to a contradiction. Since $\xi \in \text{span} \{E_2, E_3\}$, we can write

\begin{align*}
\xi &= \cos \theta E_2 + \sin \theta E_3, \tag{3.68}
\end{align*}

where $\theta = \theta(s)$ is the angle function between $\xi$ and $E_2$. Differentiating (3.68), we find

\begin{align*}
\kappa_3 &= \frac{-\alpha g(\varphi E_1, E_4)}{\sin \theta},
\end{align*}

which gives us $\alpha \neq 0$. Since $\dim M \geq 5$, this contradicts Proposition 2.1. \qed
4. SLANT CURVES WITH C-PROPER MEAN CURVATURE VECTOR FIELD

We consider the following cases:

**Case I.** The osculating order \( r = 2 \).

For this case, we have the following theorems:

**Theorem 4.1.** Let \( \gamma : I \subseteq \mathbb{R} \rightarrow M \) be a non-geodesic slant curve of order 2 with contact angle \( \alpha_0 \) in a trans-Sasakian manifold. Then \( \gamma \) has C-proper mean curvature vector field if and only if \( \alpha = 0 \) and \( \beta \neq 0 \) along the curve and

i) \( \gamma \) is a Legendre circle with \( \kappa_1 = \mp \beta = \text{constant}, \ \xi = \pm E_2, \ \lambda = -\beta^3 \); or

ii) \( \gamma \) is a non-Legendre slant curve with

\[
\begin{align*}
\kappa_1 & = \mp \beta \sin \alpha_0, \\
\kappa''_1 - \kappa^3_1 & = \pm 3\kappa'_1 \kappa_1 \tan \alpha_0, \\
\xi & = \cos \alpha_0 E_1 \pm \sin \alpha_0 E_2 \\
\lambda & = \frac{3\kappa'_1 \kappa_1}{\cos \alpha_0}. 
\end{align*}
\]

(4.1)

and

(4.2)

**Proof.** Let \( \gamma \) have C-proper mean curvature vector field. From (3.7), we have

\[
3\kappa_1 \kappa'_1 E_1 + (\kappa^3_1 - \kappa''_1) E_2 = \lambda \xi.
\]

(4.3)

Thus, \( \xi \in \text{span} \{E_1, E_2\} \). So we can write

\[
\xi = \cos \alpha_0 E_1 \pm \sin \alpha_0 E_2.
\]

(4.4)

Differentiating (4.4) and using (3.12), we find

\[
-\alpha \varphi E_1 + \beta \sin^2 \alpha_0 E_1 \mp \beta \cos \alpha_0 \sin \alpha_0 E_2 = \mp \kappa_1 \sin \alpha_0 \kappa_1 E_1 + \kappa_1 \cos \alpha_0 E_2.
\]

(4.5)

(4.4) and (4.5) give us \( \alpha = 0 \) along the curve. We have \( \beta \neq 0 \), since \( \kappa_1 = \mp \beta \sin \alpha_0 \).

If \( \alpha_0 = \frac{\pi}{2} \), then \( \gamma \) is a Legendre curve with \( \kappa_1 = \mp \beta = \text{constant}, \ \xi = \pm E_2, \ \lambda = -\beta^3 \).

Let \( \alpha_0 \neq \frac{\pi}{2} \). Then, by the use of (4.3) and (4.4), we obtain (4.1) and (4.2). \( \square \)

In the normal bundle, we can state the following theorem:

**Theorem 4.2.** Let \( \gamma : I \subseteq \mathbb{R} \rightarrow M \) be a non-geodesic slant curve of order 2 with contact angle \( \alpha_0 \) in a trans-Sasakian manifold. Then \( \gamma \) has C-proper mean curvature vector field in the normal bundle if and only if it is a Legendre curve with

\[
\begin{align*}
\kappa_1 & = \mp \beta, \\
\xi & = \pm E_2, \\
\lambda & = \beta'', 
\end{align*}
\]

(4.6)

and \( \beta(s) \neq as + b \), where \( a \) and \( b \) are arbitrary constants. In this case, \( \alpha = 0 \) along the curve.

**Proof.** Let \( \gamma \) have C-proper mean curvature vector field in the normal bundle. From (3.9) and Theorem 3.1 \( \gamma \) is a Legendre curve with

\[
-\kappa''_1 E_2 = \lambda \xi.
\]

So we have

\[
\lambda = \pm \kappa'_1
\]
and
\[ \xi = \pm E_2. \]  
(4.7)

Differentiating (4.7), we find
\[ -\alpha \varphi E_1 + \beta E_1 = \mp \kappa_1 E_1. \]  
(4.8)

(4.8) gives us (4.6) and \( \alpha = 0 \) along the curve, which completes the proof. □

Case II. The osculating order \( r = 3 \).

For this case, we have the following theorems:

**Theorem 4.3.** Let \( \gamma : I \subseteq \mathbb{R} \to M \) be a non-geodesic slant curve of order 3 with contact angle \( \alpha_0 \) in a trans-Sasakian manifold. Then \( \gamma \) has \( C \)-proper mean curvature vector field if and only if

i) it satisfies
\[
\kappa_2 = \kappa_1 + \alpha,
\]
\[
2\kappa_1^3 - \kappa''_1 = 0,
\]
\[
\alpha_0 = \frac{\pi}{4},
\]
\[
\xi = \frac{\sqrt{2}}{2} (E_1 - E_3),
\]
\[
\lambda = 3\sqrt{2} \kappa_1 \kappa'_1
\]
and
\( \kappa_1 \neq \text{constant}, \)

(in this case, \( M \) becomes an \( \alpha \)-Sasakian or a cosymplectic manifold); or

ii) it satisfies
\[
3\kappa_1 \kappa'_1 = \lambda \cos \alpha_0,
\]
\[
\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa''_1 = \lambda \eta(E_2),
\]
\[
-(2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) = \lambda \eta(E_3)
\]
and
\[
\eta(E_2)^2 + \eta(E_3)^2 = \sin^2 \alpha_0.
\]

(In this case, \( M \) becomes an \( (\alpha, \beta) \)-trans-Sasakian or a \( \beta \)-Kenmotsu manifold.)

**Proof.** Let \( \gamma \) have \( C \)-proper mean curvature vector field. Then, from (3.7), we have
\[
3\kappa_1 \kappa'_1 E_1 + (\kappa_1^3 + \kappa_1 \kappa_2^2 - \kappa''_1) E_2 - (2\kappa'_1 \kappa_2 + \kappa_1 \kappa'_2) E_3 = \lambda \xi. \]  
(4.9)

Now, let us assume that \( \beta = 0 \). Then we have \( \eta(E_2) = 0 \), so we can write
\[
\xi = \cos \alpha_0 E_1 - \sin \alpha_0 E_3. \]  
(4.10)

We cannot choose \( \eta(E_3) = \sin \alpha_0 \), because it leads to a contradiction. Differentiating (4.10), we have
\[
-\alpha \varphi E_1 = (\kappa_1 \cos \alpha_0 - \kappa_2 \sin \alpha_0) E_2, \]  
(4.11)

which gives us
\[
\kappa_2 = \kappa_1 \cot \alpha_0 + \alpha. \]  
(4.12)
Since $\alpha$ is a constant, we obtain

$$\kappa_2' = \kappa_1' \cot \alpha_0. \quad (4.13)$$

From (4.9), we can write

$$3\kappa_1\kappa_1' = \lambda \cos \alpha_0, \quad (4.14)$$

$$\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = 0 \quad (4.15)$$

and

$$2\kappa_1\kappa_2' + \kappa_1\kappa_2 = \lambda \sin \alpha_0. \quad (4.16)$$

By the use of (4.12) in (4.15), we get

$$\kappa_1^3 - \sin^2 \alpha_0 \kappa_1'' = 0 \quad (4.17)$$

So we have $\kappa_1 \neq \text{constant}$ and $\alpha_0 \neq \frac{\pi}{2}$. In view of (4.12), (4.13), (4.14) and (4.16), we find $\cos 2\alpha_0 = 0$, which means that $\alpha_0 = \frac{\pi}{4}$. Hence, taking $\alpha_0 = \frac{\pi}{4}$ in above equations, the proof is done for $\alpha$-Sasakian and cosymplectic manifolds.

Now, let us assume that $\beta \neq 0$. (4.9) gives us $\xi \in \text{span} \{E_1, E_2, E_3\}$. So we can write

$$\xi = \cos \alpha_0 E_1 + \sin \alpha_0 \{\cos \theta E_2 + \sin \theta E_3\}, \quad (4.18)$$

where $\theta = \theta(s)$ is the angle function between $E_2$ and the orthogonal projection of $\xi$ onto $\text{span} \{E_2, E_3\}$. Using (4.9) and (4.18), the proof is completed. $\square$

In the normal bundle, we can give the following result:

**Theorem 4.4.** Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order 3 with contact angle $\alpha_0$ in a trans-Sasakian manifold. Then $\gamma$ has $C$-proper mean curvature vector field in the normal bundle if and only if it is a Legendre curve with

i) $$\kappa_1 = c_1 e^{\alpha s} + c_2 e^{-\alpha s}, \quad (4.19)$$

$$\kappa_2 = \alpha,$$

$$\xi = E_3, \quad \varphi E_1 = E_2$$

and

$$\lambda = -2\alpha^2 (c_1 e^{\alpha s} - c_2 e^{-\alpha s}), \quad (4.20)$$

where $c_1$ and $c_2$ are arbitrary constants, (in this case, $M$ becomes an $\alpha$-Sasakian manifold); or

ii) $$\lambda = \frac{\kappa_1\kappa_1'' - \kappa_1^3 \kappa_2}{\beta}, \quad (4.21)$$

$$\xi = \frac{\beta}{\kappa_1} E_2 \pm \frac{\sqrt{\kappa_1^2 - \beta^2}}{\kappa_1} E_3 \quad (4.22)$$

and

$$\pm (\kappa_1'' - \kappa_1^3 \kappa_2) \sqrt{\kappa_1^2 - \beta^2} = 2\kappa_1\kappa_2' + \kappa_1\kappa_2'. \quad (4.23)$$

In this case, $M$ becomes an $(\alpha, \beta)$-trans-Sasakian or a $\beta$-Kenmotsu manifold.
Proof. Let $\gamma$ have $C$-proper mean curvature vector field in the normal bundle. From (3.9) and Theorem 3.1, $\gamma$ is a Legendre curve with
\[(\kappa_1\kappa_2^2 - \kappa_1'' \cdot E_2 - (2\kappa_1'\kappa_2 + \kappa_1\kappa_2') \cdot E_3 = \lambda \xi). \tag{4.24}\]
Let $\beta = 0$. Then we find $\eta(E_2) = 0$, which gives us
\[
\kappa_1\kappa_2^2 - \kappa_1'' = 0, \tag{4.25}
\]
and
\[
\xi = E_3, \tag{4.26}
\]
Differentiating (4.26), we have
\[
\kappa_2 = \alpha \tag{4.28}
\]
and
\[
\varphi E_1 = E_2. \tag{4.29}
\]
Since $\alpha$ is a non-zero constant, by the use of (4.25) and (4.28), we find (4.19). Using (4.19), (4.27) and (4.28), we obtain (4.20).

Now, let $\beta \neq 0$. Then (3.11) and (4.24) give us (4.21). Since the unit vector field $\xi \in \text{span}\{E_2, E_3\}$, using (3.11), we find (4.22). By the use of (4.21), (4.22) and (4.24), we obtain (4.23). Since $\beta \neq 0$, $M$ is an $(\alpha, \beta)$-trans-Sasakian or a $\beta$-Kenmotsu manifold.

Case III. The osculating order $r \geq 4$.

In this case, we can state the following theorem:

**Theorem 4.5.** Let $\gamma : I \subseteq \mathbb{R} \rightarrow M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle $\alpha_0$ in a trans-Sasakian manifold with $\dim M \geq 5$. Then $\gamma$ has $C$-proper mean curvature vector field if and only if it satisfies
\[
3\kappa_1\kappa_1' = \lambda \cos \alpha_0, \\
\kappa_1^3 + \kappa_1\kappa_2^2 - \kappa_1'' = \lambda \eta(E_2), \\
-(2\kappa_1'\kappa_2 + \kappa_1\kappa_2') = \lambda \eta(E_3), \\
-\kappa_1\kappa_2\kappa_3 = \lambda \eta(E_4)
\]
and
\[
\eta(E_2)^2 + \eta(E_3)^2 + \eta(E_4)^2 = \sin^2 \alpha_0,
\]
where $\lambda$ is a non-zero differentiable function on $I$.

**Proof.** Since $\xi$ is a unit vector field, by the use of (3.7) and (3.10), the proof is completed. □

In the normal bundle, we can give the following theorem:
Theorem 4.6. Let $\gamma : I \subseteq \mathbb{R} \to M$ be a non-geodesic slant curve of order $r \geq 4$ with contact angle $\alpha_0$ in a trans-Sasakian manifold $\dim M \geq 5$. Then $\gamma$ has $C$-proper mean curvature vector field in the normal bundle if and only if it is a Legendre curve with

\[
\kappa_1 \kappa_2^2 - \kappa_1'' = 0,
\]

\[
\kappa_2 = \alpha g(\varphi E_1, E_2),
\]

\[
\kappa_3 = -\alpha g(\varphi E_1, E_4),
\]

\[
\kappa_2^2 + \kappa_3^2 = \alpha,
\]

\[
\lambda = -2\kappa_1' \kappa_2 - \kappa_1 \kappa_2',
\]

\[
\xi = E_3, \quad \alpha \neq 0
\]

and

\[
\varphi E_1 = \frac{\kappa_2}{\alpha} E_2 - \frac{\kappa_3}{\alpha} E_4.
\]

In this case, $M$ becomes an $\alpha$-Sasakian manifold.

Proof. The proof is similar to the proof of Theorem 3.7. $\square$

5. Examples

Example 1. Let us consider the 3-dimensional manifold

\[
M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\},
\]

where $(x, y, z)$ are the standard coordinates on $\mathbb{R}^3$ and the metric tensor field on $M$ is given by

\[
g = \frac{1}{z^2} (dx^2 + dy^2 + dz^2).
\]

The vector fields

\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}
\]

are $g$-orthonormal vector fields in $\chi(M)$. Let $\varphi$ be the $(1, 1)$-tensor field defined by

\[
\varphi e_1 = -e_2, \quad \varphi e_2 = e_1, \quad \varphi e_3 = 0.
\]

Let us define a 1-form $\eta(Z) = g(Z, e_3)$, for all $Z \in \chi(M)$ and the characteristic vector field $\xi = e_3$. In ([9], [13]), it was proved that $(M, \varphi, \xi, \eta, g)$ is a Kenmotsu manifold. Thus, it is a trans-Sasakian manifold with $\alpha = 0, \beta = 1$.

The curve $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in $M$ with contact angle $\alpha_0$ if and only if the following equations are satisfied:

\[
(\gamma_1')^2 + (\gamma_2')^2 = \sin^2 \alpha_0 (\gamma_3)^2,
\]

\[
\gamma_3 = c e^{-s \cos \alpha_0},
\]

where $c > 0$ is an arbitrary constant.

Let $\gamma : I \subseteq \mathbb{R} \to M$, $\gamma(s) = (as + b, ms + n, c)$ where $a, b, m, n, c \in \mathbb{R}$, $c > 0$, $a^2 + m^2 = c^2$ and $s$ is the arc-length parameter on open interval $I$. The unit tangent vector field $T$ along $\gamma$ is

\[
T = \frac{a}{c} e_1 + \frac{m}{c} e_2.
\]
Then \( \gamma \) is a Legendre curve since \( \eta(T) = 0 \), that is, \( \alpha_0 = \frac{\pi}{2} \). Using Koszul’s formula, we get \( \nabla_T T = -e_3 \), which gives us \( \kappa_1 = 1 \), \( E_2 = -e_3 \). After simple calculations, we find \( \nabla_T E_0 = -T \), that is, \( \kappa_2 = 0 \). Then \( \gamma \) is of osculating order \( r = 2 \). From Theorem 4.1 i), \( \gamma \) has \( C \)-proper mean curvature vector field in the tangent bundle with \( \kappa_1 = \beta = 1 \), \( \xi = -E_2 \), \( \lambda = -\beta^3 = -1 \). Hence, an explicit example of Theorem 4.1 i) in the given manifold \( M \) is \( \gamma(s) = (3s, 4s, 5s) \).

In the above example, if we take \( e_3 = z\frac{\partial}{\partial z}, \xi = e_3 \) and define the other structures in the same way, we have a trans-Sasakian manifold with \( \alpha = 0 \), \( \beta = -1 \) which was given in ([10], [13]). In this manifold, \( \gamma(s) = (s, 0, 1) \) is another example of Theorem 4.1 i) with \( \kappa_1 = -\beta = 1 \), \( \xi = E_2 \), \( \lambda = -\beta^3 = 1 \).

We will use the following trans-Sasakian manifold given in [5] to construct new examples.

Let \( M = N \times (a, b) \) where \( N \) is an open connected subset of \( \mathbb{R}^2 \) and \( (a, b) \) is an open interval in \( \mathbb{R} \). Let \( (x, y, z) \) be the coordinate functions on \( M \). Now let us take the functions

\[
\omega_1, \omega_2 : N \to \mathbb{R}, \quad \sigma, f : M \to \mathbb{R}^*_+.
\]

The normal almost contact metric structure \( (\varphi, \xi, \eta, g) \) on \( M \) is given by

\[
\varphi = \begin{bmatrix} 0 & 1 & -\omega_2 \\ -1 & 0 & \omega_1 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\xi = \frac{\partial}{\partial z}, \quad \eta = dz + \omega_1 dx + \omega_2 dy,
\]

\[
g = \begin{bmatrix} \omega_1^2 + \sigma e^{2f} & \omega_1 \omega_2 & \omega_1 \\ \omega_1 \omega_2 & \omega_2^2 + \sigma e^{2f} & \omega_2 \\ \omega_1 & \omega_2 & 1 \end{bmatrix}.
\]

Let us choose \( g \)-orthonormal frame fields as follows:

\[
H_1 = \frac{e^{-f}}{\sqrt{\sigma}} \left[ \frac{\partial}{\partial x} - \omega_1 \frac{\partial}{\partial z} \right], \quad H_2 = \frac{e^{-f}}{\sqrt{\sigma}} \left[ \frac{\partial}{\partial y} - \omega_2 \frac{\partial}{\partial z} \right], \quad H_3 = \xi = \frac{\partial}{\partial z}.
\]

It is seen that \( M \) is a trans-Sasakian manifold with

\[
\alpha = \frac{e^{-2f}}{2\sigma} \left( \frac{\partial \omega_1}{\partial y} - \frac{\partial \omega_2}{\partial x} \right), \quad \beta = \frac{1}{2\sigma} \frac{\partial \sigma}{\partial z} + \frac{\partial f}{\partial z}.
\]

In [5], it is shown that \( \gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \) is a slant curve in \( M \) with contact angle \( \alpha_0 \) if and only if

\[
(\gamma_1')^2 + (\gamma_2')^2 = \frac{\sin^2 \alpha_0}{\sigma} e^{-2f},
\]

\[
\omega_1 \gamma_1' + \omega_2 \gamma_2' + \gamma_3' = \cos \alpha_0.
\]

Using this method, we have the following examples:
Example 2. Let us consider the Legendre helix $\gamma(s) = (0, \frac{s}{2}, 2)$ in $(M, \varphi, \xi, \eta, g)$ where $\omega_1 = f = 0$, $\omega_2 = 2x$ and $\sigma = 2z$. Then $M$ is a trans-Sasakian manifold of type $\left(\frac{1}{2z}, \frac{1}{2z}\right)$, that is,

$$\alpha = -\frac{1}{2z} = -\beta.$$ 

It was shown that $\kappa_1 = \kappa_2 = \frac{1}{4}$ (see [19]). Let us show that $\gamma$ has $C$-proper mean curvature vector field in the tangent and normal bundles. After direct calculations, we obtain $T = H_2$, $\nabla_T T = -\frac{1}{4}H_3$. Then we have $\xi = H_3 = -E_2$. Finally, we get $\nabla_T E_2 = -\frac{1}{4}T + \frac{1}{4}H_1$. Hence $E_3 = H_1$. By the use of Theorems 4.3 and 4.4 respectively, we find that $\gamma$ is a curve with $C$-proper mean curvature vector field in the tangent bundle with $\lambda = -\frac{1}{32}$ and in the normal bundle with $\lambda = -\frac{1}{64}$. Furthermore, in [2], the authors proved that $\gamma$ has proper mean curvature vector field (in the tangent bundle) with $\lambda = \frac{1}{8}$.

Example 3. Let us choose $\omega_1 = f = 0$, $\omega_2 = -y$ and $\sigma = z$. So $\alpha = 0$ and $\beta = \frac{1}{2z}$. Thus $M$ is a $\beta$-Kenmotsu manifold. Then $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in $M$ if and only if

$$(\gamma_1')^2 + (\gamma_2')^2 = \frac{\sin^2 \alpha_0}{\gamma_3},$$ 

$$-\gamma_2 \gamma_2' + \gamma_3' = \cos \alpha_0.$$ 

Let us take $\gamma(s) = (0, \frac{3^2}{4}z, \sqrt{2}s)$ in $M$. We find $\alpha_0 = \frac{7}{2}$, that is, $\gamma$ is a Legendre curve. After some calculations, using Theorem 3.3, we find that $\gamma$ is of osculating order $r = 2$ and it has $C$-parallel mean curvature vector field in the normal bundle with $\kappa_1 = \beta = \frac{\sqrt{2}}{4z}$, $\xi = -E_2$ and $\lambda = -\beta' = \frac{\sqrt{2}}{4z^2}$. Moreover, $\gamma$ has $C$-proper mean curvature vector field in the normal bundle with $\lambda = \beta'' = \frac{\sqrt{2}}{4z^3}$ which verifies Theorem 4.2.

Example 4. Let us choose $\omega_1 = f = 0$, $\omega_2 = y$ and $\sigma = z$. Then $\alpha = 0$ and $\beta = \frac{1}{2z}$. Hence $M$ is a $\beta$-Kenmotsu manifold. Then $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s))$ is a slant curve in $M$ if and only if

$$(\gamma_1')^2 + (\gamma_2')^2 = \frac{\sin^2 \alpha_0}{\gamma_3},$$ 

$$\gamma_2 \gamma_2' + \gamma_3' = \cos \alpha_0.$$ 

Let us consider the non-Legendre slant curve $\gamma(s) = (\frac{4}{105}7^{3/4}\sqrt{30s}, 0, \frac{\sqrt{7}s}{15})$ in $M$ with contact angle $\alpha_0 = \arccos(\frac{\sqrt{7}}{15}) = \arcsin(\frac{2\sqrt{2}}{15})$. After some straightforward calculations, using Theorem 4.1(ii), we find that $\gamma$ has $C$-proper mean curvature vector field (in the tangent bundle) with

$$\kappa_1 = \frac{\sqrt{14}}{15s},$$ 

$$\xi = \frac{\sqrt{7}}{15}E_1 - \frac{2\sqrt{2}}{15}E_2,$$
\[ \beta = \frac{15\sqrt{7}}{14s}, \]

and

\[ \lambda = \frac{-90\sqrt{7}}{49s^3}. \]

It is easy to check that \( \kappa_1 \) satisfies

\[ \kappa''_1 - \kappa^3_1 = -3\kappa'_1 \kappa_1 \tan \alpha_0. \]

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