A NOTE ON THE CLASSIFICATION OF GAMMA FACTORS

ROMÁN SASYK

Abstract. One of the earliest invariants introduced in the study of finite von Neumann algebras is the property Gamma of Murray and von Neumann. The set of separable II\(_1\) factors can be split in two disjoint subsets: those that have the property Gamma and those that do not have it, called full factors by Connes. In this note we prove that it is not possible to classify separable II\(_1\) factors satisfying the property Gamma up to isomorphism by a Borel measurable assignment of countable structures as invariants. We also show that the same holds true for the full II\(_1\) factors.

1. Introduction

In this note we continue with the line of research initiated by the author in collaboration with A. Törnquist in [19], [20] and [21], where we applied the notion of Borel reducibility from descriptive set theory to study the complexity of the classification problem of several different classes of separable von Neumann algebras.

Recall that if \(E\) and \(F\) are equivalence relations on standard Borel spaces \(X\) and \(Y\), respectively, we say that \(E\) is Borel reducible to \(F\) if there is a Borel function \(f : X \to Y\) such that

\[
(\forall x, x' \in X) x Ex' \iff f(x) F f(x'),
\]

and if this is the case we write \(E \leq_B F\). Thus if \(E \leq_B F\) then the points of \(X\) can be classified up to \(E\) equivalence by a Borel assignment of invariants that we may think of as \(F\)-equivalence classes. \(E\) is smooth if it is Borel reducible to the equality relation on \(\mathbb{R}\). While smoothness is desirable, it is most often too much to ask for. A more generous class of invariants which seems natural to consider are countable groups, graphs, fields, or other countable structures, considered up to isomorphism. Thus, following [14], we will say that an equivalence relation \(E\) is classifiable by countable structures if there is a countable language \(\mathcal{L}\) such that \(E \leq_B \simeq_{\text{Mod}(\mathcal{L})}\), where \(\simeq_{\text{Mod}(\mathcal{L})}\) denotes isomorphism in \(\text{Mod}(\mathcal{L})\), the Polish space of countable models of \(\mathcal{L}\) with universe \(\mathbb{N}\).

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In [20] it was proved that the isomorphism relation in the set of finite von Neumann algebras is not classifiable by countable structures. Nonetheless, it can certainly be the case that some subclasses of finite factors are possible to classify by countable structures. For instance, Connes’ celebrated theorem [23], says that the set of infinite dimensional injective finite factors has only one element on its isomorphism class, namely the hyperfinite II$_1$ factor $R$. In contrast with the injective case, in this note we show that it is not possible to obtain a reasonable classification up to isomorphisms for a well studied family of finite factors that includes $R$.

In order to state our results we observe first that the set of finite factors can be split in two disjoint subsets: those who satisfy the property $\Gamma$ of Murray and von Neumann and those who are full. The first set contains the hyperfinite II$_1$ factor $R$ and more generally, the class of McDuff factors, i.e. those factors of the form $M \otimes R$ for $M$ a II$_1$ factor. On the other hand the set of full factors contains the free group factors $L(F_n)$. In this article we show that the II$_1$ factors constructed in [20] are full. As a consequence Theorem 7 in [20] strengthens to prove:

**Theorem 1.1.** The isomorphism relation for full type II$_1$ factors is not classifiable by countable structures.

It remained then to analyze the complexity of the classification of II$_1$ factors with the property $\Gamma$. In this note we address this problem by showing that:

**Theorem 1.2.** The isomorphism relation for McDuff factors is not classifiable by countable structures.

An immediate consequence is:

**Corollary 1.3.** The isomorphism relation for type II$_1$ factors satisfying the property $\Gamma$ of Murray and von Neumann is not classifiable by countable structures.

We end this introduction by mentioning that the study of the connections between logic and operator algebras has recently attracted many researchers from both fields. As a consequence, in the past five years there has been a burst of activity in proving results along the lines of the ones presented in this note and first unveiled in [19], [20] and [21]. We refer the reader who wants to learn more on these exciting new developments to the recent survey of I. Farah [9].

2. Gamma factors

We start by recalling the definitions of the objects we study in this article. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$, which we give the weak topology. A separable von Neumann algebra is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. The set of von Neumann algebras acting on $\mathcal{H}$ is denoted $\text{vN}(\mathcal{H})$. A von Neumann algebra $M$ is said to be finite if it admits a finite faithful normal tracial state, i.e. a linear functional $\tau : M \to \mathbb{C}$ such that: $\tau(x^*x) \geq 0$, $\tau(x^*x) = 0$ iff $x = 0$, $\tau(1) = 1$, $\tau(xy) = \tau(yx)$ and the unit ball of $M$ is complete with respect to the norm given by the trace $\|x\|_2 = \tau(x^*x)$. If a finite von Neumann algebra is also a factor, i.e. its center is trivial, then it has a unique such trace. A finite von Neumann factor
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that is not a matrix algebra is called a type II_1 factor. This terminology is due to the general classification of von Neumann algebras according to types (see [4, Chapter 5.1] for an historical account of the theory of types).

In this note we will be interested in II_1 factors arising from the so called group-measure space construction, that we proceed to describe. For that, let G be a countably infinite discrete group which acts in a measure preserving way on a Borel probability space (X,μ). For each g ∈ G and ζ ∈ L^2(X,μ) the formula

σ_g(ζ)(x) = ζ(g^{-1}·x)

defines a unitary operator on L^2(X,μ).

We identify the Hilbert space H = L^2(G,L^2(X,μ)) with the Hilbert space of formal sums ∑_{g∈G} ζ_gξ_g, where the coefficients ζ_g are in L^2(X,μ) and satisfy ∑_g ∥ζ_g∥_L^2(X,μ) < ∞, and ξ_g are indeterminates indexed by the elements of G. The inner product on H is given by

⟨∑_{g∈G} ζ_g(x)ξ_g, ∑_{g∈G} ζ'_g(x)ξ_g⟩ = ∑_{g∈G} ⟨ζ_g,ζ'_g⟩_L^2(X,μ).

Both L^∞(X,μ) and G act by left multiplication on H by the formulas

f(ζ_g(x)ξ_g) = (f(x)ζ_g(x))ξ_g,

u_h(ζ_g(x)ξ_g) = σ_h(ζ_g(x))ξ_{hg},

where f ∈ L^∞(X,μ), ζ_g(x) ∈ L^2(X,μ) and g,h ∈ G. Thus if we denote by FS the set of finite sums,

FS = \{ ∑_{g∈G} f_g u_g : f_g ∈ L^∞(X,μ), f_g = 0, except for finitely many g \},

then each element in FS defines a bounded operator on H. Moreover, multiplication and involution in FS satisfy the formulas

(f_g u_g)(f_h u_h) = f_g σ_g(f_h) u_{gh}

and

(f u_g)^* = σ_{g^{-1}}(f^*) u_{g^{-1}},

and so FS is a *-algebra. By definition, the group-measure space von Neumann algebra is the weak operator closure of FS on B(H) and it is denoted by L^∞(X,μ) ⋊_σ G. The trace on FS, defined by

τ(∑_{g∈G} f_g u_g) = ∫_X f_e dμ,

extends to a faithful normal tracial state in L^∞(X) ⋊_σ G by the formula τ(T) = ⟨T(ξ_e),ξ_e⟩, where e represents the identity of G.

**Definition 2.1** (Murray-von Neumann [15]). A finite von Neumann algebra M has the property Γ if given x_1,\ldots,x_n ∈ M, and ε > 0 there exists u ∈ U(M), τ(u) = 0 such that

\|x_i u - u x_i\|_2 < ε, \quad \text{for all } 1 \leq i \leq n.
It follows immediately from its definition that the hyperfinite II$_1$ factor $R$ is a Γ-factor. Moreover, it is clear that any finite factor of the form $M \otimes N$ with $N$ a Γ-factor is also a Γ-factor. In particular, McDuff factors, i.e. factors of the form $M \otimes R$, are Γ-factors. The paradoxical decomposition of the free groups $F_n$, $n \geq 2$, is the key ingredient [15, Lemmas 6.2.1, 6.2.2] to show that the corresponding group von Neumann factors $L(F_n)$, $n \geq 2$, do not have the property Γ. More generally, Effros showed in [8] that if $G$ is a discrete ICC group and $L(G)$ has the property Γ, then $G$ is inner amenable (so, in particular, free groups are not inner amenable). The paradoxical decomposition of the free groups $F_n$, $n \geq 2$, is the key ingredient [15, Lemmas 6.2.1, 6.2.2] to show that the corresponding group von Neumann factors $L(F_n)$, $n \geq 2$, do not have the property Γ. More generally, Effros showed in [8] that if $G$ is a discrete ICC group and $L(G)$ has the property Γ, then $G$ is inner amenable (so, in particular, free groups are not inner amenable). The paradoxical decomposition of the free groups $F_n$, $n \geq 2$, is the key ingredient [15, Lemmas 6.2.1, 6.2.2] to show that the corresponding group von Neumann factors $L(F_n)$, $n \geq 2$, do not have the property Γ. More generally, Effros showed in [8] that if $G$ is a discrete ICC group and $L(G)$ has the property Γ, then $G$ is inner amenable (so, in particular, free groups are not inner amenable). That the converse of Effros’ theorem is false is a recent result of Vaes [24].

If $M$ is a finite von Neumann algebra with trace $\tau$, $\text{Aut}(M,\tau)$, the set of $\tau$-preserving automorphisms of $M$, is a Polish group. A basis for that topology is given by the sets $V_{T,a_1,...,a_n,\epsilon} = \{ S \in \text{Aut}(M,\tau) : \|S(a_i) - T(a_i)\|_2 \leq \epsilon, \forall 1 \leq i \leq n, a_i \in M \}$. $\text{Inn}(M)$ denotes the set of inner automorphisms of $M$, i.e. those of the form $\text{Ad}(u)$, $u \in U(M)$.

**Definition 2.2** (Connes [2]). A finite von Neumann algebra $M$ is full if $\text{Inn}(M)$ is closed in $\text{Aut}(M)$.

In [2, Corollary 3.8], Connes showed that a II$_1$ factor $M$ is full if and only if $M$ does not have the property Γ. It follows that for each $n \geq 2$, $L(F_n)$ is a full factor. In order to discern when group measure space von Neumann algebras are full we need the following:

**Definition 2.3** (Schmidt [22]). Let $G$ be a discrete group and $\sigma$ an ergodic measure preserving action of $G$ on a probability space $(X,\mu)$. A sequence $(B_n)_{n \in \mathbb{N}}$ of measurable subsets of $X$ is asymptotically invariant if

$$\mu(B_n \triangle \sigma_g(B_n)) \to 0,$$

for all $g \in G$. The sequence is trivial if

$$\mu(B_n)(1 - \mu(B_n)) \to 0.$$

The action $\sigma$ is strongly ergodic if every asymptotically invariant sequence is trivial.

The relation between strong ergodicity and fullness has been studied by several authors. For the purpose of this note, it is enough to mention the following theorem of Choda [1]:

**Theorem 2.4.** Let $G$ be a discrete group that is not inner amenable, and $\sigma$ a strongly ergodic measure preserving action of $G$ on a probability space $(X,\mu)$. Then $L^\infty(X,\mu) \rtimes G$ is a full factor.

**Remark 2.5.** The condition that is really used in the proof of Theorem 2.4 is that $L(G)$ is full.

It is known that a group is amenable if and only if it does not admit strongly ergodic actions [22], while a group has the property (T) of Kazdhan if and only if every m.p. ergodic action of it is strongly ergodic [6]. We describe now a concrete example of a strongly ergodic action of $\mathbb{F}_2$ that we will use in this work. Since

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1ICC stands for infinite conjugacy classes. $G$ is ICC if and only if $L(G)$ is a factor.
F₂ can be identified with the finite index subgroup of SL(2, ℤ) generated by the matrices \(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\) (see [7, II.B.25]), it follows that F₂ naturally acts on T². Let us denote such action by σ and by Tₐ, T₅ the automorphisms corresponding to the generators a, b of F₂. This action is clearly measure preserving, and one of the main results in [22] is that σ is strongly ergodic. Inspired by earlier work of Gaboriau and Popa and Törnquist ([12], [23]), in [20] we used this action as the starting point for showing that II₁ factors are not classifiable by countable structures. More precisely, the set Ext(σ) = \{S ∈ Aut(T², µ) : Tₐ, T₅, S generates a free action of F₃\} was shown in [23, §3] to be a dense Gδ subset of Aut(T², µ). Thus Ext(σ) is a standard Borel space. For each S ∈ Ext(σ) denote by σS the corresponding F₃ action and MS ∈ vN(L^∞(F₃, L^2(T², µ))) the corresponding group-measure space von Neumann algebra

\[M_S = L^∞(T²) ⋊_{σ_S} F₃.\]

In [20] the author and Törnquist showed:

**Theorem 2.6.** The equivalence relation on Ext(σ) given by S \(\sim_{F_{II₁}} S'\) if MS is isomorphic to M_S' is not classifiable by countable structures.

Also in [20] it was shown that S \(\rightarrow\) MS is a Borel map from Ext(σ) to vN(L²(F₃, L²(T², µ))). Thus Theorem 1.1 is an immediate consequence of the previous theorem and the next lemma.

**Lemma 2.7.** For each S ∈ Ext(σ), MS is a full factor.

**Proof.** Let (Bₙ)ₙ∈ℕ be an asymptotically invariant sequence for the F₃-action σS. Then (Bₙ)ₙ∈ℕ is an asymptotically invariant sequence for the action restricted to the subgroup generated by \{Tₐ, T₅\}. By construction, this is the F₂-action σ described above, it is then strongly ergodic by [22, §4]. It follows that (Bₙ)ₙ∈ℕ is trivial and then σS is strongly ergodic. Since F₃ is not inner amenable, the result now follows from Theorem 2.4.

In order to prove Theorem 1.2 we require the following theorem of Popa ([13, Theorem 5.1]):

**Theorem 2.8.** If M₁ and M₂ are full type II₁ factors such that M₁ ⊗ R is isomorphic to M₂ ⊗ R then there exists t ∈ ℝ > 0 such that M₁ is isomorphic to M₁ᵗ.

[Remark 2.9. By interchanging the roles of M₁ and M₂ one can assume that t ∈ (0, 1], in which case M₁ᵗ is by definition the type II₁ factor pmM₂p, where p ∈ P(M₂) is any projection of trace equal to t in M₂.

**Theorem 2.10.** The assignment MS → MS ⊗ R is a Borel reduction of \(\sim_{F_{II₁}}\) to isomorphism of McDuff factors.

**Proof.** It is fairly straightforward to prove that the map MS → MS ⊗ R is a Borel assignment (see for instance [13, Corollary 3.8]). We are left to show that if MS ⊗ R is isomorphic to MS' ⊗ R, then MS is isomorphic to MS'.
Let us fix $S, S' \in \text{Ext}(\sigma)$. Lemma 2.7 shows that $M_S$ and $M_{S'}$ are full factors. By Theorem 2.8, $M_S \otimes R$ is isomorphic to $M_{S'} \otimes R$ if and only if there exists $t > 0$ such that $M_{S'}$ is isomorphic to $(M_S)^t$. The proof is over once we show that $t = 1$. For this we make use of the celebrated theorem of Popa on II$_1$ factors with trivial fundamental group [16], [17] (see also Connes’s account in the Bourbaki Séminaire [5]). Indeed by [17, Proposition], there exists a projection $p \in P(L^\infty(T^2))$, $\tau(p) = t$, such that the inclusion of von Neumann algebras $pL^\infty(T^2) \subset p(L^\infty(T^2) \rtimes_{\sigma'} F_3)$. Feldman–Moore’s theorem [10] applies to conclude that the action $\sigma_S$ is stably orbit equivalent to the action $\sigma_{S'}$, with compression constant $c = t$. Since $F_3$ has non trivial Atiyah’s $\ell^2$-Betti numbers, Gaboriau’s theorem on $\ell^2$-Betti numbers for orbit equivalence relations [11, Theorem 3.12] then implies that $t = 1$.

**Proof of Theorem 1.2.** Since $M_S \rightarrow M_S \otimes R$ is a Borel reduction of $\simeq_{F_{II_1}}$ to isomorphism of McDuff factors and the equivalence relation $\simeq_{F_{II_1}}$ is not classifiable by countable structures, it follows that the equivalence relation of isomorphism of McDuff factors is not classifiable by countable structures. □

**References**


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Román Sasyk
Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina, and
Instituto Argentino de Matemática - CONICET
Saavedra 15, Piso 3, 1083 Buenos Aires, Argentina
rsasyk@dm.uba.ar

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