REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 56, No. 2, 2015, Pages 95–117
Published online: September 17, 2015

SOME RESULTS ON SPECTRAL DISTANCES OF GRAPHS

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Abstract. Spectral distances of graphs related to some graph operations and modifications are considered. We show that the spectral distance between two graphs does not depend on their common eigenvalues, up to multiplicities. The spectral distance between some graphs with few distinct eigenvalues is computed and due to the results obtained one of the previously posed conjectures related to the A-spectral distances of graphs has been disproved. In order to investigate the cospectrality for special classes of graphs, we computed spectral distances and distance related graph parameters on several specially constructed sets of graphs.

1. Introduction

Some problems related to the spectral distances of graphs have been presented at the Aveiro Workshop on Graph Spectra held at the University of Aveiro, Portugal, 2006 (see [14]).

Let G_1 and G_2 be two nonisomorphic graphs on n vertices whose spectra with respect to the matrix M are $m_1(G_i) \geq m_2(G_i) \geq \cdots \geq m_n(G_i)$, $i = 1, 2$. The *M-spectral distance* between G_1 and G_2 is (see [14]):

$$\sigma_M(G_1, G_2) = \sum_{i=1}^{n} |m_i(G_1) - m_i(G_2)|\,.$$

Thus we can say that graphs G_1 and G_2 are ϵ^M-cospectral if $\sigma_M(G_1, G_2) \leq \epsilon$ holds for some non-negative number ϵ. In this paper the spectral distances are considered with respect to the adjacency matrix A, the Laplacian $L = D - A$ and the signless Laplacian matrix $Q = D + A$, where D is the diagonal matrix of vertex degrees. The eigenvalues of G_1 and G_2 with respect to the named matrices we denote as $\lambda_i(G_j)$, $\mu_i(G_j)$ and $\kappa_i(G_j)$, for $i = 1, 2, \ldots, n$ and $j = 1, 2$, respectively, and, as usual, we suppose that they have non-increasing order.

However the computation and evaluation of the *distance related graph parameters* on some specially constructed sets of graphs of equal orders is of particular

2010 *Mathematics Subject Classification.* 05C50.

Key words and phrases. spectral distance, distance related graph parameters, strongly regular graph, graph energy.

The work is supported by the Serbian Ministry of Education, Science and Technological Development, Projects 174033 and III45003.

interest. We will often call these parameters the *extremal spectral distances* because of their definition. Let X be an arbitrary set of n-vertex graphs. The *M-cospectrality* of $G_1 \in X$ is defined by (see [14]):

$$\mathrm{cs}_X^M(G_1) = \min\{\sigma_M(G_1, G_2) : G_2 \in X,\ G_2 \neq G_1\}.$$

Thus, $\mathrm{cs}_X^M(G_1) = 0$ if and only if G_1 has a M-cospectral mate. The *M-cospectrality measure* on the set X is

$$\mathrm{cs}^M(X) = \max\{\mathrm{cs}_X^M(G_1) : G_1 \in X\}.$$

In [10] two additional distance related graph parameters are introduced. The *M-spectral eccentricity* of G_1 is

$$\mathrm{secc}_X^M(G_1) = \max\{\sigma_M(G_1, G_2) : G_2 \in X\},$$

while the *M-spectral diameter* on the set X is given by:

$$\mathrm{sdiam}^M(X) = \max\{\mathrm{secc}_X^M(G_1) : G_1 \in X\}.$$

This function measures how large the spectral distance of graphs from the set X can be.

We will use the notation that is common in the spectral graph theory. Thus, C_n is the cycle of order n, while K_n is the complete graph on n vertices. The complete multipartite graph on k parts and n_i, $1 \leq i \leq k$, vertices in each of them we will denote as $K_{n_1,n_2,\ldots,n_k}$, and similarly, if $n_1 = n_2 = \cdots = n_k = m$ holds, we will use the label $K_{k\times m}$. The energy of a n-vertex graph G is defined as (see [8]): $E(G) = \sum_{i=1}^n |\lambda_i(G)|$. The Laplacian energy of G is (see [9]): $LE(G) = \sum_{i=1}^n |\mu_i(G) - \frac{2m}{n}|$, where m is the number of edges of G, while the signless Laplacian energy is given by: $QE(G) = \sum_{i=1}^n |\kappa_i(G) - \frac{2m}{n}|$. For the remaining notation and terminology we refer the reader to [3], [4] or [2].

The paper is organized as follows. In Section 2 we consider spectral distances related to certain graph operations and expose some particular results that yield to consideration of the spectral distances between graphs with common eigenvalues in Section 3. In Section 4 we compute spectral distances between graphs with few distinct eigenvalues, in particular between a strongly regular graph and its complement. Due to these results we disproved one of the conjectures related to the A-spectral distances of graphs (see [10]). Besides that, some results from [11] are extended. The conjectures about the spectral distances of graphs related to the various graph matrices are considered in Section 5. One of the research tasks exposed at the Aveiro Workshop [14] is related to the investigation of the cospectrality of some graph from the special class of graphs. So, in Section 6 we estimate distance related graph parameters on the specially constructed sets of graphs of equal orders.

2. Spectral distances related to some graph operations and modifications

Originally, the graph operation called NEPS (i.e. *non-complete extended p-sum*) did not include existence of a basis with all-zero n-tuple. Precisely, we state the definition of this graph operation as it is done in [4, page 44]:

Let B be a set of non-zero binary n-tuples, i.e. $B \subseteq \{0,1\}^n/\{(0,\dots,0)\}$. The NEPS of graphs $G_1,\dots,G_n$ with basis B is the graph with vertex set $V(G_1)\times\cdots\times V(G_n)$, in which two vertices, say $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$, are adjacent if and only if there exists an n-tuple $\beta=(\beta_1,\dots,\beta_n)\in B$ such that $x_i=y_i$ whenever $\beta_i=0$, and x_i is adjacent to y_i (in G_i) whenever $\beta_i=1$.

If we permit the existence of all-zero n-tuple, we will get a graph with loops.

Proposition 2.1. *Let $G=\mathrm{NEPS}(G_1,G_2,\dots,G_n;B_1)$ and $H=\mathrm{NEPS}(H_1,H_2,\dots,H_n;\,B_2)$, such that G_i and H_i, $i=1,2,\dots,n$ are arbitrary graphs. If $G'=\mathrm{NEPS}(G_1,G_2,\dots,G_n;B_1')$ and $H'=\mathrm{NEPS}(H_1,H_2,\dots,H_n;B_2')$, where $B_1'=B_1\cup\{(0,0,\dots,0)\}$ and $B_2'=B_2\cup\{(0,0,\dots,0)\}$, then $\sigma_A(G',H')=\sigma_A(G,H)$.*

Proof. According to Theorem 2.5.4 from [4] the spectrum of the graph G', and analogously H', consists of all possible values $\Lambda_{i_1,\dots,i_n}$ such that $\Lambda_{i_1,\dots,i_n}(G')=\sum_{\beta\in B_1'}\lambda_{1_{i_1}}^{\beta_1}\cdots\lambda_{n_{i_n}}^{\beta_n}$, for $i_h=1,\dots,k_h$ and $h=1,\dots,n$, where $\lambda_{i_1},\dots,\lambda_{i_{k_i}}$ are the eigenvalues of graphs G_i, $i=1,2,\dots,n$, respectively. Hence we have that

$$\Lambda_{i_1,\dots,i_n}(G')=\sum_{\beta\in B_1}\lambda_{1_{i_1}}^{\beta_1}\cdots\lambda_{n_{i_n}}^{\beta_n}+\lambda_{1_{i_1}}^{0}\cdots\lambda_{n_{i_n}}^{0}=\Lambda_{i_1,\dots,i_n}(G)+1,$$

where $\Lambda_{i_1,\dots,i_n}(G)$ are the eigenvalues of G. Therefore, the spectral distance between G' and H' is

$$\sigma_A(G',H')=\sum_{i=1}^{2^n}|\lambda_i(G)+1-\lambda_i(H)-1|=\sigma_A(G,H). \qquad\square$$

There is a slightly more general result:

Proposition 2.2. *Let G_1 and G_2 be two arbitrary graphs. Let G_1' and G_2' denote graphs that are obtained from G_1 and G_2 by adding a loop at each vertex. Then $\sigma_A(G_1',G_2')=\sigma_A(G_1,G_2)$ holds.*

Proof. The adjacency matrix A' of the graph G' which is obtained by adding a loop at each vertex of the graph G is equal to $A'=A+E$, where A is the adjacency matrix of G, while E is the identity matrix. So, by applying the Courant-Weyl inequalities (see Theorem 1.3.15 from [4]) we obtain the following: $\lambda_i(G')\le\lambda_j(G)+1$, for $1\le j\le i\le n$ and $\lambda_i(G')\ge\lambda_j(G)+1$, for $1\le i\le j\le n$. This means that $\lambda_i(G')=\lambda_i(G)+1$, for $i=1,2,\dots,n$, wherefrom the proof follows. $\square$

Proposition 2.3. *Let $G=\mathrm{NEPS}(K_2,K_2,\dots,K_2;B)$. Let $\Lambda^{(k)}(G)=\Lambda^{(k)}_{i_1,\dots,i_n}(G)$ $=\sum_{\beta\in B}\lambda_{1_{i_1}}^{\beta_1}\cdots\lambda_{n_{i_n}}^{\beta_n}$, $i_h=1,\dots,k_h$, $h=1,\dots,n$ be the distinct eigenvalues of the graph G, such that k $(0\le k\le n)$ of the eigenvalues $\lambda_{i_1},\dots,\lambda_{i_{k_i}}$ of graphs K_2 are* *equal to* -1. *Let us suppose that the basis* B *is such that* $\Lambda^{(k)}(G) \geq \Lambda^{(k+1)}(G)+2$, *for each* $0 \leq k \leq n-1$. *Let* $G' = \mathrm{NEPS}(K_2, K_2, \ldots, K_2; B')$, *where* $B' = B \cup \{(1,1,\ldots,1)\}$. *Then* $\sigma_A(G,G') = 2^n$.

Proof. The spectrum of G' consists of all eigenvalues $\Lambda(G')$ such that (see Theorem 2.5.4 from [4]):

$$\Lambda(G') = \sum_{\beta \in B'} \lambda_{1_{i_1}}^{\beta_1} \cdots \lambda_{n_{i_n}}^{\beta_n} = \sum_{\beta \in B} \lambda_{1_{i_1}}^{\beta_1} \cdots \lambda_{n_{i_n}}^{\beta_n} + \lambda_{1_{i_1}} \cdots \lambda_{n_{i_n}} = \Lambda(G) + \lambda_{1_{i_1}} \cdots \lambda_{n_{i_n}},$$

for $i_h = 1, \ldots, k_h$, $h = 1, \ldots, n$, where $\Lambda(G)$ are the eigenvalues of G. If there are k of the eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_{k_i}}$ of graphs K_2 that are equal to -1, we have:

$$\Lambda^{(k)}(G') = \Lambda^{(k)}(G) + (-1)^k,$$

where $0 \leq k \leq n$. The multiplicity of the eigenvalue $\Lambda^{(k)}(G)$, and similarly $\Lambda^{(k)}(G')$, corresponds to the number of ways in which one can choose k of the n eigenvalues that are equal to -1, i.e. $\binom{n}{k}$. According to the assumptions from the statement, we get:

$$\Lambda^{(k)}(G') = \Lambda^{(k)}(G) + (-1)^k \geq \Lambda^{(k+1)}(G) + (-1)^{k+1} = \Lambda^{(k+1)}(G'),$$

for all $0 \leq k \leq n-1$. So, the spectral distance is

$$\sigma_A(G,G') = \sum_{k=0}^{n} \binom{n}{k} \left| \Lambda^{(k)}(G) + (-1)^k - \Lambda^{(k)}(G) \right| = 2^n. \qquad \square$$

Example 2.1. Let $G = \mathrm{NEPS}(K_2, K_2, \ldots, K_2; B)$, where the basis B consists of binary n-tuples such that each of them has exactly one component equal to 1, i.e. $B = \{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,0,\ldots,1)\}$. The graph G is n-cube whose spectrum is $[n-2i]^{\binom{n}{i}}$, $0 \leq i \leq n$. Let us denote $G' = \mathrm{NEPS}(K_2, K_2, \ldots, K_2; B')$, where $B' = B \cup \{(1,1,\ldots,1)\}$. Since the inequality $\Lambda^{(i)}(G) = n-2i \geq n-2\,(i+1)+2 = \Lambda^{(i+1)}(G)+2$ holds for every selection of i of the eigenvalues of graphs K_2 that are equal to -1, $0 \leq i \leq n-1$, we have:

$$\Lambda^{(i)}(G') = \Lambda^{(i)}(G) + (-1)^i = n-2i+(-1)^i \geq n-2(i+1)+(-1)^{i+1} = \Lambda^{(i+1)}(G').$$

So the spectral distance is

$$\sigma_A(G,G') = \sum_{i=0}^{n} \binom{n}{i} |n-2i+(-1)^i-(n-2i)| = 2^n.$$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $|V_1| = n_1$ and $|V_2| = n_2$. Two graph operations, the S_{vertex} *join* and the S_{edge} *join*, are introduced in [7]. The S_{vertex} *join* of graphs G_1 and G_2, denoted by $G_1 \dot{\vee} G_2$, is the graph that is obtained from $S(G_1)$ and G_2 by joining each vertex from the set $V(G_1) = V_1$ with each vertex from the set V_2. Here, $S(G)$ means the *subdivision graph* of the graph G (for details see [3] or [4]). The S_{edge} *join* of G_1 and G_2, denoted by $G_1 \underline{\vee} G_2$, is the graph that is obtained from $S(G_1)$ and G_2 by joining each vertex from the set $E(G_1)$, i.e. $V(S(G_1))/V(G_1)$ with each vertex from the set V_2. In [7] A-spectra of these graphs when G_i are r_i-regular graphs, $i = 1, 2$, are given.

Proposition 2.4. *Let G_i, $i = 1, 2$ be two connected r_i-regular graphs of orders n_i, respectively. Let the adjacency spectra of these graphs be: $r_i = \lambda_1(G_i) \geq \lambda_2(G_i) \geq \ldots \geq \lambda_{n_i}(G_i)$, $i = 1, 2$, such that $\lambda_{n_2}(G_2) \geq -\sqrt{\lambda_1(G_1) + r_1}$. Then*

$$\sigma_A(G_1\dot{\vee}G_2, G_1\underline{\vee}G_2) = \sum_{i\in\{1,n_1+m_1+n_2\}} |\lambda_i(G_1\dot{\vee}G_2) - \lambda_i(G_1\underline{\vee}G_2)| + |\lambda(G_1\dot{\vee}G_2) - \lambda'(G_1\underline{\vee}G_2)|,$$

where m_1 is the number of edges of G_1, while $\lambda(G_1\dot{\vee}G_2)$ and $\lambda'(G_1\underline{\vee}G_2)$ are the roots of the polynomials $h_1(x) = x^3 - r_2x^2 - (n_1n_2 + 2r_1)x + 2r_1r_2$ and $h_2(x) = x^3 - r_2x^2 - (m_1n_2 + 2r_1)x + 2r_1r_2$, respectively, that are different from $\lambda_1(G_1\dot{\vee}G_2)$ and $\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2)$, apropos $\lambda_1(G_1\underline{\vee}G_2)$ and $\lambda_{n_1+m_1+n_2}(G_1\underline{\vee}G_2)$.

Proof. According to Theorems 1.1 and 1.2 from [7], the spectra of $G_1\dot{\vee}G_2$ and $G_1\underline{\vee}G_2$ differ in exactly three eigenvalues. The remaining $n_1 + m_1 + n_2 - 3$ eigenvalues are $\pm\sqrt{\lambda_i(G_1) + r_1}$, for $i = 2, 3, \ldots, n_1$, $\lambda_i(G_2)$, for $i = 2, 3, \ldots, n_2$, and $[0]^{m_1-n_1}$. Let us consider these six different eigenvalues of $G_1\dot{\vee}G_2$ and $G_1\underline{\vee}G_2$ (three from each of the spectra).

Since $S(G_1)$ is an induced subgraph of the graph $G_1\dot{\vee}G_2$ ($G_1\underline{\vee}G_2$) from the *Interlacing Theorem* (see Theorem 0.10 from [3]) we get: $\lambda_1(G_1\dot{\vee}G_2) \geq \lambda_1(S(G_1)) = \sqrt{\lambda_1(G_1) + r_1} \geq \sqrt{\lambda_2(G_1) + r_1}$.

On the other hand, since G_2 is an induced subgraph of $G_1\dot{\vee}G_2$ ($G_1\underline{\vee}G_2$), again by use of the *Interlacing Theorem* we obtain: $\lambda_1(G_1\dot{\vee}G_2) \geq \lambda_1(G_2) \geq \lambda_2(G_2)$.

From these inequalities we can conclude that one of the roots of the polynomial $h_1(x)$ ($h_2(x)$) is just the index of the graph $G_1\dot{\vee}G_2$ ($G_1\underline{\vee}G_2$). Analogously, we have:

$$\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2) \leq \lambda_{n_1+m_1}(S(G_1)) = -\sqrt{\lambda_1(G_1) + r_1},$$

$$\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2) \leq \lambda_{n_2}(G_2),$$

and also

$$\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2) \leq \min\left\{\lambda_{n_2}(G_2), -\sqrt{\lambda_1(G_1) + r_1}\right\} = -\sqrt{\lambda_1(G_1) + r_1} \leq -\sqrt{\lambda_2(G_1) + r_1},$$

wherefrom we found that one of the two remaining roots of the polynomial $h_1(x)$ ($h_2(x)$) is the eigenvalue $\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2)$ ($\lambda_{n_1+m_1+n_2}(G_1\underline{\vee}G_2)$). The third non-null root of the polynomial $h_1(x)$ ($h_2(x)$) we can denote by λ (λ'). Without loss of generality, let us suppose that the position of the eigenvalue λ in the spectrum of $G_1\dot{\vee}G_2$ is x, while the position of the eigenvalue λ' in the spectrum of $G_1\underline{\vee}G_2$

is y, and that $x \leq y$ holds. Then the A-spectral distance becomes:

$$\begin{aligned}
&\sigma_A(G_1\dot{\vee}G_2, G_1\underline{\vee}G_2) = |\lambda_1(G_1\dot{\vee}G_2) - \lambda_1(G_1\underline{\vee}G_2)| \\
&\quad + \sum_{i=2}^{x-1} |\lambda_i(G_1\dot{\vee}G_2) - \lambda_i(G_1\underline{\vee}G_2)| \\
&\quad + |\lambda_x(G_1\dot{\vee}G_2) - \lambda_x(G_1\underline{\vee}G_2)| + \sum_{i=x+1}^{y-1} |\lambda_i(G_1\dot{\vee}G_2) - \lambda_i(G_1\underline{\vee}G_2)| \\
&\quad + |\lambda_y(G_1\dot{\vee}G_2) - \lambda_y(G_1\underline{\vee}G_2)| + \sum_{i=y+1}^{n_1+m_1+n_2-1} |\lambda_i(G_1\dot{\vee}G_2) - \lambda_i(G_1\underline{\vee}G_2)| \\
&\quad + |\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2) - \lambda_{n_1+m_1+n_2}(G_1\underline{\vee}G_2)| = |\lambda_1(G_1\dot{\vee}G_2) - \lambda_1(G_1\underline{\vee}G_2)| \\
&\quad + (\lambda - \lambda_x(G_1\underline{\vee}G_2)) + \lambda_{x+1}(G_1\dot{\vee}G_2) - \lambda_y(G_1\dot{\vee}G_2) + \lambda_y(G_1\dot{\vee}G_2) - \lambda' \\
&\quad + |\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2) - \lambda_{n_1+m_1+n_2}(G_1\underline{\vee}G_2)| = |\lambda_1(G_1\dot{\vee}G_2) - \lambda_1(G_1\underline{\vee}G_2)| \\
&\quad + (\lambda - \lambda') + |\lambda_{n_1+m_1+n_2}(G_1\dot{\vee}G_2) - \lambda_{n_1+m_1+n_2}(G_1\underline{\vee}G_2)|.
\end{aligned}$$

□

3. Spectral distances of graphs with common eigenvalues

By Proposition 2.4 from the previous section we demonstrated that if the spectra of two graphs of equal orders differ in exactly three eigenvalues, then the A-spectral distance between them will depend on only these six different eigenvalues (three from each of the spectra). In this section we will give a slightly more general result.

Let G_1 and G_2 be two graphs of order n whose spectra with respect to the matrix M are $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$, respectively. Let us denote by $z_1 > z_2 > \ldots > z_{n_1}$ and $t_1 > t_2 > \ldots > t_{n_2}$ the distinct eigenvalues of these graphs, respectively.

Definition 3.1. Given two graphs G_1 and G_2, let λ be the common eigenvalue of these graphs such that $\lambda = z_{l_1}$, for some $l_1 \in \{1, 2, \ldots, n_1\}$ and $\lambda = t_{l_2}$, for some $l_2 \in \{1, 2, \ldots, n_2\}$. Let $i_1, i_2 \in \{1, 2, \ldots, n\}$ be the least indices such that $x_{i_1} = \lambda$ and $y_{i_2} = \lambda$ holds, and similarly, let $j_1, j_2 \in \{1, 2, \ldots, n\}$ be the largest indices such that $x_{j_1} = \lambda$ and $y_{j_2} = \lambda$ holds. The M-partial spectral distance $P_{M,\lambda}(G_1, G_2)$ of graphs G_1 and G_2 that corresponds to the common eigenvalue λ is

$$P_{M,\lambda}(G_1, G_2) = \sum_{k=\min\{i_1,i_2\}}^{\max\{j_1,j_2\}} |x_k - y_k|. \tag{1}$$

It is obvious that $i_1 \leq j_1$ and $i_2 \leq j_2$. Besides that, if I_λ stands for the set of indices of the eigenvalues of graphs G_1 and G_2 that belong to the interval $[\min\{i_1, i_2\}, \max\{j_1, j_2\}]$ for each common eigenvalue λ, we have

$$\sigma_M(G_1, G_2) = \sum_\lambda P_{M,\lambda}(G_1, G_2) + \sum_{k\in\{1,2,\ldots,n\}/\bigcup_\lambda I_\lambda} |x_k - y_k| = \sum\nolimits_1 + \sum\nolimits_2, \tag{2}$$

where the first summation goes over all common eigenvalues λ of G_1 and G_2. If there are no common eigenvalues of G_1 and G_2, we simply have $\sigma_M(G_1, G_2) = \sum_2$.

Definition 3.2. Given two graphs G_1 and G_2 the partial realization $RP_{M,\lambda}(G_1, G_2)$ of the M-partial spectral distance $P_{M,\lambda}(G_1, G_2)$ that corresponds to the common eigenvalue λ of these graphs is the arithmetic expression obtained from the sum (1) in which for each $k = \min\{i_1, i_2\}, \ldots, \max\{j_1, j_2\}$ the term $|x_k - y_k|$ is replaced by $x_k + (-y_k)$ if $x_k \geq y_k$, and by $-x_k + y_k$ if $x_k < y_k$.

Definition 3.3. The reduced partial realization $RRP_{M,\lambda}(G_1, G_2)$ of the M-partial spectral distance $P_{M,\lambda}(G_1, G_2)$ of given graphs G_1 and G_2 is the arithmetic expression obtained from $RP_{M,\lambda}(G_1, G_2)$ by deleting all pairs of opposite terms of the form $a, -a$.

Of course, we have

$$P_{M,\lambda}(G_1, G_2) = RP_{M,\lambda}(G_1, G_2) = RRP_{M,\lambda}(G_1, G_2).$$

Proposition 3.1. *Let G_1 and G_2 be two graphs of order n with given spectra with respect to the matrix M. If λ is the common eigenvalue of these graphs with multiplicities m_1 and m_2, respectively, then the reduced partial realization $RRP_{M,\lambda}(G_1, G_2)$ of the M-partial spectral distance $P_{M,\lambda}(G_1, G_2)$ contains at most $|m_1 - m_2|$ terms that equal λ or $-\lambda$. In the special case, if the spectra of both graphs G_1 and G_2 contain the eigenvalue λ with the same multiplicity, then the reduced partial realization $RRP_{M,\lambda}(G_1, G_2)$ does not contain the term λ nor $-\lambda$.*

Proof. Without loss of generality, we will analyse only the case when $m_1 > m_2 > 1$ holds. All remaining cases can be considered in a similar way. With this assumption, we can distinguish the following relations between the positions i_1, i_2, j_1, and j_2 of the common eigenvalue λ in the spectra of G_1 and G_2 (we use the previously introduced notation).

If $i_1 = i_2 < j_2 < j_1$, then the M-partial spectral distance is $P_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{j_1} |x_k - y_k|$. The partial realization is $RP_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{j_2} (\lambda - \lambda) + \sum_{k=j_2+1}^{j_1} (\lambda - y_k)$, so the reduced partial realization is $RRP_{M,\lambda}(G_1, G_2) = (m_1 - m_2)\lambda - \sum_{k=j_2+1}^{j_1} y_k$.

Let $i_1 < i_2 < j_2 = j_1$ holds. Then we have $P_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{j_1} |x_k - y_k|$, so the partial realization is $RP_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{i_2-1} (y_k - \lambda) + \sum_{k=i_2}^{j_2} (\lambda - \lambda)$. Finally, we get $RRP_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{i_2-1} y_k + (m_1 - m_2)(-\lambda)$.

If $i_1 < i_2 < j_1 < j_2$, then $P_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{j_2} |x_k - y_k|$, so $RP_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{i_2-1} (y_k - \lambda) + \sum_{k=i_2}^{j_1} (\lambda - \lambda) + \sum_{k=j_1+1}^{j_2} (\lambda - x_k)$. So, we get $RRP_{M,\lambda}(G_1, G_2) = \sum_{k=i_1}^{i_2-1} y_k - \sum_{k=j_1+1}^{j_2} x_k + (m_1 - m_2)(-\lambda)$.

The case when $i_1 < i_2 = j_1 < j_2$ can be analysed like the previous one.

Let $i_2 < i_1 < j_2 < j_1$ holds. Then $P_{M,\lambda}(G_1, G_2) = \sum_{k=i_2}^{j_1} |x_k - y_k|$, so the partial realization becomes: $RP_{M,\lambda}(G_1, G_2) = \sum_{k=i_2}^{i_1-1} (x_k - \lambda) + \sum_{k=i_1}^{j_2} (\lambda - \lambda) + \sum_{k=j_2+1}^{j_1} (\lambda - y_k)$. The reduced partial realization is $RRP_{M,\lambda}(G_1, G_2) = \sum_{k=i_2}^{i_1-1} x_k - \sum_{k=j_2+1}^{j_1} y_k + (m_1 - m_2)\lambda$.

The case when $i_2 < i_1 = j_2 < j_1$ can be analysed like the previous one.

If $i_1 < j_1 < i_2 < j_2$, then $P_{M,\lambda}(G_1,G_2) = \sum_{k=i_1}^{j_2} |x_k - y_k|$, so $RP_{M,\lambda}(G_1,G_2) = \sum_{k=i_1}^{j_1}(y_k-\lambda)+\sum_{k=j_1+1}^{i_2-1}(y_k-x_k)+\sum_{k=i_2}^{j_2}(\lambda-x_k)$. The reduced partial realization is $RRP_{M,\lambda}(G_1,G_2) = \sum_{k=i_1}^{i_2-1} y_k - \sum_{k=j_1+1}^{j_2} x_k + (m_1-m_2)(-\lambda)$.

Let $i_2 < j_2 < i_1 < j_1$. The partial distance is $P_{M,\lambda}(G_1,G_2) = \sum_{k=i_2}^{j_1} |x_k - y_k|$, so we have $RP_{M,\lambda}(G_1,G_2) = \sum_{k=i_2}^{j_2}(x_k-\lambda) + \sum_{k=j_2+1}^{i_1-1}(x_k-y_k) + \sum_{k=i_1}^{j_1}(\lambda-y_k)$. The reduced partial realization is $RRP_{M,\lambda}(G_1,G_2) = \sum_{k=i_2}^{i_1-1} x_k - \sum_{k=j_2+1}^{j_1} y_k + (m_1-m_2)\lambda$.

If $i_1 < i_2 < j_2 < j_1$, then $P_{M,\lambda}(G_1,G_2) = \sum_{k=i_1}^{j_1} |x_k - y_k|$, so $RP_{M,\lambda}(G_1,G_2) = \sum_{k=i_1}^{i_2-1}(y_k-\lambda)+\sum_{k=i_2}^{j_2}(\lambda-\lambda)+\sum_{k=j_2+1}^{j_1}(\lambda-y_k)$. Now, we get $RRP_{M,\lambda}(G_1,G_2) = \sum_{k=i_1}^{i_2-1} y_k - \sum_{k=j_2+1}^{j_1} y_k - (j_2 - j_1 + i_2 - i_1)\lambda$. ☐

From Proposition 3.1 we can see that if the multiplicities of the common eigenvalue λ in the spectra of graphs whose partial spectral distance we compute are the same, then the reduced partial realization does not contain the term λ nor $-\lambda$. In the case when these multiplicities are different, according to the previously introduced notation, the following corollary holds:

Corollary 3.1. *If the M-spectra of both graphs G_1 and G_2 contain the eigenvalue λ with multiplicities $m_1 \neq m_2$, respectively such that*

$$i_2 - i_1 = j_1 - j_2, \quad \text{for } i_1 < i_2 \leq j_2 < j_1 \text{ and } m_1 > m_2, \tag{3}$$

or

$$i_1 - i_2 = j_2 - j_1, \quad \text{for } i_2 < i_1 \leq j_1 < j_2 \text{ and } m_1 < m_2, \tag{4}$$

holds, then the reduced partial realization $RRP_{M,\lambda}(G_1,G_2)$ of the M-partial spectral distance $P_{M,\lambda}(G_1,G_2)$ does not contain the term λ nor $-\lambda$.

Proof. It follows directly from the last case that is considered in the proof of Proposition 3.1. ☐

As the corresponding eigenvalues that make the addends in the sum $\sum_2$ in formula (2) are mutually different, from formula (2) and Proposition 3.1 we get the following theorem:

Theorem 3.1. *The M-spectral distance between two graphs of equal orders does not depend on their common eigenvalues up to multiplicities.*

Theorem 3.2. *Let the M-spectra of both n-vertex graphs G_1 and G_2 contain eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_m$, with multiplicities $m_{11}, m_{12}, \ldots, m_{1m}$ and $m_{21}, m_{22}, \ldots, m_{2m}$, respectively, such that $m_{1l} \neq m_{2l}$, for each $l \in \{1,2,\ldots,m\}$. If for each pair m_{1l}, m_{2l}, according to Definition 3.1, relation (3) or (4) holds, then the M-spectral distance between graphs G_1 and G_2 does not depend on their common eigenvalues at all.*

4. Spectral distances between graphs with few distinct eigenvalues

Recall from [3] that if $r = \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ is the A-spectrum of a r-regular graph G, then the A-spectrum of its complement $\overline{G}$ is $n-1-r \geq -1-\lambda_n(G) \geq \cdots \geq -1-\lambda_2(G)$, so the A-spectral distance between these two graphs is

$$\sigma_A(G,\overline{G}) = |n-2r-1| + \sum_{i=2}^{n} |\lambda_i(G) + \lambda_{n-i+2}(G) + 1|.$$

From the last equality we can notice that it is not easy to exactly compute the A-spectral distance between a regular graph and its complement in the general case. Some results on spectral distances between a regular graph and its complement are exposed in [11]. In this section we will extend the existing results by computing and comparing spectral distances between graphs with few distinct eigenvalues, precisely between a strongly regular graph and its complement.

A *strongly regular graph* with parameters (n,r,e,f) (see, for example, [4]) is a r-regular graph on n vertices in which any two adjacent vertices have exactly e common neighbours and any two non-adjacent vertices have exactly f common neighbours. It can be checked that if G is strongly regular graph with parameters (n,r,e,f), then its complement $\overline{G}$ is strongly regular with parameters $(\overline{n},\overline{r},\overline{e},\overline{f})$, where $\overline{n}=n$, $\overline{r}=n-r-1$, $\overline{e}=n-2-2r+f$, and $\overline{f}=n-2r+e$. We say that a strongly regular graph G is *primitive* (see [4]) if both G and $\overline{G}$ are connected.

Proposition 4.1. *Let G be a primitive strongly regular graph with parameters (n,r,e,f). Then*

$$\sigma_A(G,\overline{G}) = |n-2r-1| + (n-1)\,|e+1-f| \\ + \left(1 - \frac{2}{\sqrt{\Delta}}\,|e-f+1|\right) \times |2r+(n-1)(e-f)|,$$

where $\Delta = (e-f)^2 + 4(r-f)$.

Proof. According to Theorem 3.6.5 from [4], the spectrum of G consists of three distinct eigenvalues: $[r], [s]^k$ and $[t]^l$, where $s,t = \frac{1}{2}((e-f) \pm \sqrt{\Delta})$ and $k,l = \frac{1}{2}\left(n-1 \mp \frac{2r+(n-1)(e-f)}{\sqrt{\Delta}}\right)$. From Theorem 2.1.2 (see [4]), one can conclude that the spectrum of $\overline{G}$ is $[n-r-1], [-t-1]^l$ and $[-s-1]^k$. So, we have

$$\begin{aligned}\sigma_A(G,\overline{G}) &= \sum_{i=1}^{n} \left|\lambda_i(G) - \lambda_i(\overline{G})\right| \\ &= |n-2r-1| + \sum_{i=2}^{\min\{k,l\}+1} |s+t+1| + \sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} |\lambda_i(G) - \lambda_i(\overline{G})| \\ &\quad + \sum_{i=\max\{k,l\}+2}^{n} |s+t+1|\end{aligned}$$

$$= |n-2r-1| + (\min\{k,l\} - \max\{k,l\} + n - 1)\,|s+t+1| + \sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} |\lambda_i(G) - \lambda_i(\overline{G})|.$$

The last addend in the previous equality depends on the mutual relationship between the multiplicities k and l; it is

$$\sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} |2s+1| = \sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} \left|\sqrt{\Delta} - (f-e-1)\right|,$$

if $k > l$, and

$$\sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} |2t+1| = \sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} \left|\sqrt{\Delta} + (f-e-1)\right|,$$

if $k < l$. If $k > l$, then we have $r < \frac{n-1}{2}(f-e)$, i.e. $f > e$, but if $k < l$ holds, then $f \leq e+1$ (because if $f > e+1$ holds, let us suppose, for example, that $f = e+2$; then we have $2r + (n-1)(e-f) = 2r - 2(n-1) \leq 0$, since $r \leq n-1$, which contradicts the assumption). So, we have

$$\sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} |\lambda_i(G) - \lambda_i(\overline{G})| = (\max\{k,l\} - \min\{k,l\}) \left|\sqrt{\Delta} - |f-e-1|\right|.$$

Since $|f-e-1|^2 = (f-e-1)^2 = (f-e)^2 + 2(e-f) + 1 \leq (f-e)^2 + 2(r-f) + 1 \leq \Delta$, it follows that

$$\sum_{i=\min\{k,l\}+2}^{\max\{k,l\}+1} |\lambda_i(G) - \lambda_i(\overline{G})| = \frac{1}{\sqrt{\Delta}} |2r + (n-1)(e-f)| \left(\sqrt{\Delta} - |f-e-1|\right),$$

i.e.

$$\sigma_A(G,\overline{G}) = |n-2r-1| + (n-1)|e-f+1| + \left(1 - \frac{2}{\sqrt{\Delta}}|e-f+1|\right) |2r+(n-1)(e-f)|.$$

In the case when the multiplicities k and l are equal, i.e. $2r + (n-1)(e-f) = 0$, the graphs G and $\overline{G}$ have the same parameters (see, for example, [13]): $(n,r,e,f) = (4t+1, 2t, t-1, t)$, for some $t \geq 1$. Then, their spectra consist of: $\left[\frac{n-1}{2}\right]$, $\left[\frac{-1+\sqrt{n}}{2}\right]^{\frac{n-1}{2}}$ and $\left[\frac{-1-\sqrt{n}}{2}\right]^{\frac{n-1}{2}}$, so the spectral distance is $\sigma_A(G,\overline{G}) = |n-2r-1| + (n-1)(f-e-1) = 0$. □

We proved (see Proposition 2.3 from [11]) that for a r-regular graph G such that its degree is not greater than the degree of its complement $\overline{G}$ the following inequality between A- and L- spectral distances holds: $\sigma_L(G,\overline{G}) \leq \sigma_A(G,\overline{G}) + (n-2)(n-2r-1)$. We also noticed that equality in this relation is attained for, for example, the Petersen graph. By the following proposition we will prove that the mentioned equality is attained for all the other strongly regular graphs with parameters (n,r,e,f), such that $r \leq \frac{n-1}{2}$ and $f = e+1$.

Proposition 4.2. *Let G be a primitive strongly regular graph with parameters (n, r, e, f) such that $r \leq \frac{n-1}{2}$ and $f = e + 1$. Then*

$$\sigma_L(G, \overline{G}) = \sigma_A(G, \overline{G}) + (n-2)(n-2r-1).$$

Proof. The L-spectrum of the graph G consists of $[r-t]^l$, $[r-s]^k$ and $[0]$, while the L-spectrum of its complement is $[n-r+s]^k$, $[n-r+t]^l$ and $[0]$, where $s, t = \frac{1}{2}\left(-1 \pm \sqrt{\Delta}\right)$ and $\Delta = 1 + 4(r-e-1)$.

Since $2r + (n-1)(e-f) = 2r - (n-1) \leq 2\frac{n-1}{2} - (n-1) = 0$, we have $k \geq l$. Let us suppose that $k > l$. Then the L-spectral distance is

$$\begin{aligned}
\sigma_L(G, \overline{G}) &= \sum_{i=1}^{l} |r-t-n+r-s| + \sum_{i=l+1}^{k} |n-r+s-r+s| \\
&\quad + \sum_{i=k+1}^{n-1} |r-s-n+r-t| \\
&= 2l\,|n-2r-1| + (k-l)\left|n-2r-1+\sqrt{\Delta}\right| \\
&= 2l(n-2r-1) + (k-l)(n-2r-1+\sqrt{\Delta}) \\
&= (n-2r-1)(l+k) + \sqrt{\Delta}\left(-\frac{2r+(n-1)(-1)}{\sqrt{\Delta}}\right) \\
&= n(n-2r-1).
\end{aligned}$$

In the case when $k = l$, G and $\overline{G}$ have the same parameters, so $\sigma_L(G, \overline{G}) = 0$.

From the previous proposition we have that the A-spectral distance between $G(n, r, e, e+1)$ and its complement $\overline{G}$ is $\sigma_A(G, \overline{G}) = 2(n-2r-1)$, when $k > l$, and similarly, $\sigma_A(G, \overline{G}) = 0$, for $k = l$. Therefore we have

$$\sigma_A(G, \overline{G}) + (n-2)(n-2r-1) = n(n-2r-1) = \sigma_L(G, \overline{G}). \qquad \square$$

By using the previously obtained results, one can easily prove the following statement:

Proposition 4.3. *Let G be a primitive strongly regular graph with parameters (n, r, e, f). Then we have*

(1) *If $f = e$ and $r = \frac{n-1}{2}$, then $\sigma_L(G, \overline{G}) = \sigma_A(G, \overline{G}) + (n-2)(n-2r-1)$.*
(2) *If $f = e + a$, $a > 1$ and $r \leq \frac{n-a}{2}$, then there is no strongly regular graph with given parameters such that $\sigma_L(G, \overline{G}) = \sigma_A(G, \overline{G}) + (n-2)(n-2r-1)$.*
(3) *Suppose that $e = f + b$, $b \geq 1$ and $r \leq \frac{n-b}{2}$. If $n - 2r + 1 < \sqrt{\Delta}$, then $\sigma_L(G, \overline{G}) = \sigma_A(G, \overline{G}) + (n-2)(n-2r-1)$ for $r = \frac{n-1}{2}$. There is no other strongly regular graph with given parameters such that the A- and L-spectral distances between it and its complement satisfy the given equality.*

5. About conjectures related to spectral distances of graphs

Some conjectures related to the A-spectral distances of graphs have been posed in [10]. Among them of particular interest are the following four:

Conjecture 5.1. *The A-spectral distance between any two graphs of order n does not exceed*

$$E_n^{\max} = \max\{E(G) : G \text{ is a graph of order } n\}.$$

Conjecture 5.2. *If R_1 and R_2 are graphs having the maximal A-spectral distance among connected regular graphs of order n, then one of them is K_n.*

Conjecture 5.3. *If B_1 and B_2 are graphs having the maximal A-spectral distance among connected bipartite graphs of order n, then one of them is $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$.*

Conjecture 5.4. *The A-spectral distance between any two trees of order n does not exceed $\sigma_A(P_n, K_{1,n-1})$.*

Proposition 4.1 inspired us to try to find a pair of graphs whose A-spectral distance is greater than the maximal energy $E_n^{\max}$ on the set of all graphs of order n. As for a graph G on n vertices the following inequality holds [12]: $E(G) \le \frac{n}{2}(1+\sqrt{n})$, with equality if and only if G is a strongly regular graph with parameters $(n, (n+\sqrt{n})/2, (n+\sqrt{n})/4, (n+\sqrt{n})/4)$, we concluded that for a strongly regular graph G with parameters (n, r, e, f) such that $f \approx \sqrt{n}$ and e is a small positive number, $\sigma_A(G, \overline{G}) = O(n^{3/2})$ holds.

Precisely, let us consider the following graphs. The generalized quadrangle $G_1 = GQ(2, 4)$ (more about this kind of graphs the reader can find in, for example, [6, p. 81]) is the strongly regular graph with parameters $(27, 10, 1, 5)$. The A-spectrum of this graph consists of $10, [1]^{20}, [-5]^6$, so we can easily compute $\sigma_A(G_1, \overline{G_1}) = 84$. As $E_n^{\max} \le \frac{n}{2}(1+\sqrt{n}) \approx 83.648$, by this example we have disproved Conjecture 5.1.

The situation is similar with the generalized quadrangle $G_2 = GQ(3, 9)$, which is also the strongly regular graph with parameters $(112, 30, 2, 10)$. As the A-spectrum of this graph consists of $30, [2]^{90}, [-10]^{21}$, we found $\sigma_A(G_2, \overline{G_2}) = 690 > 648, 648 = \frac{n}{2}(1+\sqrt{n}) \ge E_n^{\max}$, i.e. we found another example that disproves Conjecture 5.1. Besides that, the energies of G_2 and $\overline{G_2}$ are also smaller then the A-spectral distance between them, i.e. $E(G_2) = 420$ and $E(\overline{G_2}) = 540$.

Now, we will consider Conjecture 5.2 on some subset of all connected regular graphs of order n, but with respect to the L-spectrum.

Proposition 5.1. *Let G_R be the set of connected regular graphs R of order n with the adjacency spectrum $\lambda_1(R) \ge \lambda_2(R) \ge \cdots \ge \lambda_n(R) \ge -2$. Then*

$$\sigma_L(R_1, R_2) \le \sigma_L(C_n, K_n),$$

where R_1 and R_2 are r_i-regular graphs, $i = 1, 2$ from the set G_R such that $r_i \ge 2$, $i = 1, 2$ and $r_1 + r_2 \le n - 3$.

Proof. According to Proposition 3.1 from [11], we find that

$$\sigma_L(C_n, K_n) = 2\left(\frac{n(n-1)}{2} - n\right) = n(n-3),$$

and also

$$\sigma_L(R, K_n) = 2\left(\frac{n(n-1)}{2} - \frac{nr}{2}\right) = n(n-1-r),$$

for some r-regular graph R from the set G_R.

Since the L-spectrum of an arbitrary r-regular graph R from the set G_R is $\mu_1(R) = r - \lambda_n(R) \geq \mu_2(R) = r - \lambda_{n-1}(R) \geq \cdots \geq \mu_n(R) = r - \lambda_1(R) = 0$, we have

$$\begin{aligned}
\sigma_L(R_1, R_2) &= \sum_{i=1}^{n} |\mu_i(R_1) - \mu_i(R_2)| \\
&= \sum_{i=1}^{n} |(r_1 - \lambda_i(R_1)) - (r_2 - \lambda_i(R_2))| \leq \sum_{i=1}^{n} (\max\{r_1, r_2\} + 2) \\
&\leq n\,(n - 3 - \min\{r_1, r_2\} + 2) = \sigma_L\left(R_{\min\{r_1,r_2\}}, K_n\right) \\
&\leq n(n-1-2) = \sigma_L(C_n, K_n). \qquad \square
\end{aligned}$$

We saw that there are pairs of graphs of order n whose A-spectral distance is greater than the maximal energy on the set of all graphs of order n. The computational results (see Appendix) show that there is no pair of trees of order $5 \leq n \leq 11$ whose (L-) Q-spectral distance is greater than the Laplacian energy $LE(K_{1,n-1})$ of the star $K_{1,n-1}$ (i.e. the signless Laplacian energy $QE(K_{1,n-1})$). So, we can pose the following conjecture:

Conjecture 5.5. *If T_1 and T_2 are arbitrary trees of order n, then $\sigma_L(T_1, T_2) \leq LE(K_{1,n-1})$ and $\sigma_Q(T_1, T_2) \leq QE(K_{1,n-1})$.*

With the additional assumptions, we can prove the following statement:

Lemma 5.1. *Let T_1 and T_2 be trees of order $n \geq 3$. Let us suppose that for the Laplacian eigenvalues of T_1 and T_2 the following inequalities hold: $|\mu_1(T_1) - \mu_1(T_2)| \leq \min\{\mu_1(T_1), \mu_1(T_2)\} - 2$ and $|\mu_i(T_1) - \mu_i(T_2)| \leq \min\{\mu_i(T_1), \mu_i(T_2)\}$, for $i = 2, 3, \ldots, n$. Then we have*

$$\sigma_L(T_1, T_2) \leq LE(K_{1,n-1}).$$

Proof. According to the assumptions, we get

$$\sigma_L(T_1, T_2) \leq (\mu_1(T_1) - 2) + \sum_{i=2}^{n} \mu_i(T_1) = 2(n-1) - 2 < 2n - 4 + \frac{4}{n} = LE(K_{1,n-1}). \quad \square$$

6. Spectral distances in some specially constructed sets of graphs

In order to investigate the cospectrality on the special classes of graphs [14] and to study the conjectures about the spectral distances of graphs [10], in this section we will calculate spectral distances and estimate distance related graph parameters on the sets of specially chosen graphs of the same order. We will consider three classes of graphs. The first of them consists of some complete multipartite graphs; the second class comprehends the graphs that have $2K_n$ as the common subgraph, while the graphs in the third class are results of NEPS that is applied on n graphs K_2.

6.1. **Complete multipartite graphs.** Let $X = \{G_1, G_2, \dots, G_p\}$ be the set of complete multipartite graphs $G_i = K_{k_i \times m_i}$ of order n, where $1 \le i \le p$. Here, p signifies the number of factors (or divisors) of n. Each graph G_i from the set X has k_i parts (where k_i is one of the divisors of n), and each of these parts has equal number of vertices $m_i = k_{p-i+1}$, for $1 \le i \le p$. We also assume that for graphs G_i and G_j from the set X such that $i < j$, $k_i < k_j$ holds. Thus, G_1 is the graph that consists of isolated vertices ($k_1 = 1$ and $m_1 = k_p = n$), $G_2 = K_{2\times m_2} = K_{2\times k_{p-1}}$ is the complete bipartite graph, while G_p ($k_p = n$ and $m_p = k_1 = 1$) is the complete graph. The spectrum of the graph $G_i = K_{k_i\times m_i}$ consists of (see [3, page 73]): $[(k_i - 1)m_i]$, $[0]^{k_i(m_i-1)}$, and $[-m_i]^{k_i-1}$, but because $k_i \cdot m_i = n$ holds for each i, we have: $[n - m_i]$, $[0]^{n-k_i}$ and $[-m_i]^{k_i-1}$.

Proposition 6.1. *Let X be the previously defined set of graphs. Then*

(1) $\mathrm{cs}^A_X(G_i) = \min\{2(n - k_i m_{i+1}), 2(n - k_{i-1} m_i)\}$, *for* $2 \le i \le n - 1$*;*
(2) $\mathrm{cs}^A(X) = 2(n - m_2) = 2(n - k_{p-1})$*;*
(3) $\mathrm{secc}^A_X(G_i) = \max\{2(n - k_i), 2(n - m_i)\}$, *for* $1 \le i \le n$*;*
(4) $\mathrm{sdiam}^A(X) = E(K_n) = 2(n - 1)$.

Proof. Let us determine the A-spectral distance between graphs $G_i = K_{k_i\times m_i}$ and $G_j = K_{k_j\times m_j}$ from the set X, such that $1 \le i < j \le p$, which means $k_i < k_j$ and $m_i > m_j$. Since $\lambda_1(G_i) < \lambda_1(G_j)$ and $|\lambda_n(G_i)| > |\lambda_n(G_j)|$, we have

$$\begin{aligned}\sigma_A(G_i, G_j) &= |\lambda_1(G_i) - \lambda_1(G_j)| + \sum_{k=2}^{n-k_j+1} |\lambda_k(G_i) - \lambda_k(G_j)| \\ &\quad + \sum_{k=n-k_j+2}^{n-k_i+1} |\lambda_k(G_i) - \lambda_k(G_j)| + \sum_{k=n-k_i+2}^{n} |\lambda_k(G_i) - \lambda_k(G_j)| \\ &= |(n - m_i) - (n - m_j)| + \sum_{k=n-k_j+2}^{n-k_i+1} | - m_j| \\ &\quad + \sum_{k=n-k_i+2}^{n} | - m_i + m_j| = k_i\, m_i + (k_j - 2k_i)m_j = 2(n - k_i\, m_j).\end{aligned}$$

Now, we can determine distance related graph parameters on the set X. According to the definition, for the cospectrality we have

$$\mathrm{cs}^A_X(G_i) = \min\{\sigma_A(G_i, G_j) : G_j \in X, j \ne i\} = \min\{2(n - k_i m_{i+1}), 2(n - k_{i-1} m_i)\},$$

for $2 \leq i \leq n-1$. In other words, the graphs that are nearest to the chosen graph G_i from the set X are its neighbours G_j such that the difference between the numbers of parts k_i and k_j is minimal.

Since in the set X there are graphs with only one part and also with the parts of cardinality equal to one, we can conclude that:

$$\mathrm{cs}^A(X) = \max\{2(n - 1 \cdot m_2), 2(n - k_{p-1} \cdot 1)\} = 2(n - m_2),$$

because $m_2 = k_{p-1}$ holds.

The maximal spectral distance is achieved with the empty graph or with the complete graph, i.e.

$$\mathrm{secc}_X^A(G_i) = \max\{\sigma_A(G_i, G_j) : G_j \in X, j \neq i\} = \max\{2(n - k_i \cdot 1), 2(n - 1 \cdot m_i)\}.$$

So, the spectral diameter is $\mathrm{sdiam}^A(X) = \sigma_A(G_1, G_p) = E(K_n) = 2(n-1)$. $\square$

Example 6.1. For $n = 12$ the set X comprehends the following six graphs: $G_1 = 12\,K_1$, $G_2 = K_{2\times 6}$, $G_3 = K_{3\times 4}$, $G_4 = K_{4\times 3}$, $G_5 = K_{6\times 2}$, and $G_6 = K_{12}$. The cospectrality of each graph from the set X is $\mathrm{cs}(G_1) = \sigma_A(G_1, G_2) = 12$, $\mathrm{cs}(G_2) = \sigma_A(G_2, G_3) = 8$, $\mathrm{cs}(G_3) = \sigma_A(G_3, G_4) = 6$, $\mathrm{cs}(G_4) = \sigma_A(G_3, G_4) = 6$, $\mathrm{cs}(G_5) = \sigma_A(G_4, G_5) = 8$, and $\mathrm{cs}(G_6) = \sigma_A(G_5, G_6) = 12$, so the cospectrality measure is $\mathrm{cs}(X) = \sigma_A(G_1, G_2) = \sigma_A(G_5, G_6) = 12$. The spectral eccentricities are $\mathrm{secc}(G_1) = \sigma_A(G_1, G_6) = 22$, $\mathrm{secc}(G_2) = \sigma_A(G_2, G_6) = 20$, $\mathrm{secc}(G_3) = \sigma_A(G_3, G_6) = 18$, $\mathrm{secc}(G_4) = \sigma_A(G_4, G_1) = 18$, $\mathrm{secc}(G_5) = \sigma_A(G_5, G_1) = 20$, and $\mathrm{secc}(G_6) = \sigma_A(G_1, G_6) = 22$, and the spectral diameter is $\mathrm{sdiam}(X) = E(G_6) = 22$.

6.2. KK_n^j **graphs.** The graph KK_n^j is defined in [5] as the graph that is obtained from two copies of K_n by adding j edges between one vertex of a copy of K_n and j vertices of the other copy, for $1 \leq j \leq n$. Its A-characteristic polynomial for $n \geq 3$ and $1 \leq j \leq n$ is also exposed in [5]: $P_{KK_n^j}(x) = (x+1)^{(2n-4)} h(x)$, where $h(x)$ is the polynomial whose degree is equal to four. We will first show that the roots of $h(x)$ are just the first, second, third and $2n$-th eigenvalue of the graph KK_n^j.

The adjacency matrix A^j of the graph KK_n^j is $A^j = A^0 + M_j$. Here, A^0 is the adjacency matrix of $2K_n$, whence M_j is the matrix of the following form:

$$M_j = \begin{pmatrix} O & X_j \\ X_j^T & O \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

where O is the null matrix, while X_j is the matrix of order n that in only one row (which corresponds to the vertex that is chosen in one of the two copies of K_n) has j entries that are equal to one, and all others equal to zero.

By applying the row transformations on the matrix M_j, we get the matrix with just two non-null rows. So, the spectrum of the matrix M_j consists of $2n-2$ eigenvalues that are equal to zero. Since the corresponding graph is bipartite, remaining two eigenvalues are symmetric with respect to zero, and it can be checked that they are equal to $\pm\sqrt{j}$, for $1 \leq j \leq n$.

Let us now outline the order of eigenvalues in the spectrum of KK_n^j. If we apply the Courant-Weyl inequalities (see Theorem 1.3.15 from [4]) to A^0 and M_j, we get the following relations:

$$\begin{gathered}
\lambda_l(KK_n^j) \geq \lambda_{l+1}(2K_n) + \lambda_{2n-1}(M_j) = \lambda_{l+1}(2K_n) = -1, \quad l = 4, 5, \dots, 2n-1; \\
\lambda_l(KK_n^j) \leq \lambda_{l-1}(2K_n) + \lambda_2(M_j) = \lambda_{l-1}(2K_n) = -1, \quad l = 4, 5, \dots, 2n-1; \\
\lambda_{2n}(KK_n^j) \leq \lambda_{2n-1}(2K_n) + \lambda_2(M_j) = \lambda_{2n-1}(2K_n) = -1; \\
\lambda_3(KK_n^j) \geq \lambda_4(2K_n) + \lambda_{2n-1}(M_j) = \lambda_4(2K_n) = -1,
\end{gathered}$$

since the spectrum of $2K_n$ consists of $[n-1]^2$ and $[-1]^{2n-2}$. Therefore the roots of the polynomial $h(x)$ are $\lambda_1(KK_n^j), \lambda_2(KK_n^j), \lambda_3(KK_n^j)$, and $\lambda_{2n}(KK_n^j)$.

Theorem 6.1. *Let G be the set of graphs $G = \{G_0, G_1, \dots, G_n\}$ such that $G_0 = 2K_n$ and $G_i = KK_n^i$, for $1 \leq i \leq n$. Then*

$$\sigma_A(G_i, G_j) \leq \lambda_1(G_j) - \lambda_1(G_i) + 3\sqrt{j-i},$$

for $0 \leq i < j \leq n$ and $n \geq 3$.

Proof. The A-spectral distance between $G_i = KK_n^i$ and $G_j = KK_n^j$, for $0 \leq i < j \leq n$ is

$$\begin{aligned}
\sigma_A(G_i, G_j) = \sum_{l=1}^{2n} |\lambda_l(G_i) - \lambda_l(G_j)| &= |\lambda_1(G_i) - \lambda_1(G_j)| \\
&+ |\lambda_2(G_i) - \lambda_2(G_j)| + |\lambda_3(G_i) - \lambda_3(G_j)| + |\lambda_{2n}(G_i) - \lambda_{2n}(G_j)|.
\end{aligned}$$

Since the increase of the number of edges does not decrease the maximal eigenvalue (see Theorem 0.7 from [3]), we have

$$\begin{aligned}
\sigma_A(G_i, G_j) = \lambda_1(G_j) - \lambda_1(G_i) &+ |\lambda_2(G_i) - \lambda_2(G_j)| \\
&+ |\lambda_3(G_i) - \lambda_3(G_j)| + |\lambda_{2n}(G_i) - \lambda_{2n}(G_j)|.
\end{aligned}$$

For the adjacency matrices A^i and A^j of KK_n^i and KK_n^j, respectively, $A^j = A^i + M_{j-i}$ holds. Here, M_{j-i} is, as before, the matrix of order $2n$, with $j-i+1$ non-null rows. By applying the Courant-Weyl inequalities we get:

$$\begin{gathered}
\lambda_l(KK_n^j) \leq \lambda_l(KK_n^i) + \lambda_1(M_{j-i}) = \lambda_l(KK_n^i) + \sqrt{j-i}, \quad l = 2, 3, 2n, \\
\lambda_l(KK_n^j) \geq \lambda_l(KK_n^i) + \lambda_{2n}(M_{j-i}) = \lambda_l(KK_n^i) - \sqrt{j-i}, \quad l = 2, 3, 2n,
\end{gathered}$$

which means: $|\lambda_l(G_i) - \lambda_l(G_j)| \leq \sqrt{j-i}$, for $l = 2, 3, 2n$. Finally, we have

$$\sigma_A(G_i, G_j) \leq \lambda_1(G_j) - \lambda_1(G_i) + 3\sqrt{j-i}. \qquad \square$$

Remark 6.1. In the general case, the distance related graph parameters cannot be computed and estimated exactly. For example, on the set G for $G_0 = 2K_7$ we have $\mathrm{cs}(G) = \sigma_A(G_0, G_1) = 2$, while $\mathrm{sdiam}(G) = \sigma_A(G_0, G_5) = 3.68$. This value of the spectral diameter is different from the corresponding values on some other (see [10] and [11]) previously considered sets of graphs with specially chosen structure and interconnections, where the spectral diameter was equal to the spectral distance between, we can say, extremal graphs, i.e. graphs with the least and the largest number of edges.

Remark 6.2. Since the graph G_i is $(j-i)$-edge deleted subgraph of the graph G_j, for $0 \leq i < j \leq n$, according to Proposition 3.1 from [11], L-, and similarly, Q-spectral distance between these graphs is equal to

$$\sigma_L(G_i, G_j) = \sigma_Q(G_i, G_j) = 2(j-i).$$

Now, the distance related graph parameters can be exactly computed:

- $\mathrm{cs}^L_G(G_i) = \min\{\sigma_L(G_i, G_j) : G_j \in G, j \neq i\} = 2$;
- $\mathrm{cs}^L(G) = 2$;
- $\mathrm{secc}^L_G(G_i) = \max\{\sigma_L(G_i, G_j) : G_j \in G, j \neq i\} = \max\{2i, 2(n-i)\}$;
- $\mathrm{sdiam}^L(G) = 2n$.

6.3. **Some graphs obtained by NEPS.** We will now continue with another family of graphs. Namely, let G_k be the graph $G_k = \mathrm{NEPS}(\underbrace{K_2, K_2, \ldots, K_2}_{n}; B_k)$, where the basis B_k consists of $1 \leq k \leq n$ binary n-tuples with exactly one entry equal to one. The graph G_k is regular, integral [1] and bipartite [15]. If B_k denotes the matrix whose rows are vectors of the basis B_k, one can easily found that the rank of such a matrix in the field GF_2 is k. So, G_k has 2^{n-k} connected components [16]. All of these components are isomorphic, and each of them is isomorphic to the graph $G'_k = \mathrm{NEPS}(\underbrace{K_2, K_2, \ldots, K_2}_{k}; B'_k)$, for the basis B'_k such that $B'_k = P(B_k)$. Here, P is an isomorphism between the linear subspace $L(B_k)$, generated by the basis B_k, and the vector space GF_2^k. Since the graph G'_k is k-cube, whose spectrum is $[k-2i]^{\binom{k}{i}}$, for $0 \leq i \leq k$, the spectrum of the graph G_k is $[k-2i]^{2^{n-k}\binom{k}{i}}$, for $0 \leq i \leq k$ and $1 \leq k \leq n$. We will consider the spectral distances and distance related graph parameters on the set of graphs $F = \{G_1, G_2, \ldots, G_n\}$, each of whose graphs G_j, for $1 \leq j \leq n$, is one of the previously described graphs G_k.

Proposition 6.2. *The A-spectral distance between graphs G_k and G_{k+1}, for $1 \leq k \leq n-1$, from the set F is*

$$\sigma_A(G_k, G_{k+1}) = 2^n.$$

Proof. We will first prove some inequalities that are related to the multiplicities of the distinct eigenvalues of graphs G_k and G_{k+1}.

From $\binom{k+1}{i+1} \geq \binom{k}{i}$ and $\binom{k+1}{i} \geq \binom{k}{i}$, we get $2^{n-k}\binom{k}{i} \leq 2^{n-k-1}\left(\binom{k+1}{i} + \binom{k+1}{i+1}\right)$, for all $0 \leq i \leq k$.

As $2(k+1-i) > k+1$ holds for all $i \leq \lfloor k/2 \rfloor$, we have $2^{n-k}\binom{k}{i} > 2^{n-k-1}\binom{k+1}{i}$, for all $0 \leq i \leq \lfloor k/2 \rfloor$.

Since graphs G_k and G_{k+1} are bipartite, each of the distinct eigenvalues of G_k will contribute to the value of the spectral distance $\sigma_A(G_k, G_{k+1})$ with two corresponding distinct eigenvalues of G_{k+1}, i.e. $|(k-2i)-(k+1-2i)| = 1$ and $|(k-2i)-(k+1-2(i+1))| = 1$. So, the spectral distance is

$$\sigma_A(G_k, G_{k+1}) = \sum_{i=0}^{k} 2^{n-k}\binom{k}{i} \cdot 1 = 2^n. \qquad \square$$

Proposition 6.3. *The A-spectral distance between graphs G_k and G_{k+2} from the set F for $1 \leq k \leq n-2$ and k-odd is*

$$\sigma_A(G_k, G_{k+2}) = 2^{n-k+2}\left(\sum_{j=0}^{\lfloor k/2 \rfloor} 2^{-2}\sum_{i=0}^{j}\binom{k+2}{i} - \sum_{j=1}^{\lfloor k/2 \rfloor}\sum_{i=0}^{j-1}\binom{k}{i}\right).$$

Proof. We begin with some inequalities related to the multiplicities of the eigenvalues of graphs G_k and G_{k+2}, when k is odd.

We have $2^{n-k}\binom{k}{i} > 2^{n-k-2}\binom{k+2}{i}$, for $0 \leq i \leq \lfloor k/2 \rfloor$. To see that this holds, it is enough to show that $4 > \frac{(k+2)(k+1)}{(k+2-i)(k+1-i)}$. Since $i \leq \lfloor k/2 \rfloor$, we have $\frac{(k+2)(k+1)}{(k+2-i)(k+1-i)} < \frac{(k+2)(k+1)}{(k+2-\frac{k-1}{2})(k+1-\frac{k-1}{2})} = 4\,\frac{(k+2)(k+1)}{(k+5)(k+3)} < 4$, which proves the inequality.

We claim that $2^{n-k}\binom{k}{i} \leq 2^{n-k-2}\left(\binom{k+2}{i} + \binom{k+2}{i+1}\right)$, for $1 \leq k \leq n$ and $0 \leq i \leq \lfloor k/2 \rfloor$, i.e.

$$4\binom{k}{i} - \binom{k+2}{i} - \binom{k+2}{i+1} \leq 0. \tag{5}$$

It is a matter of routine to check that (5) holds for $i = 0$. For $1 \leq i \leq \lfloor k/2 \rfloor$ inequality (5) becomes:

$$\frac{k!}{i!(k-i)!}\left(4 - \frac{(k+1)(k+2)(k+3)}{(i+1)(k-i+1)(k-i+2)}\right) \leq 0.$$

Since $(k+2)(k+3) \geq 4(i+1)(k-i+1)$, we have

$$\frac{(k+1)(k+2)(k+3)}{(i+1)(k-i+1)(k-i+2)} \geq \frac{(k+1)4(i+1)(k-i+1)}{(i+1)(k-i+1)(k-i+2)} \geq 4,$$

which proves (5).

Now, the spectral distance is

$$\begin{aligned}\sigma_A(G_k, G_{k+2}) = 2 \cdot \Bigg\{ &\sum_{i=1}^{2^{n-(k+2)}} |k-(k+2)| + \sum_{i=2^{n-(k+2)}+1}^{2^{n-k}} |k-k| \\ &+ \sum_{i=2^{n-k}+1}^{2^{n-(k+2)}\cdot(1+(k+2))} |(k-2)-k| \\ &+ \sum_{i=2^{n-(k+2)}\cdot(1+(k+2))+1}^{2^{n-k}\cdot(1+k)} |(k-2)-(k-2)| + \cdots \\ &+ \sum_{2^{n-(k+2)}\left(\binom{k+2}{0}+\binom{k+2}{1}+\cdots+\binom{k+2}{\lfloor k/2 \rfloor}\right)+1}^{2^{n-1}} |1-1| \Bigg\}.\end{aligned}$$

Furthermore we have

$$\begin{aligned}\sigma_A(G_k, G_{k+2}) = 4 \Bigg\{ &2^{n-(k+2)} + 2^{n-(k+2)}\left(1+\binom{k+2}{1}\right) - 2^{n-k} \\ &+ 2^{n-(k+2)}\left(1+\binom{k+2}{1}+\binom{k+2}{2}\right) - 2^{n-k}\left(1+\binom{k}{1}\right) \\ &+ 2^{n-(k+2)}\left(1+\binom{k+2}{1}+\binom{k+2}{2}+\binom{k+2}{3}\right) - 2^{n-k}\left(1+\binom{k}{1}+\binom{k}{2}\right) \\ &+ \cdots + 2^{n-(k+2)}\left(1+\binom{k+2}{1}+\binom{k+2}{2}+\cdots+\binom{k+2}{\lfloor k/2 \rfloor}\right) \\ &- 2^{n-k}\left(1+\binom{k}{1}+\binom{k}{2}+\cdots+\binom{k}{\lfloor k/2 \rfloor - 1}\right)\Bigg\}.\end{aligned}$$

Hence, by grouping the corresponding addends we get the relation from the statement. □

Remark 6.3. We can notice that the computation of the A-spectral distances between graphs from the set F becomes more complicated as the differences between the cardinalities of the corresponding basis increase. On that way, it is not easy to exactly compute and evaluate A-distance related graph parameters on this set.

On the other hand, L- or Q-distance related graph parameters on the set F can be exactly computed. Namely, each graph G_k is an edge-deleted subgraph of the graph G_{k+1}, for $1 \le k \le n-1$. Precisely, we have:

Proposition 6.4. *Let F be the previously described set of graphs. Then*

(1) $\mathrm{cs}_F^L(G_k) = 2^n$, *for* $1 \le k \le n$*;*
(2) $\mathrm{cs}^L(F) = 2^n$*;*
(3) $\mathrm{secc}_F^L(G_k) = \max\{2^n(k-1), 2^n(n-k)\}$, *for* $1 \le k \le n$*;*
(4) $\mathrm{sdiam}^L(F) = 2^n(n-1)$.

Proof. Each binary n-tuple with exactly one entry equal to one implies the existence of 2^{n-1} edges in the corresponding graph. Since each of the basis B_l is originated from the basis B_k, for $k < l$, by adding some binary n-tuples with exactly one entry equal to one, each graph G_k is a edge-deleted subgraph of the graph G_l. So, the L-spectral distance between arbitrary graphs G_k and G_l, $1 \leq k < l \leq n$, from the set F, according to Proposition 3.1 from [11], is

$$\sigma_L(G_k, G_l) = 2\,(2^{n-1}\,l - 2^{n-1}\,k) = 2^n\,(l-k).$$

The values of the distance related graph parameters from the statement can be obtained by direct computation according to the definition. □

Appendix

In the following tables we give the computational results related to the energies and spectral distances of graphs with respect to the Laplacian and the signless Laplacian matrix. The first column of all subsequent tables contains the order of the considered graphs. The values of the maximal (L-) Q-energy are given in the second column, while the third column contains the number of pairs of graphs whose (L-) Q-spectral distance is greater than the maximal (L-) Q-energy. The number of all graphs that have the maximal (L-) Q-energy are exposed in the fourth column. The graphs with the maximal (L-) Q-energy are outlined in the last column. Observe that R_r in the last column stands for some r-regular graph. Numerical results for L- and Q-energies and spectral distances are the same on the set of connected bipartite graphs and the set of trees.

Table 1. Q-energies and Q-spectral distances of all n-vertex graphs

n	$\max QE(G)$	$\sigma \geq \max QE(G)$	No.	G
3	4	3	1	K_3
4	6	18	1	K_4
5	8	148	1	K_5
6	10.93	1573	1	$CS_{2,4}$
7	14.49	34439	1	$CS_{2,5}$
8	18.80	2212476	1	$CS_{3,5}$

TABLE 2. Q-energies and Q-spectral distances of connected regular graphs

n	max $QE(G)$	$\sigma \geq$ max $QE(G)$	No.	G
6	10	4	1	K_6
7	12	5	1	K_7
8	14	51	2	K_8, R_5
9	16.47	105	1	R_6
10	20	3938	1	R_6
11	22.45	2534	1	R_6

TABLE 3. Q-energies and Q-spectral distances of connected bipartite graphs

n	max $QE(G)$	$\sigma \geq$ max $QE(G)$	No.	G
4	5	0	1	$K_{1,3}$
5	6.8	0	1	$K_{1,4}$
6	9.33	0	1	$K_{2,4}$
7	12.57	0	1	$K_{2,5}$
8	16	103	1	$K_{2,6}$
9	19.56	587	1	$K_{2,7}$
10	23.2	11771	1	$K_{2,8}$
11	27.82	196914	1	$K_{3,8}$

TABLE 4. Q-energies and Q-spectral distances of trees

n	max $QE(G)$	$\sigma \geq$ max $QE(G)$	No.	G
5	6.8	0	1	$K_{1,4}$
6	8.67	0	1	$K_{1,5}$
7	10.57	0	1	$K_{1,6}$
8	12.5	0	1	$K_{1,7}$
9	14.44	0	1	$K_{1,8}$
10	16.4	0	1	$K_{1,9}$
11	18.36	0	1	$K_{1,10}$

TABLE 5. L-energies and L-spectral distances of all n-vertex graphs

n	max $LE(G)$	$\sigma \geq$ max $LE(G)$	No.	G
3	4	3	1	K_3
4	6	18	4	$PA_{3,1}, K_3 \cup K_1, K_2 \nabla 2K_1, K_4$
5	9.6	84	1	$K_4 \cup K_1$
6	13.33	907	1	$K_5 \cup K_1$
7	17.14	19868	2	$K_6 \cup K_1, K_5 \cup 2K_1$
8	22.5	686232	1	$K_6 \cup 2K_1$

TABLE 6. L-energies and L-spectral distances of connected regular graphs

n	max $LE(G)$	$\sigma \geq$ max $LE(G)$	No.	G
6	10	4	1	K_6
7	12	5	1	K_7
8	14	51	2	K_8, R_5
9	16.47	105	1	R_6
10	20	3938	1	R_6
11	22.45	2396	1	R_6

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Received: May 25, 2014
Accepted: March 25, 2015