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AN APPROXIMATION FORMULA FOR EULER–MASCHERONI'S CONSTANT

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Abstract. A fast approximation formula for Euler–Mascheroni's constant based on a certain integral due to Peter Borwein is given. This asymptotic formula for γ involves harmonic numbers, $\ln 2$ and rational numbers whose denominators are powers of 2.

1. Introduction and results

There are many results concerning the arithmetic nature of Euler–Mascheroni's constant $\gamma := \lim_{N\to\infty} 1 + 1/2 + 1/3 + \cdots + 1/N - \ln N$ (or approximations of it). P. Bundschuh (see [7]) proved that either γ is transcendental or certain roots of transcendental equations involving the gamma function and its derivative are irrational. J. Sondow (see [12]) proved some neat criteria for the irrationality of γ; many formulae for γ were discovered by him and can be found online in his personal homepage. Extensions and refinements of these results are given in [8] and [9]. A. Aptekarev was the first to prove that either γ or the so called Gompertz's constant is irrational (see [1]). T. Rivoal has recently extended this result in [11]. T. Rivoal and A. Aptekarev constructed rapidly converging sequences to γ or combinations of γ and logarithms (see [2, 10, 11]). S. Ramanujan has given many formulae for γ which can be found in [4]; these have been proved thanks to B. Berndt's work. Also Wolfram's website contains many other formulae.

Sondow's construction is based on the famous integrals introduced by F. Beukers for a simplified proof of Apery's result that $\zeta(3)$ is irrational (see also [13]). One of Sondow's criteria is the following: *If*

$$S_n := \prod_{k=1}^{n} \prod_{i=0}^{\min(k-1,n-k)} \prod_{j=i+1}^{n-i} (n+k)^{2\ \mathrm{LCM}(1,\ldots,2n)/j\binom{n}{i}^2},$$

and $\{\ln S_n\} \geq 1/2^n$ *infinitely often, then* γ *is irrational. Here* $\{x\}$ *means the fractional part of* x.

Numerical computations suggest that $\{\ln S_n\}$ is dense in $(0,1)$.

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In this note we prove, in Theorem A below, a fast approximation formula for γ. The idea is to have an approximation formula involving the logarithms of *fixed numbers*. We show in Theorem B, which follows immediately from Theorem A, that this is possible in some sense (with a price to be paid). In Theorem B a formula depending on a parameter n of the form:

$$Z_1\,\gamma + Z_2\,\ln 2 + \sum r,$$

tending to zero as $n \to \infty$, is given. Here Z_1, Z_2 are integer numbers (tending to infinity as $n \to \infty$) and the expression $\sum r$ means a finite sum of certain rational numbers involving harmonic numbers, i.e. $\sum_{j=1}^{n} 1/j$, and rational numbers with denominators a power of two.

In the present paper we use ideas from P. Borwein [6], where a very nice proof of the irrationality of certain series is given using a certain integral which we modify slightly to construct an approximation to Euler's constant. Specifically we define the following function for $n \geq 2$:

$$F_n(z) := \frac{1}{2\pi i}\int_{|t|=1} \prod_{k=1}^{n-1} \frac{(1-2^k/t)}{(1-2^k t)}\,\frac{(-1/t)}{(1-2^n t)} \sum_{h=\theta(n)}^{\infty} \frac{1}{(z+2^h/t)} dt, \qquad (1)$$

where $\theta(n)$ is a function from the set of positive integers to itself such that $n \leq \theta(n)$.

Our proof is based on the integral

$$\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} F_n(z)\,dz\,d\tau, \qquad (2)$$

where the rational function $p_n(\tau)$ is defined recursively by

$$p_0(\tau) = 1/(1+\tau), \quad (\tau p_{r-1}(\tau))' = p_r(\tau),$$

and β is a curve which, roughly speaking, encloses the negative integers and goes from $-\infty + i\infty$ to $-\infty - i\infty$. Specifically we take $\beta = \beta_1 \cup \beta_2 \cup \beta_3$, where with $\alpha := 3\pi/4$ one defines β_2 (resp. $-\beta_1$) $:= \{\exp(-i\alpha)t$ (resp. $\exp(i\alpha)t$), $1/2 \leq t \leq \infty\}$. The curve β_3 is the arc of the circle given by the equation $\{\exp(it)/2;\ \alpha \leq t \leq 2\pi - \alpha\}$.

Recall the notation for the q-binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{\Pi_{s=n-m+1}^{n}(1-q^s)}{\Pi_{s=1}^{m}(1-q^s)}, \quad \text{if } 0 < m < n,$$

$\left[\frac{n}{0}\right]_q = \left[\frac{n}{n}\right]_q := 1$, and defined to be zero otherwise (see [5]).

Definition 1. *Define, for $2 \leq n$,*

$$q_{m,n} := (-1)^{m-1} 2^{m(m-1)/2} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2 \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_2,$$

$$a_n := (-1)^n \sum_{m=1}^{n} q_{m,n},$$

$$
\begin{aligned}
b_n &:= -\sum_{m=1}^{n} q_{m,n} \sum_{h=1}^{\theta(n)+m-1} \frac{1}{2^{hn}} \sum_{i=0}^{n-1} (-1)^{n-1-i} 2^{h(n-1-i)} p_i(1), \\
c_n &:= (-1)^{n+1} \sum_{m=1}^{n} q_{m,n} \Big(\theta(n)+m-1\Big), \\
d_n &:= \sum_{k=1}^{n} (-1)^k \frac{p_{k-1}(1)}{2^k-1}, \\
e_n &:= (-1)^n \sum_{m=1}^{n} q_{m,n} \sum_{j=1}^{2^{\theta(n)+m-1}} \frac{1}{j}.
\end{aligned}
$$

Observe that $a_n, c_n \in \mathbb{Z}$, e_n is a sum of harmonic numbers, and by Lemma 3 (a) the coefficient $p_n(1)$ can be written in closed form:

$$
p_n(1) = \frac{1}{2^{n+1}} \sum_{i=0}^{n} (-1)^i \left\langle {n \atop i} \right\rangle,
$$

for $n \geq 1$, and $p_0(1) = 1/2$, where $\left\langle {n \atop i} \right\rangle$ are the Eulerian numbers. Recall that the Eulerian number $\left\langle {n \atop k} \right\rangle$ gives the number of permutations of $\{1, 2, \ldots, n\}$ having k permutation ascents, and one has the formula

$$
\left\langle {n \atop k} \right\rangle = \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k-j+1)^n.
$$

Our main result is the following explicit approximation formula.

Theorem A. *Let $\theta(n)$ be any function from the set of positive integers to itself such that $n \leq \theta(n)$. Then*

$$
\begin{aligned}
&-\gamma a_n + a_n d_n + b_n + e_n + c_n \ln 2 \\
&\qquad = O\Big(\Big(\frac{0.792(n+1)}{\ln(n+2)}\Big)^{n+1} n! \max\Big(\frac{\theta(n) 2^{n(n+1)/2}}{2^{n\theta(n)+\theta(n)}}, \frac{1}{2^{\theta(n)n+n}}\Big)\Big).
\end{aligned}
$$

Note: the constant implied in the O-symbol does not depend on n nor the function $\theta(n)$.

For example taking $\theta(n) = n$ the above O-term is

$$
O\Big(\frac{1}{2^{\frac{n^2}{2} - cn \ln n}}\Big),
$$

where $c > 0$ is some positive constant. The proof of Theorem A is given in Section 2.

Definition 2. *For fixed $0 < \rho < \frac{1}{8}$, set*

$$
\begin{aligned}
g_n &:= 2^{[n^2\rho]} \, \Pi_{k=[n/2]}^{n} (2^k - 1), \\
b'_n &:= (-1)^{n+1} \sum_{m=1}^{n} q_{m,n} \sum_{i=0}^{n-1} \frac{(-1)^i p_i(1)}{(2^{i+1}-1) 2^{(i+1)(n+m-1)}}.
\end{aligned}
$$

Next we specialize $\theta(n) = n$. Indeed one has the following theorem.

Theorem B. *Assume $\theta(n) = n$, $0 < \rho < \frac{1}{8}$. For some constant $0 < c'$,*

$$- \gamma a_n g_n + b'_n g_n + e_n g_n + c_n g_n \ln 2 = \gamma Z_1 + b'_n g_n + e_n g_n + Z_2 \ln 2 = O(2^{-(\frac{1}{8}-\rho)n^2 + c' n \ln n}),$$

where Z_i are integer numbers. Moreover $e_n g_n$ is a sum of harmonic numbers and $b'_n g_n$ is a rational number whose denominator is a power of 2.

Theorem B is proved in Section 3. We make some remarks concerning the last theorem.

One has the following corollary, where $\{x\}$ means the fractional part of x.

Corollary. *If γ is a rational number then:*

a) The sequence $\{b'_n g_n + e_n g_n + c_n g_n \ln 2\}$ cannot be dense in $[A, B] \subset [0, 1]$, $A < B$.

b) The sequence $\{n!(b'_n g_n + e_n g_n + c_n g_n \ln 2)\}$ has $\{0, 1\}$ as its only possible limit points.

The proof, which follows from Theorem B, is given in Section 3.

Remark 1. Notice that $|a_n| \approx 2^{(\frac{3}{2})(n^2-n)}$ as $n \to \infty$, which together with Lemma 7 (iii) gives

$$|a_n g_n| \approx 2^{(\frac{15}{8}+\rho)n^2 + O(n)} \quad \text{as } n \to \infty.$$

Also if $[n^2\rho] \geq n$, then $p_i(1)2^{[n^2\rho]} \in \mathbb{Z}$, $i = 1, \ldots, n-1$, and therefore $b'_n g_n 2^{2n^2-n} \in \mathbb{Z}$.

Remark 2. The coefficient a_n has some similitude with a particular case of the so called q-little Legendre polynomials defined by:

$$P_n(x, q) = \sum_{m=0}^{n} (-1)^m q^{m(m+1)/2 - mn} \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n+m \\ m \end{bmatrix}_q x^m.$$

Notice that

$$a_{n+1} = (-1)^{n+1} \sum_{m=0}^{n} (-1)^m 2^{m(m+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_2 \begin{bmatrix} n+m \\ m \end{bmatrix}_2 \Big(2^n + \frac{2^n - 1}{2^{m+1} - 1}\Big).$$

2. Outline and proof of Theorem A

The proof of Theorem A is rather technical so we give first an outline and the ideas behind it.

In Lemma 1 we decompose the function (1) as

$$F_n(z) = A_n(z) + B_n(z) + C_n(z),$$

and apply the formula (2) to this equality (Lemma 2 is an auxiliary result). That is,

$$\frac{1}{2\pi i}\int_0^1\int_\beta \frac{\tau^{-z}p_n(\tau)}{(-z)^n}F_n(z)dzd\tau$$
$$=\frac{1}{2\pi i}\int_0^1\int_\beta \frac{\tau^{-z}p_n(\tau)}{(-z)^n}\Big(A_n(z)+B_n(z)+C_n(z)dz\Big)d\tau.$$

Integrals along $F_n(z)$ and $C_n(z)$ are estimated in Lemmas 5 and 6 and they give the O-term of Theorem A; these integrals are small in magnitude. The term $B_n(z)$ is a polynomial so the integral corresponding to it is zero. The integral corresponding to $A_n(z)$ gives the linear form $-\gamma a_n + a_n d_n + b_n + e_n + c_n \ln 2$. This is proved in Lemma 4 with the help of Lemma 3. This is basically the proof of Theorem A.

We explain some ideas behind our theorem. A particular case of [6] gives a proof of the irrationality of $\sum_{h=1}^{\infty}\frac{1}{(z+2^h)}$ for rational fixed z using an integral of the form (see [6, formula (1)])

$$\tilde{F}_n(z) := \frac{1}{2\pi i}\int_{|t|=1}\prod_{k=1}^{n-1}\frac{(1+2^k/zt)}{(1-2^kt)}\,\frac{(-1/t)}{(1-2^nt)}\sum_{h=1}^{\infty}\frac{1}{(z+2^h/t)}dt,$$

which is slightly similar to our $F_n(z)$. The integral $\tilde{F}_n(z)$, which can be shown to be small for fixed z and large n, is of the form

$$\tilde{F}_n(z) = \mathrm{Rac}_1(z)\sum_{h=1}^{\infty}\frac{1}{(z+2^h)} + \mathrm{Rac}_2(z),$$

where $\mathrm{Rac}_i(z)$, $i=1,2$, are rational functions in z. On the other hand,

$$1-\gamma = \int_0^1 \frac{\sum_{h=1}^{\infty}\tau^{2^h}}{(1+\tau)}d\tau = \frac{1}{2\pi i}\int_0^1\int_\beta \frac{\tau^{-z}}{1+\tau}\sum_{h=1}^{\infty}\frac{1}{(z+2^h)}dzd\tau$$
$$= \frac{1}{2\pi i}\int_0^1\int_\beta \tau^{-z}p_0(\tau)\sum_{h=1}^{\infty}\frac{1}{(z+2^h)}dzd\tau;$$

the first equality is well-known (we prove it below) and the second one follows from shifting the curve β to the left picking the residues at the poles $z=-2^h$. So this suggests that the integral

$$\frac{1}{2\pi i}\int_0^1\int_\beta \tau^{-z}p_0(\tau)\tilde{F}_n(z)\,dzd\tau$$

is small. But this is not a linear form in γ and $\ln 2$. Indeed other constants of apparently different type appear (for example $\int_0^1 \frac{\sum_{h=1}^{\infty}\frac{\tau^{2^h}}{2^h}}{(1+\tau)}d\tau$).

But the integral

$$\frac{1}{2\pi i}\int_0^1\int_\beta \tau^{-z}p_0(\tau)F_n(z)\,dzd\tau$$

is a linear form in γ and $\ln 2$ only. On the other hand, estimates for this last integral are not good enough because now z is a variable and one needs roughly a factor $1/z^n$ (when z is large) which is not present. This can be corrected with Definition 2 introducing a polynomial $p_n(\tau)$ which is not too large (and, of course, without losing the fact that this is a linear form in γ and $\ln 2$).

Lemma 1. *Let F_n be defined by* (1)*. Then if $n \geq 2$,*

$$F_n(z) = A_n(z) + B_n(z) + C_n(z),$$

where

$$A_n(z) = \sum_{h=1}^{\infty} \frac{1}{(z+2^h)} \sum_{m=1}^{n} (-1)^{m-1} 2^{m(m-1)/2} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2 \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_2 - \sum_{m=1}^{n} (-1)^{m-1} 2^{m(m-1)/2} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2 \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_2 \sum_{h=1}^{\theta(n)+m-1} \frac{1}{(z+2^h)},$$

and $B_n(z)$ is a polynomial in z.

Also $C_n(z) = 0$ if $2^{\theta(n)} > |z|$ and if z belongs to the ring shaped region $\{2^{\theta(n)} \leq 2^{m_0} < |z| < 2^{m_0+1}\}$, $m_0 \in N$, then

$$C_n(z) = -\sum_{h=\theta(n)}^{m_0} \prod_{k=1}^{n-1} \frac{(1+2^{k-h}z)}{(z+2^{k+h})} \frac{z^{n-1}}{(z+2^{n+h})}.$$

Proof. The formula follows using the theory of residues: the product in (1) has simple poles at $t = \frac{1}{2}, \ldots, \frac{1}{2^n}$, which gives

$$\sum_{h=1}^{\infty} \frac{1}{(z+2^h)} \sum_{m=1}^{n} \frac{\Pi_{k=1}^{n-1}(1-2^{k+m})}{\Pi_{k=1,k\neq m}^{n}(1-2^{k-m})} - \sum_{m=1}^{n} \frac{\Pi_{k=1}^{n-1}(1-2^{k+m})}{\Pi_{k=1,k\neq m}^{n}(1-2^{k-m})} \sum_{h=1}^{\theta(n)+m-1} \frac{1}{(z+2^h)},$$

where we have used that

$$\sum_{h=1}^{\infty} \frac{1}{(z+2^h 2^m)} = \sum_{h=1}^{\infty} \frac{1}{(z+2^h)} - \sum_{h=1}^{m} \frac{1}{(z+2^h)}.$$

The term $A_n(z)$ is obtained if one notices that

$$\frac{\Pi_{k=1}^{n-1}(1-2^{k+m})}{\Pi_{k=1,k\neq m}^{n}(1-2^{k-m})} = (-1)^{m-1} 2^{m(m-1)/2} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2 \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_2. \tag{3}$$

The integrand in formula (1) has a pole of order $n-1$ at $t = 0$, which gives the polynomial $B_n(z)$ whose precise form we shall not need.

The term $C_n(z)$ is present due to the possible poles (in the variable t) of $\sum_{h=\theta(n)}^{\infty} \frac{1}{(z+2^h/t)}$: if $2^{\theta(n)} > |z|$ there are no poles in $\{|t| < 1\}$ and therefore $C_n \equiv 0$. If $2^{\theta(n)} \leq 2^{m_0} < |z| < 2^{m_0+1}$ then in the last sum the only poles inside

$\{|t|<1\}$ are those corresponding to $h=\theta(n),\ldots,m_0$. Using residues the formula follows. □

Also we will need an estimate of

$$I_m(z) := \frac{1}{2\pi i}\int_{|t|=1}\prod_{k=1}^{n-1}\frac{(1-2^k/t)}{(1-2^k t)}\frac{(-1/t)}{(1-2^n t)}\frac{1}{(z+2^m/t)}dt, \tag{4}$$

for one has

$$F_n(z) = \sum_{m=\theta(n)}^{\infty} I_m(z). \tag{5}$$

Lemma 2. *Let I_m be defined as above. Then $I_m(z)=0$ if $2^m<|z|$ and*

$$I_m(z) = \prod_{k=1}^{n-1}\frac{(1+2^{k-m}z)}{(z+2^{k+m})}\frac{z^{n-1}}{(z+2^{n+m})},$$

if $2^m>|z|$.

Proof. Let $\delta=\{|t|=1\}$ and $\delta_m=\{|t|=|2^m/z|+1\}$. Then

$$I_m(z) = \frac{1}{2\pi i}\int_{\delta\cup\delta_m}\prod_{k=1}^{n-1}\frac{(1-2^k/t)}{(1-2^k t)}\frac{(-1/t)}{(1-2^n t)}\frac{1}{(z+2^m/t)}dt,$$

for we can change the curve δ_m to an arbitrarily large circle and therefore the integral over δ_m is zero. If $2^m>|z|$ then one uses residues for the pole at $t=-2^m/z$ in the open annulus formed by the two curves δ,δ_m and the result follows (if $2^m<|z|$ there is no pole inside this open annulus). □

Lemma 3. *Set $p_0(\tau):=\frac{1}{1+\tau}$ and define $p_r(\tau)$ recursively by*

$$(\tau p_{r-1}(\tau))' = p_r(\tau).$$

Then:

a) If $n\geq 2$ then $p_{n-1}(\tau)=\frac{e_1}{(1+\tau)^2}+\cdots+\frac{e_n}{(1+\tau)^n}$ with $e_i\in\mathbb{Z}$. Also

$$\max_{\tau\in[0,1]}|p_n(\tau)| \leq \left(\frac{0.792(n+1)}{\ln(n+2)}\right)^{n+1} n!\,,$$

and one has the closed formula

$$p_n(1) = \frac{1}{2^{n+1}}\sum_{i=0}^{n}\left\langle {n \atop i}\right\rangle(-1)^i,$$

for $n\geq 1$, where $\left\langle {n \atop i}\right\rangle$ are the Eulerian numbers.

b) If $n\geq 2$ then

$$\frac{1}{2\pi i}\int_0^1\int_\beta \frac{\tau^{-z}p_n(\tau)}{(-z)^n}\sum_{h=1}^{\infty}\frac{1}{(z+2^h)}dz d\tau = \sum_{k=1}^{n}(-1)^{n-k}\frac{p_{k-1}(1)}{2^k-1}+(-1)^n(1-\gamma).$$

c) If $n \geq 2$ then

$$\int_0^1 p_n(\tau)\tau^{2^h} d\tau$$
$$= \sum_{i=0}^{n-1} (-1)^{n-1-i} 2^{h(n-1-i)} p_i(1) + (-1)^n 2^{nh} \ln 2 + (-1)^{n+1} 2^{nh} \sum_{j=2^{h-1}+1}^{2^h} \frac{1}{j}.$$

Proof. a) The first part follows by induction and is left as an easy exercise to the reader. We prove the inequality stated. First observe that

$$(\tau(\tau(\tau \ldots (\tau p_0(\tau))' \ldots)')')' = p_n(\tau),$$

where the number of derivatives is n. But

$$(\tau(\tau(\tau \ldots (\tau p_0(\tau))' \ldots)')')' = \sum_{k=1}^{n+1} S(n+1,k)\tau^{k-1} p_0^{(k-1)}(\tau),$$

where $S(n,k)$ are the Stirling numbers of the second kind (recall that these numbers have the property $S(1,n) = S(n,n) = 1$ and

$$S(n,k) = kS(n-1,k) + S(n-1,k-1),$$

from which the above formula follows by iteration of this last formula). Using these identities we have if $\tau \in [0,1]$

$$|p_n(\tau)| \leq \sum_{k=1}^{n+1} S(n+1,k) \max_{i=0,\ldots,n} |p_0^{(i)}(\tau)| \leq \sum_{k=1}^{n+1} S(n+1,k) n! = B(n+1)n!\,,$$

where $B(n)$ is the n-th Bell number, see [14]. The inequality in the lemma follows from a result in [3], namely: if $n \in N$ then

$$B(n) < \Big(\frac{0.792n}{\ln(n+1)}\Big)^n.$$

Finally, the closed form formula is proved as follows: using Cauchy's formula in the above formulae gives

$$p_n(1) = \frac{1}{2\pi i} \int_{|z-1|=1} \sum_{k=1}^{n+1} S(n+1,k)(k-1)!\, \frac{1}{(z-1)^k(1+z)}\, dz.$$

But if $n \geq 1$ then

$$(-1)^{n+1} Li_{-n}(z) = \sum_{k=1}^{n+1} S(n+1,k)(k-1)!\, \frac{1}{(z-1)^k},$$

where $Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ is the polylogarithm function, which gives

$$p_n(1) = \frac{(-1)^{n+1}}{2\pi i} \int_{|z-1|=1} \frac{Li_{-n}(z)}{1+z} dz = \frac{(-1)^n}{2\pi i} \int_{|z+1|=1} \frac{Li_{-n}(z)}{1+z} dz.$$

The proof of the last equality (and the closed form formula) follow from here and the known identity

$$Li_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{i=0}^{n} \left\langle \frac{n}{i} \right\rangle z^{n-i}.$$

Hint: $\int_{|z|=R} \frac{Li_{-n}(z)}{1+z} dz = 0$, if $R > 2$, (say) for $|\frac{Li_{-n}(z)}{1+z}| = O(1/|z|^2)$ as $|z| \to \infty$. Now use residues.

b) From the well-known formula

$$\int_0^1 \frac{\sum_{h=1}^{\infty} \tau^{2^h}}{(1+\tau)} d\tau = 1 - \gamma,$$

and, for $0 < \tau < 1$,

$$\frac{1}{2\pi i} \int_\beta \frac{\tau^{-z}}{(z+2^h)} dz = \tau^{2^h}$$

(this last formula follows by deforming the curve β, moving it arbitrarily to a left half plane and taking into account the pole at $z = -2^h$), one has

$$\frac{1}{2\pi i} \int_0^1 \int_\beta \tau^{-z} p_0(\tau) \sum_{h=1}^{\infty} \frac{1}{(z+2^h)} dz d\tau$$
$$= \frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{-z}}{1+\tau} \sum_{h=1}^{\infty} \frac{1}{(z+2^h)} dz d\tau = \int_0^1 \frac{\sum_{h=1}^{\infty} \tau^{2^h}}{(1+\tau)} d\tau = 1 - \gamma. \quad (6)$$

Also

$$\int_0^1 \frac{\tau^{2^h}}{1+\tau} d\tau = \ln 2 - \sum_{j=1}^{2^h} \frac{(-1)^{j+1}}{j}. \quad (7)$$

(Remark: the above formula for $1-\gamma$ follows from (7) and the identity

$$\sum_{j=2^{h-1}+1}^{2^h} \frac{1}{j} = \sum_{j=1}^{2^h} \frac{(-1)^{j+1}}{j}. \quad (8)$$

Indeed adding (7) for $h = 1, \dots, N$ and using (8) we get $\int_0^1 \frac{\sum_{h=1}^{N} \tau^{2^h}}{(1+\tau)} d\tau = N \ln 2 - \sum_{j=2}^{2^N} 1/j$, and the formula follows.)

Now observe that integrating by parts and using $(\tau p_{r-1}(\tau))' = p_r(\tau)$ yield

$$\int_0^1 p_r(\tau) \sum_{h=1}^{\infty} \frac{\tau^{2^h}}{2^{hr}} d\tau = \frac{1}{2^r - 1} p_{r-1}(1) - \int_0^1 p_{r-1}(\tau) \sum_{h=1}^{\infty} \frac{\tau^{2^h}}{2^{h(r-1)}} d\tau.$$

The lemma follows iterating this last formula, using (6) and noting that

$$\frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{-z} p_r(\tau)}{(-z)^r} \sum_{h=1}^{\infty} \frac{1}{(z+2^h)} dz d\tau = \int_0^1 p_r(\tau) \sum_{h=1}^{\infty} \frac{\tau^{2^h}}{2^{rh}} d\tau,$$

(same proof as (6)). This proves part b).

c) Integrating by parts and using $(\tau p_{r-1}(\tau))' = p_r(\tau)$ yield:

$$\int_0^1 p_n(\tau)\tau^{2^h} d\tau = p_{n-1}(1) - 2^h \int_0^1 p_{n-1}(\tau)\tau^{2^h} d\tau.$$

Iterating this one gets

$$\int_0^1 p_n(\tau)\tau^{2^h} d\tau = \sum_{i=0}^{n-1} (-1)^{n-1-i} 2^{h(n-1-i)} p_i(1) + (-1)^n 2^{nh} \int_0^1 \frac{\tau^{2^h}}{1+\tau} d\tau.$$

Now use (7) and (8) to obtain (c). □

The following lemma is a kind of linear form for γ.

Lemma 4. *The following formula holds: if $2 \le n$ then*

$$\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} F_n(z) dz d\tau = -a_n\gamma + a_n d_n + b_n + e_n + c_n \ln 2$$
$$+ \frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} C_n(z) dz d\tau,$$

where $C_n(z)$ is defined in Lemma 1, γ is Euler's constant, and a_n, b_n, c_n, d_n, e_n are as in Theorem A.

Proof. Recall that from Lemma 1 one has

$$F_n(z) = A_n(z) + B_n(z) + C_n(z).$$

First note that $\int_\beta \tau^{-z} z^m dz = 0$ if $m \in \mathbb{Z}$ (again deform arbitrarily the curve to a left half-plane), and therefore

$$\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} B_n(z)\, dz d\tau = 0.$$

So we are left with the integral over A_n. From the definition of $A_n(z)$ in Lemma 1 and Lemma 3 (b) one gets:

$$\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} A_n(z) dz d\tau$$
$$= a_n(1-\gamma) + a_n d_n - \sum_{m=1}^{n} q_{m,n} \sum_{h=1}^{\theta(n)+m-1} \frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} \frac{1}{(z+2^h)} dz d\tau,$$

where $a_n, d_n, q_{m,n}$ are defined in Theorem A. Now notice that

$$\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} \frac{1}{(z+2^h)} dz d\tau = \frac{1}{2^{hn}} \int_0^1 p_n(\tau)\tau^{2^h} d\tau,$$

and use Lemma 3 (c). This gives (here $e_n' := (-1)^n \sum_{m=1}^n q_{m,n} \sum_{j=2}^{2^{\theta(n)+m-1}} \frac{1}{j}$)

$$\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} A_n(z) dz d\tau = a_n(1-\gamma) + a_n d_n + b_n + e_n' + c_n \ln 2,$$

where b_n, c_n are defined in Theorem A. The lemma follows by noticing that $a_n + e'_n = e_n$, where e_n is defined in Theorem A. □

In the following lemmas (5 and 6) we estimate the integrals that appear in Lemma 4; we remark that the constant implied in the O-symbol in these lemmas is absolute.

Lemma 5. *If* $2 \le n$ *then*

$$\Big|\frac{1}{2\pi i}\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} F_n(z)\, dz d\tau\Big| = O\Big(\frac{1}{2^{\theta(n)n+n}} \max_{\tau\in[0,1]} |p_n(\tau)|\Big).$$

Proof. We will use formula (5); so we first estimate $I_m(z)$ with $m \ge \theta(n) \ge n$. Recall that by Lemma 2 one has $I_m(z) = 0$ if $|z| > 2^m$. We chop the curve β in pieces: let β'_r be the part of the curve β in $\{2^{r-1} \le |z| < 2^r\}$, $r = 1, 2, \dots, m$. Then using Lemma 2, taking absolute values in the product expression and writing $(1 + 2^a) = 2^a(1 + 2^{-a})$ whenever $a \ge 1$, one gets

$$\max_{z\in\beta'_r}\Big|\frac{I_m(z)}{(-z)^n}\Big| \le O\Big(\frac{2^{(n-1-m+r)(n-m+r)/2}}{2^{mn+n(n+1)/2}2^r}\Big) \le O\Big(\frac{1}{2^{mn+n+r}}\Big),$$

if $1 \le n-1-m+r$, where the constant implied in the O-symbol is absolute. The last inequality follows because $2^{(n-1-m+r)(n-m+r)/2} \le 2^{(n-1)n/2}$ if $r = 1, \dots, m$.

If $n - 1 - m + r \le 0$ then

$$\max_{z\in\beta'_r}\Big|\frac{I_m(z)}{(-z)^n}\Big| \le O\Big(\frac{1}{2^{mn+n(n+1)/2}2^r}\Big) \le O\Big(\frac{1}{2^{mn+n+r}}\Big).$$

Now if $L_{\beta'_r}(\le 2^r)$ is the length of any of the two segments which compose β'_r then

$$\begin{aligned}\int_0^1 \int_{\beta'_r} \Big|\frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z)\Big| dz d\tau &\le 2L_{\beta'_r}\int_0^1 \tau^{2^{r-1}\cos(\pi-\alpha)}|p_n(\tau)| \max_{z\in\beta'_r}\Big|\frac{I_m(z)}{(-z)^n}\Big| d\tau\\ &\le O\Big(\frac{1}{2^{mn+n+r}} \max_{\tau\in[0,1]}|p_n(\tau)|\Big),\end{aligned}$$

where α is the angle defined at the beginning in the definition of the curves β_1, β_2.

Finally let β'_0 be the part of the curve β in $|z| < 1$. It is easy to see that

$$\max_{z\in\beta'_0}\Big|\frac{I_m(z)}{(-z)^n}\Big| \le O\Big(\frac{1}{2^{mn+n(n+1)/2}}\Big),$$

and

$$\int_0^1 \int_{\beta'_0} \Big|\frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z)\Big| dz d\tau \le O\Big(\frac{1}{2^{mn+n(n+1)/2}} \max_{\tau\in[0,1]}|p_n(\tau)|\Big).$$

All this gathers to give

$$\begin{aligned}\Big|\int_0^1\int_\beta \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) dz d\tau\Big| &= \Big|\int_0^1 \int_{\bigcup_{i=0}^m \beta'_i} \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) dz d\tau\Big|\\ &\le \int_0^1 \int_{\bigcup_{i=0}^m \beta'_i} \Big|\frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z)\Big| dz d\tau = O\Big(\frac{1}{2^{mn+n}} \max_{\tau\in[0,1]}|p_n(\tau)|\Big),\end{aligned}$$

and the lemma follows using this last formula and formula (5). □

Lemma 6. *If* $2 \le n$ *then*

$$\Big|\frac{1}{2\pi i}\int_0^1\int_\beta \frac{\tau^{-z}p_n(\tau)}{(-z)^n}C_n(z)dzd\tau\Big| = O\Big(\frac{\theta(n)2^{n(n+1)/2}}{2^{n\theta(n)+\theta(n)}}\max_{\tau\in[0,1]}|p_n(\tau)|\Big).$$

Proof. Set $D_h(z) := \prod_{k=1}^{n-1}\frac{(1+2^{k-h}z)}{(z+2^{k+h})}\ \frac{z^{-1}}{(z+2^{n+h})}$. Use the definition of $C_n(z)$ as given in Lemma 1, to notice that $C_n(z)/(-z)^n = \pm\sum_{h=\theta(n)}^{m_0} D_h(z)$ for $2^{m_0} < |z| < 2^{m_0+1}$, $\theta(n) \le m_0 \in N$. Using the same notation as in the proof of Lemma 5,

$$\max_{z\in\beta'_{m_0+1}} |D_h(z)| = O\left(\frac{2^{\sum_{k=1}^{n-1}(k-h+m_0+1)}}{2^{\sum_{k=1}^{n}\max\{m_0,k+h\}}2^{m_0}}\right) = O\left(\frac{2^{(n-1)(m_0+1-h)+n(n-1)/2}}{2^{\sum_{k=1}^{n}\max\{m_0,k+h\}}2^{m_0}}\right).$$

Now we have two cases. One case is when $h \le m_0 - n$ and therefore

$$\sum_{k=1}^{n}\max\{m_0, k+h\} = m_0 n.$$

In this case

$$\max_{z\in\beta'_{m_0+1}} |D_h(z)| = O\Big(\frac{2^{-(h-1)(n-1)+n(n-1)/2}}{2^{2m_0}}\Big) = O\Big(\frac{2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{2m_0}}\Big),$$

where the last inequality follows because $\theta(n) \le h$.

The other case is when h satisfies $m_0 - n + 1 \le h \le m_0$. In this case

$$\sum_{k=1}^{n}\max\{m_0, k+h\} = m_0 n + (h-m_0+n)(h-m_0+n+1)/2.$$

Thus

$$\begin{aligned}\max_{z\in\beta'_{m_0+1}} |D_h(z)| &= O\Big(\frac{2^{-(h-1)(n-1)+n(n-1)/2-(h-m_0+n)(h-m_0+n+1)/2}}{2^{2m_0}}\Big)\\ &= O\Big(\frac{2^{-(h-1)(n-1)+n(n-1)/2}}{2^{2m_0}}\Big) = O\Big(\frac{2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{2m_0}}\Big),\end{aligned}$$

This gives

$$\max_{z\in\beta'_{m_0+1}} \Big|\frac{C_n(z)}{(-z)^n}\Big| = O\Big(\frac{m_0 2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{2m_0}}\Big).$$

Therefore one has

$$\begin{aligned}&\int_0^1\int_{\beta'_{m_0+1}}\Big|\frac{\tau^{-z}p_n(\tau)}{(-z)^n}C_n(z)\Big|dzd\tau\\ &\qquad\le 2L_{\beta'_{m_0+1}}\int_0^1 \tau^{2^{m_0}\cos(\pi-\alpha)}|p_n(\tau)|\max_{z\in\beta'_{m_0+1}}\Big|\frac{C_n(z)}{(-z)^n}\Big|d\tau\\ &\qquad\qquad= O\Big(\frac{m_0 2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{2m_0}}\max_{\tau\in[0,1]}|p_n(\tau)|\Big).\end{aligned}$$

Now using this last formula and (recall $C_n(z) = 0$ if $|z| < 2^{\theta(n)}$)

$$\int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} C_n(z) dz d\tau = \sum_{m_0=\theta(n)}^{\infty} \int_0^1 \int_{\beta'_{m_0+1}} \frac{\tau^{-z} p_n(\tau)}{(-z)^n} C_n(z) dz d\tau,$$

one gets

$$\Big| \frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} C_n(z) dz d\tau \Big| = O\Big(\frac{\theta(n) 2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{2\theta(n)}} \max_{\tau\in[0,1]} |p_n(\tau)| \Big).$$

The lemma follows simplifying the O-symbol. □

Theorem A is a direct consequence of Lemmas 3 (a), 4, 5 and 6.

3. Proofs of Theorem B and of Corollary

We need first the following lemma.

Lemma 7. *Let a_n, c_n, d_n be defined as in Theorem A. Then*

i) $a_n, c_n \in \mathbb{Z}$.

ii) $d_n 2^n \Pi_{k=[n/2]}^n (2^k - 1) \in \mathbb{Z}$.

iii) For some constant $0 < c$, $\Pi_{k=[n/2]}^n (2^k - 1) = O(2^{\frac{3}{8}n^2 + cn})$.

Proof. The proof of (i-ii) is immediate using Lemma 3 (a), i.e. $2^n p_{n-1}(1) \in \mathbb{Z}$. Part (iii) is easy and is left to the reader. □

Proof of Theorem B. We apply Theorem A in its simplest form, i.e. we take $\theta(n) = n$. In this case the O-term of Theorem A is

$$O\Big(\frac{1}{2^{\frac{n^2}{2} - cn \ln n}} \Big),$$

where $c > 0$ is some positive constant. Next one has

$$b'_n = a_n d_n + b_n.$$

Thus Theorem A yields

$$-\gamma a_n + b'_n + e_n + c_n \ln 2 = O\Big(\frac{1}{2^{\frac{n^2}{2} - cn \ln n}} \Big).$$

Multiplying this by $g_n = 2^{[n^2 \rho]} \Pi_{k=[n/2]}^n (2^k - 1)$ gives the desired result if one notices that by Lemma 7

$$g_n = O(2^{(\frac{3}{8}+\rho)n^2 + cn}),$$

and that $a_n g_n, c_n g_n \in \mathbb{Z}$. This finishes the proof of Theorem B. □

Proof of Corollary. We prove the remark for the second sequence. Note that if γ were a rational number then $\gamma n! a_n g_n \in \mathbb{Z}$ for $n_0 < n$. Also $n! = O(2^{c'' n \ln n})$ $(0 < c'')$ and therefore by Theorem B

$$n!(-\gamma a_n g_n + b'_n g_n + e_n g_n + c_n g_n \ln 2) \to 0,$$

and the assertion follows from this.

For the first sequence the proof is similar and is left to the reader. □

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