REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 55, No. 2, 2014, Pages 101–106
Published online: October 23, 2014

ON MINIMAL NON-$\mathscr{B}$-GROUPS

HUAGUO SHI AND ZHANGJIA HAN

Abstract. A finite group G is called a $\mathscr{B}$-group if every proper subgroup of G is either normal or abnormal in G. In this paper the authors classify the non-$\mathscr{B}$-groups whose proper subgroups are all $\mathscr{B}$-groups.

1. Introduction

Throughout this paper the term group always means a group of finite order. A subgroup U is said to be abnormal in G if $g \in \langle U, U^g \rangle$ for all $g \in G$. The term abnormal subgroup is due to Carter [1]. Since abnormal subgroups are selfnormalizing, abnormality is a strong form of non-normality. Clearly every maximal non-normal subgroup is abnormal, and thus finite groups without proper abnormal subgroups are nilpotent. The properties of abnormal subgroups in finite soluble groups have been investigated in detail (see, for example, [4] and [11]).

Definition 1.1. *A group G is called a $\mathscr{B}$-group if every subgroup of G is either normal or abnormal in G.*

By [5, Lemma 1] we know that G is a $\mathscr{B}$-group if and only if every primary subgroup of G is either normal or abnormal in G. In addition, if G is a $\mathscr{B}$-group, then every subgroup and every homomorphic image of G is also a $\mathscr{B}$-group.

Given a group theoretical property $\mathscr{P}$, a $\mathscr{P}$-critical group or a minimal non-$\mathscr{P}$-group is a group which is not $\mathscr{P}$-group but all of whose proper subgroups are $\mathscr{P}$-groups. There are many remarkable examples about the minimal non-$\mathscr{P}$-groups: minimal non-abelian groups (Miller and Moreno [10]), minimal non-nilpotent groups (Schmidt), minimal non-supersolvable groups ([3]), minimal non-p-nilpotent groups (Itô), minimal non-PN-groups ([12]), minimal non-MSP-groups ([6]), and minimal non-NSN-groups ([7]). In this paper, by applying the structure of $\mathscr{B}$-groups, we give the classification of minimal non-$\mathscr{B}$-groups.

Our notations are all standard. For example, we denote by $A \rtimes B$ the semidirect product of A and B; C_n always denotes a cyclic group of order n, and $\pi(G)$ denotes the set of all prime divisors of $|G|$. All unexplained notations can be found in [8] and [11].

2010 *Mathematics Subject Classification.* 20D45 (20E34).

Key words and phrases. normal subgroup; abnormal subgroup; minimal non-$\mathscr{B}$-group.

This work is supported by the National Scientific Foundation of China (No. 11301426) and Scientific Research Foundation of Sichuan Provincial Education Department (No. 14ZA0314).

2. Some preliminaries

In this section, we collect some lemmas which will be used in the following.

Lemma 2.1. [8, 7.2.2] *Suppose that the Sylow p-subgroups of G are cyclic, where p is the smallest prime divisor of $|G|$. Then G has a normal p-complement.*

Lemma 2.2. [9] *Suppose that p'-group H acts on a p-group G. Let*

$$\Omega(G) = \begin{cases} \Omega_1(G), & p > 2, \\ \Omega_2(G), & p = 2. \end{cases}$$

If H acts trivially on $\Omega(G)$, then H acts trivially on G as well.

Lemma 2.3. (Maschke's Theorem). [8, 8.4.6] *Suppose that the action of A on an abelian group G is coprime and H is an A-invariant direct factor of G. Then H has an A-invariant complement in G.*

Lemma 2.4. [14] *If G is a minimal non-abelian simple group, then G is isomorphic to one of the following simple groups:*

(1) *$PSL(2,p)$, where p is a prime with $p > 3$ and $5 \nmid p^2 - 1$.*
(2) *$PSL(2,2^q)$, where q is a prime.*
(3) *$PSL(2,3^q)$, where q is a prime.*
(4) *$PSL(3,3)$.*
(5) *The Suzuki group $Sz(2^q)$, where q is an odd prime.*

In proving our main theorem, the following result will be frequently used.

Lemma 2.5. [5, Theorem 3] *Let G be a $\mathscr{B}$-group. Then one of the following statements is true:*

(1) *G is a Dedekind group.*
(2) *$G = KQ$, $K = G'$ is an abelian q'-subgroup of G, $Q \in \mathrm{Syl}_q(G)$ is cyclic, where q is the smallest prime divisor of $|G|$. Q induces a fixed-point-free power automorphism of order q on K, $\Phi(Q) = Z(G)$.*

3. Minimal non-$\mathscr{B}$-groups

In this section, we classify finite minimal non-$\mathscr{B}$-groups. Here we first note that the classification of minimal non-Dedekind groups was given in [2] and [10]. So, it only remains to classify finite minimal non-$\mathscr{B}$-groups which are not minimal non-Dedekind groups.

In the first place, we study some basic properties of minimal non-$\mathscr{B}$-groups.

Lemma 3.1. *Let G be a minimal non-$\mathscr{B}$-group. Then G is solvable.*

Proof. Suppose that G is not solvable. Let M be a maximal normal subgroup of G. Then G/M is a non-abelian simple group and every proper subgroup of G/M is a $\mathscr{B}$-group and hence G/M is a minimal non-abelian simple group. Thus G/M is isomorphic to one of the simple groups mentioned in Lemma 2.4. It is well know that each of $PSL(2,p)$, $PSL(2,3^q)$ and $PSL(3,3)$ contains a subgroup which is isomorphic to A_4, the alternating group of degree 4. However, the group A_4 is not a $\mathscr{B}$-group. So G/M can not be isomorphic to $PSL(2,p)$ or $PSL(2,3^q)$ or $PSL(3,3)$.

Suppose that $G/M \cong Sz(q)$, where $q = 2^r$, r is an odd prime. By [13, Proposition 16], G has an abelian subgroup A with order $2^r + 2^{\frac{r+1}{2}} + 1$ such that $|N_G(A)| = 4(2^r + 2^{\frac{r+1}{2}} + 1)$ and $C_G(x) = A$ for any $x \in A$. Apparently $N_G(A)$ is a proper subgroup of G. However $N_G(A)$ is not a $\mathscr{B}$-group, a contradiction. So $G/M \ncong Sz(q)$.

Suppose that $G/M \cong PSL(2,2^r)$, r a prime. As $PSL(2,4) \cong PSL(2,5) \cong A_5$, we may assume that r is an odd prime. As we all know that $PSL(2,2^r)$ has a maximal subgroup which is a Frobenius subgroup F of order $2^r(2^r-1)$ with the Frobenius kernel K of order 2^r. It is easy to see that F is not a $\mathscr{B}$-group. So G/M can not be isomorphic to $PSL(2,2^r)$. □

Lemma 3.2. *Let G be a minimal non-$\mathscr{B}$-group. Then $|\pi(G)| \leq 3$.*

Proof. Suppose that $|\pi(G)| > 3$. Let $\{P_1, P_2, \dots, P_k, \dots, P_r\}$, $r > 3$, be a Sylow system of G, where $P_i \in \mathrm{Syl}_{p_i}(G)$, $i = 1, 2, \dots, r$. Since G is not a $\mathscr{B}$-group, without loss of generality we suppose that $P_2 \nleq N_G(P_1)$. Hence for every $P_i (i \geq 3)$, the subgroup $P_1P_2P_i$ is a $\mathscr{B}$-group described in (2) of Lemma 2.5. Hence $P_1P_2P_i = P_2P_i \rtimes P_1$. Now by the arbitrariness of P_i, we get that G itself is a $\mathscr{B}$-group, a contradiction. □

The following theorem classifies all minimal non-$\mathscr{B}$-groups whose order has just two prime divisors.

Theorem 3.3. *Let G be a minimal non-$\mathscr{B}$-group with $|\pi(G)| = 2$. Then one of the following holds.*

(1) $G = C_q \rtimes (C_{p^n} \times C_p)$, $\Phi(C_{p^n})C_p = Z(G)$.
(2) $G = C_q \rtimes Q_8$, Q_8 *induces an automorphism of order* 2 *on* C_q.
(3) $G = C_{q^n} \rtimes C_{p^m}$, $m \geq 2$, $\Phi(\Phi(P)) = Z(G)$.
(4) $G = \langle a, b, c \mid a^q = b^q = c^{p^m} = 1,\ ab = ba,\ a^c = a^i,\ b^c = b^j,\ i \not\equiv j \pmod q,\ i^p \equiv j^p \equiv 1 \pmod q \rangle$.
(5) $G = \langle a, b, c \mid a^{q^m} = b^{q^m} = 1,\ c^{p^n} = 1,\ ab = ba,\ a^c = a^u,\ b^c = b^v.\ u \not\equiv v \pmod{q^m},\ u \equiv v \pmod{q^{m-1}},\ u^p \equiv v^p \equiv 1 \pmod{q^m},\ u \not\equiv 1 \pmod q,\ v \not\equiv 1 \pmod q,\ m \geq 2 \rangle$.
(6) $G = Q \rtimes P$, *Q is an elementary abelian q-group. P is cyclic, P acts irreducibly on Q, $\Phi(P)$ induces a power automorphism of order p on Q, and* $\Phi(\Phi(P)) = Z(G)$.

Proof. Let $G = PQ$, where $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. For convenience, we always let p be the smallest prime divisor of $|G|$. Since G is solvable, there is a normal maximal subgroup M of G such that $|G : M| = p$ or q. By our assumption, M is a $\mathscr{B}$-group. Hence we can get that either P or Q must be normal in G.

Case 1. P is normal in G.

By Lemma 2.1, P is not a cyclic subgroup. Let Q_1 and Q_2 be two different maximal subgroups of Q. Then PQ_1 and PQ_2 are all $\mathscr{B}$-groups by hypothesis. By

Lemma 2.5, both PQ_1 and PQ_2 are nilpotent, which implies that G is nilpotent, a contradiction. Therefore we get that Q is cyclic. Since $\Phi(Q)P$ is a $\mathscr{B}$-group, we have that $\Phi(Q)P = \Phi(Q) \times P$. Thus $\Phi(Q) \leq Z(G)$.

Again by Lemma 2.5, we know that Q acts non-trivially on P, but acts trivially on every Q-invariant proper subgroup of P. Applying Hall–Higman's reduction theorem, we get that $\exp(P) = p$ (when $p > 2$) or $\exp(P) \leq 4$ (when $p = 2$) and $P' = \Phi(P) = \Omega_1(P) \leq Z(G)$. Moreover, we have that Q acts irreducibly on $P/\Phi(P)$.

If $\exp(P) = p$, then P is an elementary abelian p-group. Now $G = P \rtimes C_{q^m}$, Q acts irreducibly on the elementary abelian p-subgroup P, and $\Phi(C_{q^m}) = Z(G)$. By a routine check, we can get that G is a minimal non-Dedekind group, a contradiction.

If $\exp(P) = 4$, then we can easily get that P is not abelian. Hence $P = Q_8 \times E$, where E is an elementary abelian p-group. Since $Q\Omega_1(P)$ is a $\mathscr{B}$-group, we get that $Q \leq C_G(E)$ by Lemma 2.5. Thus $E = 1$ since Q acts irreducibly on $P/\Phi(P)$. As we all know that $\mathrm{Aut}(Q_8) = S_4$, so $G = Q_8 \rtimes C_{3^m}$. It is easy to see that G is a minimal non-Dedekind group, a contradiction.

Case 2. Q is normal in G.

(1) Suppose in the first place that P is non-cyclic. By the same argument as above we can get that $\exp(Q) = q$. Hence Q is an elementary abelian q-group, and the action of P on Q is irreducible. Assume that $|Q| > q$. Let P_1 and P_2 be two maximal subgroups of P. Then both P_1Q and P_2Q are $\mathscr{B}$-groups and hence P_i $(i = 1, 2)$ induces a fixed-point-free power automorphism on Q, which implies that the action of P on Q is reducible, a contradiction. Thus we have $|Q| = q$. On the other hand, let P_1 be a maximal subgroup of P. Then P_1Q is a $\mathscr{B}$-group. It follows that if P has more than one non-cyclic maximal subgroup, then G is nilpotent by Lemma 2.5, a contradiction. Hence P has at most one non-cyclic maximal subgroup. Since P is a $\mathscr{B}$-group, P must be isomorphic to $C_{p^n} \times C_p$ or the quaternion group Q_8.

If $P \cong C_{p^n} \times C_p$, then $G \cong C_q \rtimes (C_{p^n} \times C_p)$, $\Phi(C_{p^n})C_p = Z(G)$. That is, G is of type (1).

If $P \cong Q_8$, then $G \cong C_q \rtimes Q_8$, Q_8 induces an automorphism of order 2 on C_q. That is, G is of type (2).

(2) Suppose that P is cyclic and P acts reducibly on $Q/\Phi(Q)$. Let $P = \langle c \rangle$ and $Q = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$, $o(x_i) = q^{m_i}$, where $q^{m_1} = \exp(Q)$ and $x_i^c = x_i^{u_i}$. If $n \geq 3$, then the action of P on $\langle x_1, x_i \rangle$ is reducible, there exists a natural number w satisfying $x^c = x^w$ for each $x \in \langle x_1, x_i \rangle$. Hence $w \equiv u_1 \pmod{q^{m_1}}$ and $w \equiv u_i \pmod{q^{m_i}}$. Thus we get $u_i \equiv u_1 \pmod{q^{m_i}}$. It follows that P induces a fixed-point-free power automorphism of order p on Q and hence G itself is a $\mathscr{B}$-group, a contradiction. Therefore we have $n \leq 2$.

If Q is cyclic, then we have $|P| \geq p^2$ by the structure of $\mathscr{B}$-groups. Hence $G = C_{q^n} \rtimes C_{p^m}$, $m \geq 2$, $\Phi(\Phi(P)) = Z(G)$. That is, G is of type (3).

If Q is non-cyclic and $\Omega_1(Q) = Q$, then Q is an elementary abelian q-group. In this case, $G = \langle a, b, c\rangle$, $a^q = b^q = c^{p^m} = 1$, $ab = ba$, $a^c = a^i$, $b^c = b^j$, $i \not\equiv j \pmod q$, $i^p \equiv j^p \equiv 1 \pmod q$. That is, G is of type (4).

If Q is non-cyclic and $\Omega_1(Q) \neq Q$, then we can get $C_Q(P) = 1$ by Lemma 2.2. Let $Q = \langle a, b\rangle$, $a^{q^m} = 1$, $b^{q^n} = 1$ and $P = \langle c\rangle$, $a^c = a^u$, $b^c = b^v$. We claim that $m = n$. Indeed, let $m < n$, and $Q_1 = \langle a, b^q\rangle$. Then $PQ_1 = Q_1 \rtimes P$ is a $\mathscr{B}$-group. Hence there exists a natural number w such that $a^c = a^w$, $(b^q)^c = (b^p)^w$ by Lemma 2.5. It follows that $u \equiv w \pmod{q^m}$ and $qv \equiv qw \pmod{q^n}$. Thus $v \equiv w \pmod{q^{n-1}}$. Since $m < n$, we get $v \equiv w \pmod{q^m}$ and hence $u \equiv v \pmod{q^m}$. Therefore we have $x^c = x^v$ for every $x \in Q$, which implies that P induces a fixed-point-free power automorphism of order p on Q, a contradiction. Hence $m = n$ and then $G = \langle a, b, c\rangle$, $a^{q^m} = b^{q^m} = 1$, $c^{p^n} = 1$, $ab = ba$, $a^c = a^u$, $b^c = b^v$. $u \not\equiv v \pmod{q^m}$, $u \equiv v \pmod{q^{m-1}}$, $u^p \equiv v^p \equiv 1 \pmod{q^m}$, $u \not\equiv 1 \pmod q$, $v \not\equiv 1 \pmod q), m \geq 2$. That is, G is of type (5).

(3) Suppose that P is cyclic and the action of P on $Q/\Phi(Q)$ is irreducible. Then, we can easily get that either Q is cyclic or $Q = \Omega_1(Q)$. If Q is cyclic, then G is of type (3).

If $Q = \Omega_1(Q)$, then $G = Q \rtimes P$, Q is an elementary abelian q-group. P is cyclic, the action of P on Q is irreducible, $\Phi(P)$ induces a power automorphism of order p on Q, and $\Phi(\Phi(P)) = Z(G)$. That is, G is of type (6). $\square$

The following theorem classifies all minimal non-$\mathscr{B}$-groups whose order has just three prime divisors.

Theorem 3.4. *Let G be a minimal non-$\mathscr{B}$-group with $|\pi(G)| = 3$. Then one of the following holds:*

(1) *$G = (C_q \times C_r) \rtimes C_{p^m}$, and $Z(G) = C_q \times \Phi(C_{p^m})$.*
(2) *$G = (C_r \rtimes C_{q^m}) \rtimes C_p$, and $Z(G) = C_r \times \Phi(C_{q^m})$ or $Z(G) = 1$.*

Proof. Since G is solvable, we may assume that $G = PQR$, where $P \in \mathrm{Syl}_p(G), Q \in \mathrm{Syl}_q(G)$, $R \in \mathrm{Syl}_r(G)$. Without loss of generality, we always let p be the smallest prime divisor of $|G|$. If P is non-cyclic, then PQ and PR are all nilpotent by Lemma 2.5. Hence P is normal in G. Let P_1 be a maximal subgroup of P. Then P_1QR is a $\mathscr{B}$-group and hence we get that P_1QR is nilpotent by Lemma 2.5, which implies that G itself is nilpotent, a contradiction. Thus we have that P is cyclic. By Lemma 2.1, $QR \trianglelefteq G$. Since QR is a $\mathscr{B}$-group, we have by Lemma 2.5 that RQ is either a Dedekind group or a group of the type (2) in Lemma 2.5.

Case 1. RQ is a Dedekind group.

In this case both Q and R are normal in G. Since G is not a $\mathscr{B}$-group, we know that either R or Q can not normalize P. Assume that R can not normalize P for instance, and let $x \in R$ be an element of order r. If $\langle x\rangle PQ$ is a $\mathscr{B}$-group, then we have $x \in C_R(P)$, which implies that P acts trivially on $\Omega_1(R)$. Thus P acts trivially on R by Lemma 2.2, a contradiction. Therefore we get that R is of prime order. By the same argument we have that Q is of prime order too. Hence $G = (C_q \times C_r) \rtimes C_{p^m}$, and $Z(G) = C_q \times \Phi(C_{p^m})$. That is, G is of type (1).

Case 2. RQ is a group of the type (2) in Lemma 2.5.

Without loss of generality, we assume that $q < r$. Then $R \nleq N_G(Q)$. By the same reason as in Case 1, we can get that P and R are all of prime order. If P acts trivially on R, then we have that P acts trivially on $\Phi(Q)$ since $P\Phi(Q)R$ is a $\mathscr{B}$-group. Thus $G = (C_r \rtimes C_{q^m}) \rtimes C_p$, and $Z(G) = C_r \times \Phi(C_{q^m})$. That is, G is of type (2).

If P acts non-trivially on R, then either $\Phi(Q) = 1$ or P acts non-trivially on $\Phi(Q)$. Thus $G = (C_r \rtimes C_{q^m}) \rtimes C_p$, and $Z(G) = 1$. That is, G is of type (2). □

References

[1] R. W. Carter, Nilpotent self-normalizing subgroups of soluble groups, Math. Z. 75 (1961), 136–139. MR 0123603.

[2] S. Chen, Classification of finite groups whose maximal subgroups are Dedekind groups, J. Zhejiang Univ. Sci. Ed. 29 (2002), no. 2, 121–124. MR 1898787.

[3] K. Doerk, Minimal nicht überauflösbare endliche Gruppen, Math. Z. 91 (1966), 198–205. MR 0191962.

[4] K. Doerk and T. Hawkes, Finite Solvable Groups, Berlin-New York: Walter de Gruyter, 1992. MR 1169099.

[5] A. Fattahi, Groups with only normal and abnormal subgroups, J. Algebra 28 (1974), 15–19. MR 0335628.

[6] P. Guo, X. Zhang, On minimal non-MSP-groups, Ukrainian Math. J. 63 (2012), no. 9, 1458–1463. MR 3109666.

[7] Z. Han, G. Chen, and H. Shi, On minimal non-NSN-groups, J. Korean Math. Soc. 50 (2013), no. 3, 579–589. MR 3056923.

[8] H. Kurzweil and B. Stellmacher, The theory of finite groups: An introduction, New York: Springer-Verlag, 2004. MR 2014408.

[9] T. J. Laffey, A lemma on finite p-groups and some consequences, Proc. Cambridge Philos. Soc. 75 (1974), 133–137. MR 0332961.

[10] G. A. Miller and H. C. Moreno, Non-abelian groups in which every subgroup is abelian, Trans. Amer. Math. Soc. 4 (1903), 398–404. MR 1500650.

[11] D. J. S. Robinson, A course in the theory of groups, New York: Springer, 1993. MR 1261639.

[12] N. Sastry, On minimal non-PN-groups, J. Algebra 65 (1980), 104–109. MR 0578798.

[13] M. Suzuki, On a class of doubly transitive groups, Ann. of Math. (2) 75 (1962), 105–145. MR 0136646.

[14] J. G. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc. 74 (1968), 383–437. MR 0230809.

Huaguo Shi
Sichuan Vocational and Technical College, Sichuan 629000, P. R. China
`shihuaguo@126.com`

Zhangjia Han
School of Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan 610225, P. R. China
`hzjmm11@163.com`

Received: June 6, 2013
Accepted: October 2, 2013