AN APPROXIMATION FORMULA FOR
EULER–MASCHERONI’S CONSTANT

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Abstract. A fast approximation formula for Euler–Mascheroni’s constant based on a certain integral due to Peter Borwein is given. This asymptotic formula for \( \gamma \) involves harmonic numbers, \( \ln 2 \) and rational numbers whose denominators are powers of 2.

1. Introduction and results

There are many results concerning the arithmetic nature of Euler–Mascheroni’s constant \( \gamma := \lim_{N \to \infty} 1 + 1/2 + 1/3 + \cdots + 1/N - \ln N \) (or approximations of it). P. Bundschuh (see [7]) proved that either \( \gamma \) is transcendental or certain roots of transcendental equations involving the gamma function and its derivative are irrational. J. Sondow (see [12]) proved some neat criteria for the irrationality of \( \gamma \); many formulae for \( \gamma \) were discovered by him and can be found online in his personal homepage. Extensions and refinements of these results are given in [8] and [9]. A. Aptekarev was the first to prove that either \( \gamma \) or the so called Gompertz’s constant is irrational (see [1]). T. Rivoal has recently extended this result in [11]. T. Rivoal and A. Aptekarev constructed rapidly converging sequences to \( \gamma \) or combinations of \( \gamma \) and logarithms (see [2, 10, 11]). S. Ramanujan has given many formulae for \( \gamma \) which can be found in [4]; these have been proved thanks to B. Berndt’s work. Also Wolfram’s website contains many other formulae.

Sondow’s construction is based on the famous integrals introduced by F. Beukers for a simplified proof of Apery’s result that \( \zeta(3) \) is irrational (see also [13]). One of Sondow’s criteria is the following: If

\[
S_n := \prod_{k=1}^{n} \min(k-1,n-k) \prod_{i=0}^{n-i} \prod_{j=i+1}^{n} \frac{(n+k)^2 \cdot \text{LCM}(1,\ldots,2n)/j^2}{(n+i)^2},
\]

and \( \{\ln S_n\} \geq 1/2^n \) infinitely often, then \( \gamma \) is irrational. Here \( \{x\} \) means the fractional part of \( x \).

Numerical computations suggest that \( \{\ln S_n\} \) is dense in \((0,1)\).

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In this note we prove, in Theorem A below, a fast approximation formula for $\gamma$. The idea is to have an approximation formula involving the logarithms of fixed numbers. We show in Theorem B, which follows immediately from Theorem A, that this is possible in some sense (with a price to be paid). In Theorem B a formula depending on a parameter $n$ of the form:

$$Z_1 \gamma + Z_2 \ln 2 + \sum r,$$

tending to zero as $n \to \infty$, is given. Here $Z_1, Z_2$ are integer numbers (tending to infinity as $n \to \infty$) and the expression $\sum r$ means a finite sum of certain rational numbers involving harmonic numbers, i.e. $\sum_{j=1}^{n} 1/j$, and rational numbers with denominators a power of two.

In the present paper we use ideas from P. Borwein [6], where a very nice proof of the irrationality of certain series is given using a certain integral which we modify slightly to construct an approximation to Euler’s constant. Specifically we define the following function for $n \geq 2$:

$$F_n(z) := \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n-1} \frac{(1-2^k/t)}{(1-2^k t)} \frac{(-1/t)}{(1-2^n t)} \sum_{h=\theta(n)}^{\infty} \frac{1}{(z+2^h/t)} dt,$$

(1)

where $\theta(n)$ is a function from the set of positive integers to itself such that $n \leq \theta(n)$.

Our proof is based on the integral

$$\frac{1}{2\pi i} \int_{0}^{1} \int_{-\beta}^{\beta} \frac{\tau^{-z} p_n(\tau)}{(-z)^n} F_n(z) dz d\tau,$$

(2)

where the rational function $p_n(\tau)$ is defined recursively by

$$p_0(\tau) = 1/(1+\tau), \quad (\tau p_{r-1}(\tau))^' = p_r(\tau),$$

and $\beta$ is a curve which, roughly speaking, encloses the negative integers and goes from $-\infty + i\infty$ to $-\infty - i\infty$. Specifically we take $\beta = \beta_1 \cup \beta_2$ union $\beta_3$, where with $\alpha := 3\pi/4$ one defines $\beta_2$ (resp. $-\beta_1$) := $\{\exp(-i\alpha)t \ (\text{resp.} \ \exp(i\alpha)t), \ 1/2 \leq t \leq \infty\}$. The curve $\beta_3$ is the arc of the circle given by the equation $\{\exp(it)/2, \ \alpha \leq t \leq 2\pi - \alpha\}$.

Recall the notation for the $q$-binomial coefficients

$$\left[ \frac{n}{m} \right]_q := \frac{\prod_{s=m-n+1}^{n} (1-q^s)}{\prod_{s=1}^{m} (1-q^s)}, \quad \text{if } 0 < m < n,$$

$\left[ \frac{n}{n} \right]_q = [\frac{n}{n}]_q := 1$, and defined to be zero otherwise (see [5]).

**Definition 1.** Define, for $2 \leq n$,

$$q_{m,n} := (-1)^{n-1} 2^{m(m-1)/2} \left[ \frac{n-1}{m-1} \right]_q \left[ \frac{n+m-1}{m} \right]_q,$$

$$a_n := (-1)^n \sum_{m=1}^{n} q_{m,n},$$

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\[ b_n := - \sum_{m=1}^{n} q_{m,n} \sum_{h=1}^{\theta(n)+m-1} \frac{1}{2^n h^n} \sum_{i=0}^{n-1} (-1)^{n-1-i} 2^{h(n-1-i)} p_i(1), \]

\[ c_n := (-1)^{n+1} \sum_{m=1}^{n} q_{m,n} \left( \theta(n) + m - 1 \right), \]

\[ d_n := \sum_{k=1}^{n} (-1)^k p_{k-1}(1) \frac{2^k - 1}{2^k}, \]

\[ e_n := (-1)^n \sum_{m=1}^{n} q_{m,n} \sum_{j=1}^{2^{\theta(n)+m-1}} \frac{1}{j}. \]

Observe that \( a_n, c_n \in \mathbb{Z} \), \( e_n \) is a sum of harmonic numbers, and by Lemma 3 (a) the coefficient \( p_n(1) \) can be written in closed form:

\[ p_n(1) = \frac{1}{2^{n+1}} \sum_{i=0}^{n} (-1)^i \left\langle \frac{n}{i} \right\rangle, \]

for \( n \geq 1 \), and \( p_0(1) = 1/2 \), where \( \left\langle \frac{n}{k} \right\rangle \) are the Eulerian numbers. Recall that the Eulerian number \( \left\langle \frac{n}{k} \right\rangle \) gives the number of permutations of \( \{1, 2, \ldots, n\} \) having \( k \) permutation ascents, and one has the formula

\[ \left\langle \frac{n}{k} \right\rangle = \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k-j+1)^n. \]

Our main result is the following explicit approximation formula.

**Theorem A.** Let \( \theta(n) \) be any function from the set of positive integers to itself such that \( n \leq \theta(n) \). Then

\[ - \gamma a_n + a_n d_n + b_n + e_n + c_n \ln 2 \]

\[ = O\left( \frac{0.792(n+1)}{\ln(n+2)} \right)^{n+1} n! \max \left( \frac{\theta(n)2^{n(n+1)/2}}{2n^{\theta(n)+\theta(n)}}, \frac{1}{2^{\theta(n)n+n}} \right). \]

Note: the constant implied in the \( O \)-symbol does not depend on \( n \) nor the function \( \theta(n) \).

For example taking \( \theta(n) = n \) the above \( O \)-term is

\[ O\left( \frac{1}{2^{\frac{n^2}{2}} - c n \ln n} \right), \]

where \( c > 0 \) is some positive constant. The proof of Theorem A is given in Section 2.

**Definition 2.** For fixed \( 0 < \rho < \frac{1}{8} \), set

\[ g_n := 2^{[n^2 \rho]} \prod_{k=\lfloor n/2 \rfloor}^{n} (2^k - 1), \]

\[ b'_n := (-1)^{n+1} \sum_{m=1}^{n} q_{m,n} \sum_{i=0}^{n-1} \frac{(-1)^i p_i(1)}{(2^i+1 - 1)2^{i+1}(n+m-1)}. \]
Next we specialize \( \theta(n) = n \). Indeed one has the following theorem.

**Theorem B.** Assume \( \theta(n) = n, 0 < \rho < \frac{1}{8} \). For some constant \( 0 < c' \),

\[
- \gamma a_n g_n + b'_n g_n + e_n g_n + c_n g_n \ln 2 = \gamma Z_1 + b'_n g_n + e_n g_n + Z_2 \ln 2 = O(2^{-(\frac{1}{8} - \rho)n^2 + c' n \ln n}),
\]

where \( Z_i \) are integer numbers. Moreover \( e_n g_n \) is a sum of harmonic numbers and \( b'_n g_n \) is a rational number whose denominator is a power of 2.

Theorem B is proved in Section 3. We make some remarks concerning the last theorem.

One has the following corollary, where \( \{x\} \) means the fractional part of \( x \).

**Corollary.** If \( \gamma \) is a rational number then:

a) The sequence \( \{b'_n g_n + e_n g_n + c_n g_n \ln 2\} \) cannot be dense in \([A, B] \subset [0, 1], A < B\).

b) The sequence \( \{n!(b'_n g_n + e_n g_n + c_n g_n \ln 2)\} \) has \( \{0, 1\} \) as its only possible limit points.

The proof, which follows from Theorem B, is given in Section 3.

**Remark 1.** Notice that \( |a_n| \approx 2^{\left(\frac{3}{2}\right)(n^2 - n)} \) as \( n \to \infty \), which together with Lemma (iii) gives

\[
|a_n g_n| \approx 2^{\left(\frac{15}{8} + \rho\right)n^2 + O(n)} \text{ as } n \to \infty.
\]

Also if \( [n^2 \rho] \geq n \), then \( p_i(1)2^{[n^2 \rho]} \in \mathbb{Z}, i = 1, \ldots, n - 1 \), and therefore \( b'_n g_n 2^{2n^2 - n} \in \mathbb{Z} \).

**Remark 2.** The coefficient \( a_n \) has some similitude with a particular case of the so called \( q \)-little Legendre polynomials defined by:

\[
P_n(x, q) = \sum_{m=0}^{n} (-1)^m q^{m(m+1)/2 - mn} \left[ \frac{n}{m} \right] q \left[ \frac{n + m}{m} \right] q x^m.
\]

Notice that

\[
a_{n+1} = (-1)^{n+1} \sum_{m=0}^{n} (-1)^m 2^{m(m+1)/2} \left[ \frac{n}{m} \right] 2 \left[ \frac{n + m}{m} \right] 2 \left( 2^n + \frac{2^n - 1}{2^{m+1} - 1} \right).
\]

2. Outline and proof of Theorem A

The proof of Theorem A is rather technical so we give first an outline and the ideas behind it.

In Lemma we decompose the function (1) as

\[
F_n(z) = A_n(z) + B_n(z) + C_n(z),
\]
and apply the formula (2) to this equality (Lemma 2 is an auxiliary result). That is,

\[ \frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} F_n(z) \, dz \, d\tau = \frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{-z} p_n(\tau)}{(-z)^n} (A_n(z) + B_n(z) + C_n(z) \, dz) \, d\tau. \]

Integrals along \( F_n(z) \) and \( C_n(z) \) are estimated in Lemmas 5 and 6 and they give the \( O \)-term of Theorem A; these integrals are small in magnitude. The term \( B_n(z) \) is a polynomial so the integral corresponding to it is zero. The integral corresponding to \( A_n(z) \) gives the linear form \(-\gamma a_n + a_n d_n + b_n + e_n + c_n \ln 2\). This is proved in Lemma 4 with the help of Lemma 3. This is basically the proof of Theorem A.

We explain some ideas behind our theorem. A particular case of [6] gives a proof of the irrationality of \( \sum_{h=1}^\infty \frac{1}{(z+2^h)} \) for rational fixed \( z \) using an integral of the form (see [6, formula (1)])

\[ \tilde{F}_n(z) := \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n-1} \frac{1 + 2^k / z t}{1 - 2^k t} \frac{(-1/t)}{(1 - 2^n t)} \sum_{h=1}^\infty \frac{1}{(z + 2^h/t)} \, dt, \]

which is slightly similar to our \( F_n(z) \). The integral \( \tilde{F}_n(z) \), which can be shown to be small for fixed \( z \) and large \( n \), is of the form

\[ \tilde{F}_n(z) = \text{Rac}_1(z) \sum_{h=1}^\infty \frac{1}{(z + 2^h)} + \text{Rac}_2(z), \]

where \( \text{Rac}_i(z), i = 1, 2, \) are rational functions in \( z \). On the other hand,

\[ 1 - \gamma = \int_0^1 \sum_{h=1}^\infty \frac{\tau^{2^h}}{(1 + \tau)} \, d\tau = \frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{-z}}{1 + \tau} \sum_{h=1}^\infty \frac{1}{(z + 2^h)} \, dz \, d\tau \]

\[ = \frac{1}{2\pi i} \int_0^1 \int_\beta \tau^{-z} p_0(\tau) \sum_{h=1}^\infty \frac{1}{(z + 2^h)} \, dz \, d\tau; \]

the first equality is well-known (we prove it below) and the second one follows from shifting the curve \( \beta \) to the left picking the residues at the poles \( z = -2^h \). So this suggests that the integral

\[ \frac{1}{2\pi i} \int_0^1 \int_\beta \tau^{-z} p_0(\tau) \tilde{F}_n(z) \, dz \, d\tau \]

is small. But this is not a linear form in \( \gamma \) and \( \ln 2 \). Indeed other constants of apparently different type appear (for example \( \int_0^1 \frac{\sum_{h=1}^\infty \tau^{2^h}}{(1 + \tau)} d\tau \)).

But the integral

\[ \frac{1}{2\pi i} \int_0^1 \int_\beta \tau^{-z} p_0(\tau) F_n(z) \, dz \, d\tau \]
is a linear form in $\gamma$ and $\ln 2$ only. On the other hand, estimates for this last integral are not good enough because now $z$ is a variable and one needs roughly a factor $1/z^n$ (when $z$ is large) which is not present. This can be corrected with Definition [5] introducing a polynomial $p_\tau(\tau)$ which is not too large (and, of course, without losing the fact that this is a linear form in $\gamma$ and $\ln 2$).

**Lemma 1.** Let $F_n$ be defined by (1). Then if $n \geq 2$,

$$F_n(z) = A_n(z) + B_n(z) + C_n(z),$$

where

$$A_n(z) = \sum_{h=1}^{\infty} \frac{1}{(z + 2^h)} \sum_{m=1}^{n} (-1)^{m-1} 2^{m(m-1)/2} \left( \frac{n-1}{n-1} \right)_2 \left( \frac{n+m+1}{m} \right)_2$$

$$- \sum_{m=1}^{n} (-1)^{m-1} 2^{m(m-1)/2} \left( \frac{n-1}{n-1} \right)_2 \left( \frac{n+m+1}{m} \right)_2 \sum_{h=1}^{\theta(n)+m-1} \frac{1}{(z + 2^h)},$$

and $B_n(z)$ is a polynomial in $z$.

Also $C_n(z) = 0$ if $2^{\theta(n)} > |z|$ and if $z$ belongs to the ring shaped region $\{2^{\theta(n)} < 2^{m_0} < |z| < 2^{m_0+1}\}$, $m_0 \in N$, then

$$C_n(z) = - \sum_{h=\theta(n)}^{m_0} \prod_{k=1}^{n-1} \frac{1 + 2^{k-h}z}{(z + 2^{k+h})} \frac{z^{n-1}}{(z + 2^{n+h})}.$$

**Proof.** The formula follows using the theory of residues: the product in (1) has simple poles at $t = \frac{1}{2}, \ldots, \frac{1}{2^n}$, which gives

$$\sum_{h=1}^{\infty} \frac{1}{(z + 2^h)} \sum_{m=1}^{n} \Pi_{k=1}^{n-1} (1 - 2^{k+m})$$

$$- \sum_{m=1}^{n} \Pi_{k=1, k \neq m}^{n-1} (1 - 2^{k-m}) \frac{\theta(n)+m-1}{(z + 2^h)},$$

where we have used that

$$\sum_{h=1}^{\infty} \frac{1}{(z + 2^h 2^m)} = \sum_{h=1}^{\infty} \frac{1}{(z + 2^h)} - \sum_{h=1}^{m_0} \frac{1}{(z + 2^h)}.$$

The term $A_n(z)$ is obtained if one notices that

$$\frac{\Pi_{k=1}^{n-1} (1 - 2^{k+m})}{\Pi_{k=1, k \neq m}^{n} (1 - 2^{k-m})} = (-1)^{m-1} 2^{m(m-1)/2} \left( \frac{n-1}{n-1} \right)_2 \left( \frac{n+m+1}{m} \right)_2.$$  \hfill (3)

The integrand in formula (1) has a pole of order $n - 1$ at $t = 0$, which gives the polynomial $B_n(z)$ whose precise form we shall not need.

The term $C_n(z)$ is present due to the possible poles (in the variable $t$) of

$$\sum_{h=\theta(n)}^{m_0} \frac{1}{(z + 2^h)},$$

if $2^{\theta(n)} > |z|$ there are no poles in $\{|t| < 1\}$ and therefore $C_n \equiv 0$. If $2^{\theta(n)} \leq 2^{m_0} < |z| < 2^{m_0+1}$ then in the last sum the only poles inside
\{ |t| < 1 \} are those corresponding to \( h = \theta(n), \ldots, m_0 \). Using residues the formula follows.

Also we will need an estimate of

\[
I_m(z) := \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n-1} \frac{(1 - 2^k/t)}{(1 - 2^k t)} \frac{(-1/t)}{(1 - 2^n t)} \frac{1}{(z + 2^m/t)} dt,
\]

for one has

\[
F_n(z) = \sum_{m=\theta(n)}^{\infty} I_m(z).
\]

**Lemma 2.** Let \( I_m \) be defined as above. Then \( I_m(z) = 0 \) if \( 2^m < |z| \) and

\[
I_m(z) = \prod_{k=1}^{n-1} \frac{1 + 2^{k-m} z}{(z + 2^{k+m})} \frac{z^{-1}}{(z + 2^{n+m})},
\]

if \( 2^m > |z| \).

**Proof.** Let \( \delta = \{ |t| = 1 \} \) and \( \delta_m = \{ |t| = 2^m/|z| + 1 \} \). Then

\[
I_m(z) = \frac{1}{2\pi i} \int_{\delta \cup \delta_m} \prod_{k=1}^{n-1} \frac{(1 - 2^k/t)}{(1 - 2^k t)} \frac{(-1/t)}{(1 - 2^n t)} \frac{1}{(z + 2^m/t)} dt,
\]

for we can change the curve \( \delta_m \) to an arbitrarily large circle and therefore the integral over \( \delta_m \) is zero. If \( 2^m > |z| \) then one uses residues for the pole at \( t = -2^m/z \) in the open annulus formed by the two curves \( \delta, \delta_m \) and the result follows (if \( 2^m < |z| \) there is no pole inside this open annulus). \( \square \)

**Lemma 3.** Set \( p_0(\tau) := \frac{1}{1+\tau} \) and define \( p_r(\tau) \) recursively by

\[
(\tau p_{r-1}(\tau))' = p_r(\tau).
\]

Then:

a) If \( n \geq 2 \) then \( p_n-1(\tau) = \frac{e_1}{(1+\tau)^2} + \cdots + \frac{e_n}{(1+\tau)^n} \) with \( e_i \in \mathbb{Z} \). Also

\[
\max_{\tau \in [0,1]} |p_n(\tau)| \leq \left( \frac{0.792(n+1)}{\ln(n+2)} \right)^{n+1} n!,
\]

and one has the closed formula

\[
p_n(1) = \frac{1}{2^{n+1}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i,
\]

for \( n \geq 1 \), where \( \binom{n}{i} \) are the Eulerian numbers.

b) If \( n \geq 2 \) then

\[
\frac{1}{2\pi i} \int_{0}^{1} \int_{\beta} \frac{\tau^{-z} p_n(\tau)}{(-z)^n} \frac{1}{(z + 2^h)} dz d\tau = \sum_{k=1}^{n} (-1)^{n-k} \frac{p_{k-1}(1)}{2^k - 1} + (-1)^n(1 - \gamma).
\]
c) If \( n \geq 2 \) then
\[
\int_0^1 p_n(\tau)\tau^{2^h} d\tau = \sum_{i=0}^{n-1} (-1)^{n-1-i}2^{h(n-1-i)}p_i(1) + (-1)^n 2^{nh} \ln 2 + (-1)^{n+1} 2^{nh} \sum_{j=2^{h-1}+1}^{2^h} \frac{1}{j}.
\]

**Proof.** a) The first part follows by induction and is left as an easy exercise to the reader. We prove the inequality stated. First observe that
\[
(\tau(\tau \ldots (\tau p_0(\tau)') \ldots')')' = p_n(\tau),
\]
where the number of derivatives is \( n \). But
\[
(\tau(\tau \ldots (\tau p_0(\tau)') \ldots')')' = \sum_{k=1}^{n+1} S(n+1, k)\tau^{k-1}p_0^{(k-1)}(\tau),
\]
where \( S(n, k) \) are the Stirling numbers of the second kind (recall that these numbers have the property \( S(1, n) = S(n, n) = 1 \) and
\[
S(n, k) = kS(n-1, k) + S(n-1, k-1),
\]
from which the above formula follows by iteration of this last formula). Using these identities we have if \( \tau \in [0, 1] \)
\[
|p_n(\tau)| \leq \sum_{k=1}^{n+1} S(n+1, k) \max_{i=0, \ldots, n} |p_0^{(i)}(\tau)| \leq \sum_{k=1}^{n+1} S(n+1, k)n! = B(n+1)n!,
\]
where \( B(n) \) is the \( n \)-th Bell number, see \cite{14}. The inequality in the lemma follows from a result in \cite{3}, namely: if \( n \in \mathbb{N} \) then
\[
B(n) < \left( \frac{0.792n}{\ln(n+1)} \right)^n.
\]

Finally, the closed form formula is proved as follows: using Cauchy’s formula in the above formulae gives
\[
p_n(1) = \frac{1}{2\pi i} \int_{|z-1|=1} \sum_{k=1}^{n+1} S(n+1, k)(k-1)! \frac{1}{(z-1)^k(1+z)} \, dz.
\]
But if \( n \geq 1 \) then
\[
(-1)^{n+1}Li_{-n}(z) = \sum_{k=1}^{n+1} S(n+1, k)(k-1)! \frac{1}{(z-1)^k},
\]
where \( Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \) is the polylogarithm function, which gives
\[
p_n(1) = \frac{(-1)^{n+1}}{2\pi i} \int_{|z-1|=1} \frac{Li_{-n}(z)}{1+z} \, dz = \frac{(-1)^n}{2\pi i} \int_{|z+1|=1} \frac{Li_{-n}(z)}{1+z} \, dz.
\]
The proof of the last equality (and the closed form formula) follow from here and the known identity

\[ Li_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{i=0}^{n} \binom{n}{i} z^{n-i}. \]

Hint: \( \int_{|z|=R} \frac{Li_{-n}(z)}{1+z}dz = 0 \), if \( R > 2 \), (say) for \( |Li_{-n}(z)| = O(1/|z|^2) \) as \( |z| \to \infty \). Now use residues.

b) From the well-known formula

\[ \int_{0}^{1} \frac{1}{(1+\tau)} \sum_{h=1}^{\infty} \tau^{2h} \ d\tau = 1 - \gamma, \]

and, for \( 0 < \tau < 1 \),

\[ \frac{1}{2\pi i} \int_{\beta} \frac{\tau^{-z}}{(z + 2^h)} \ d\tau = \tau^{2h}, \]

(this last formula follows by deforming the curve \( \beta \), moving it arbitrarily to a left half plane and taking into account the pole at \( z = -2^h \)), one has

\[ \frac{1}{2\pi i} \int_{0}^{1} \tau^{2h} p_0(\tau) \sum_{h=1}^{\infty} \frac{1}{(z + 2^h)}dzd\tau \]

\[ = \frac{1}{2\pi i} \int_{0}^{1} \tau^{2h} \sum_{h=1}^{\infty} \frac{1}{(z + 2^h)}dzd\tau = \int_{0}^{1} \sum_{h=1}^{\infty} \tau^{2h} \frac{1}{(1+\tau)} \ d\tau = 1 - \gamma. \]  (6)

Also

\[ \int_{0}^{1} \frac{\tau^{2h}}{1+\tau} \ d\tau = \ln 2 - \sum_{j=1}^{2^h} \frac{(-1)^{j+1}}{j}. \]  (7)

(Remark: the above formula for \( 1 - \gamma \) follows from (7) and the identity

\[ \sum_{j=2^{h-1}+1}^{2^h} \frac{1}{j} = \sum_{j=1}^{2^h} \frac{(-1)^{j+1}}{j}. \]  (8)

Indeed adding (7) for \( h = 1, \ldots, N \) and using (8) we get \( \int_{0}^{1} \sum_{h=1}^{N} \frac{\tau^{2h}}{(1+\tau)} \ d\tau = N \ln 2 - \sum_{j=2}^{2^N} \frac{1}{j} \), and the formula follows.)

Now observe that integrating by parts and using \((\tau p_{r-1}(\tau))' = p_r(\tau)\) yield

\[ \int_{0}^{1} p_r(\tau) \sum_{h=1}^{\infty} \tau^{2h} 2^{hr}d\tau = \frac{1}{2^r-1} p_{r-1}(1) - \int_{0}^{1} p_{r-1}(\tau) \sum_{h=1}^{\infty} \tau^{2h} 2^{h(r-1)}d\tau. \]

The lemma follows iterating this last formula, using (6) and noting that

\[ \frac{1}{2\pi i} \int_{0}^{1} \int_{\beta} \tau^{-z} p_r(\tau) \sum_{h=1}^{\infty} \frac{1}{(z + 2^h)}dzd\tau = \int_{0}^{1} p_r(\tau) \sum_{h=1}^{\infty} \tau^{2h} 2^{hr}d\tau, \]

(same proof as (6)). This proves part b).
c) Integrating by parts and using \((\tau \rho_{r-1}(\tau))' = \rho_r(\tau)\) yield:

\[
\int_0^1 p_n(\tau)\tau^{2h} d\tau = p_{n-1}(1) - 2^h \int_0^1 p_{n-1}(\tau)\tau^{2h} d\tau.
\]

Iterating this one gets

\[
\int_0^1 p_n(\tau)\tau^{2h} d\tau = \sum_{i=0}^{n-1} (-1)^{n-i} 2^{h(n-i)} p_i(1) + (-1)^n 2^{nh} \int_0^1 \frac{\tau^{2h}}{1 + \tau} d\tau.
\]

Now use \((7)\) and \((8)\) to obtain (c). \(\square\)

The following lemma is a kind of linear form for \(\gamma\).

**Lemma 4.** The following formula holds: if \(2 \leq n\) then

\[
\frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} F_n(z) dz d\tau = -a_n\gamma + a_n d_n + b_n + e_n + c_n \ln 2
\]

\[
+ \frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} C_n(z) dz d\tau,
\]

where \(C_n(z)\) is defined in Lemma \(1\), \(\gamma\) is Euler’s constant, and \(a_n, b_n, c_n, d_n, e_n\) are as in Theorem \(A\).

**Proof.** Recall that from Lemma \(1\) one has

\[
F_n(z) = A_n(z) + B_n(z) + C_n(z).
\]

First note that \(\int_\beta \tau^{n-1} z^m dz = 0\) if \(m \in \mathbb{Z}\) (again deform arbitrarily the curve to a left half-plane), and therefore

\[
\frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} B_n(z) dz d\tau = 0.
\]

So we are left with the integral over \(A_n\). From the definition of \(A_n(z)\) in Lemma \(1\) and Lemma \(3\) (b) one gets:

\[
\frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} A_n(z) dz d\tau
\]

\[
= a_n(1 - \gamma) + a_n d_n - \sum_{m=1}^{n} q_{m,n} \frac{\theta(n)+m-1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} \frac{1}{(z+2^h)} dz d\tau,
\]

where \(a_n, d_n, q_{m,n}\) are defined in Theorem \(A\). Now notice that

\[
\frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} \frac{1}{(z+2^h)} dz d\tau = \frac{1}{2^{hn}} \int_0^1 p_n(\tau)\tau^{2h} d\tau,
\]

and use Lemma \(3\) (c). This gives (here \(e_n' := (-1)^n \sum_{m=1}^{n} q_{m,n} \sum_{j=2}^{\theta(n)+m-1} \frac{1}{j}\))

\[
\frac{1}{2\pi i} \int_0^1 \int_\beta \frac{\tau^{n-1}}{(z)^n} A_n(z) dz d\tau = a_n(1 - \gamma) + a_n d_n + b_n + e_n' + c_n \ln 2,
\]

where \( b_n, c_n \) are defined in Theorem \( A \). The lemma follows by noticing that \( a_n + e'_n = e_n \), where \( e_n \) is defined in Theorem \( A \). □

In the following lemmas (5 and 6) we estimate the integrals that appear in Lemma \( 4 \) we remark that the constant implied in the \( O \)-symbol in these lemmas is absolute.

**Lemma 5.** If \( 2 \leq n \) then

\[
\left| \frac{1}{2\pi i} \int_0^1 \int_{\beta} \frac{\tau^{-z}p_n(\tau)}{(-z)^n} F_n(z) \quad dzd\tau \right| = O\left( \frac{1}{2^{\theta(n)n^2}} \max_{\tau \in [0,1]} |p_n(\tau)| \right).
\]

**Proof.** We will use formula \( 3 \); so we first estimate \( I_m(z) \) with \( m \geq \theta(n) \geq n \). Recall that by Lemma \( 2 \), one has \( I_m(z) = 0 \) if \( |z| > 2^m \). We chop the curve \( \beta \) in pieces: let \( \beta'_r \) be the part of the curve \( \beta \) in \( \{2^{r-1} \leq |z| < 2^r\} \), \( r = 1, 2, \ldots, m \). Then using Lemma \( 2 \), taking absolute values in the product expression and writing \((1 + 2^a) = 2^a(1 + 2^{-a}) \) whenever \( a \geq 1 \), one gets

\[
\max_{z \in \beta'_r} \left| I_m(z) \right| |(-z)^n| \leq O\left( \frac{2^{(n-1-m)+r(n-m)+r/2}}{2^{mn+n(n+1)/2+r}} \right) \leq O\left( \frac{1}{2^{mn+n+r}} \right),
\]

if \( 1 \leq n-1-m+r \), where the constant implied in the \( O \)-symbol is absolute. The last inequality follows because \( 2^{(n-1-m)+r(n-m)+r/2} \leq 2^{(n-1)n/2} \) if \( r = 1, \ldots, m \).

If \( n-1-m+r \leq 0 \) then

\[
\max_{z \in \beta'_r} \left| I_m(z) \right| |(-z)^n| \leq O\left( \frac{1}{2^{mn+n(n+1)/2}} \right) \leq O\left( \frac{1}{2^{mn+n+r}} \right).
\]

Now if \( L_{\beta'_r} \leq 2^r \) is the length of any of the two segments which compose \( \beta'_r \), then

\[
\int_0^1 \int_{\beta'_r} \left| \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) \right| \quad dzd\tau \leq 2L_{\beta'_r} \int_0^1 \tau^{2^{r-1}(\cos(\pi-\alpha))} |p_n(\tau)| \max_{z \in \beta'_r} \left| I_m(z) \right| |(-z)^n| \quad d\tau
\]

\[
\leq O\left( \frac{1}{2^{mn+n+r}} \max_{\tau \in [0,1]} |p_n(\tau)| \right),
\]

where \( \alpha \) is the angle defined at the beginning in the definition of the curves \( \beta_1, \beta_2 \).

Finally let \( \beta'_0 \) be the part of the curve \( \beta \) in \( |z| < 1 \). It is easy to see that

\[
\max_{z \in \beta'_0} \left| I_m(z) \right| |(-z)^n| \leq O\left( \frac{1}{2^{mn+n(n+1)/2}} \right),
\]

and

\[
\int_0^1 \int_{\beta'_0} \left| \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) \right| \quad dzd\tau \leq O\left( \frac{1}{2^{mn+n(n+1)/2}} \max_{\tau \in [0,1]} |p_n(\tau)| \right).
\]

All this gathers to give

\[
\left| \int_0^1 \int_{\beta} \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) \quad dzd\tau \right| = \left| \int_0^1 \int_{\bigcup_{i=0}^m \beta'_i} \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) \quad dzd\tau \right|
\]

\[
\leq \int_0^1 \int_{\bigcup_{i=0}^m \beta'_i} \left| \frac{\tau^{-z}p_n(\tau)}{(-z)^n} I_m(z) \right| \quad dzd\tau = O\left( \frac{1}{2^{mn+n}} \max_{\tau \in [0,1]} |p_n(\tau)| \right).
\]
and the lemma follows using this last formula and formula (5).

Lemma 6. If \( 2 \leq n \) then

\[
\left| \frac{1}{2\pi i} \int_0^1 \tau^z p_n(\tau) C_n(z) dzd\tau \right| = O\left( \frac{\theta(n)2^{n(n+1)/2}}{2^n\theta(n)+\theta(n)} \max_{\tau \in [0,1]} |p_n(\tau)| \right).
\]

Proof. Set \( D_h(z) := \prod_{k=1}^{n} \frac{(1+z^{-1-k})}{(z+2^{-n-k})} \left( \frac{z^{-1}}{z+2^{-n-k}} \right)^{\frac{1}{2}} \). Use the definition of \( C_n(z) \) as given in Lemma 5 to notice that \( C_n(z)/(z) \) is \( = \sum_{h=0}^{m_0} D_h(z) \) for \( 2^{m_0} < |z| < 2^{m_0+1}, \theta(n) \leq m_0 \in N \). Using the same notation as in the proof of Lemma 5,

\[
\max_{z \in \beta_{m_0+1}'} |D_h(z)| = O\left( \frac{2^{\sum_{k=1}^{n} (k-h+m_0+1)}}{2^{\sum_{k=1}^{n} \max\{m_0, k+h\} 2m_0}} \right) = O\left( \frac{2^{n(n-1)+n(n-1)/2}}{2^{m_0}} \right).
\]

Now we have two cases. One case is when \( h \leq m_0 - n \) and therefore

\[
\sum_{k=1}^{n} \max\{m_0, k+h\} = m_0n.
\]

In this case

\[
\max_{z \in \beta_{m_0+1}'} |D_h(z)| = O\left( \frac{2^{-(h-1)(n-1)+n(n-1)/2}}{2^{m_0}} \right) = O\left( \frac{2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{m_0}} \right),
\]

where the last inequality follows because \( \theta(n) \leq h \).

The other case is when \( h \) satisfies \( m_0 - n + 1 \leq h \leq m_0 \). In this case

\[
\sum_{k=1}^{n} \max\{m_0, k+h\} = m_0n + (h - m_0 + n)(h - m_0 + n + 1)/2.
\]

Thus

\[
\max_{z \in \beta_{m_0+1}'} |D_h(z)| = O\left( \frac{2^{-(h-1)(n-1)+n(n-1)/2-(h-m_0+n)(h-m_0+n+1)/2}}{2^{m_0}} \right)
\]

\[
= O\left( \frac{2^{-(h-1)(n-1)+n(n-1)/2}}{2^{m_0}} \right) = O\left( \frac{2^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{m_0}} \right),
\]

This gives

\[
\max_{z \in \beta_{m_0+1}'} \left| \frac{C_n(z)}{(z)^n} \right| = O\left( \frac{m_02^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{m_0}} \right).
\]

Therefore one has

\[
\int_0^1 \int_{\beta_{m_0+1}'} |\frac{\tau^z p_n(\tau)}{(-z)^n} C_n(z)| dzd\tau
\]

\[
\leq 2L_{\beta_{m_0+1}'} \int_0^1 \tau^{2m_0} \cos(\pi-\alpha) |p_n(\tau)| \max_{z \in \beta_{m_0+1}'} \left| \frac{C_n(z)}{(-z)^n} \right| d\tau
\]

\[
= O\left( \frac{m_02^{-(\theta(n)-1)(n-1)+n(n-1)/2}}{2^{m_0}} \right) \max_{\tau \in [0,1]} |p_n(\tau)|.
\]
Now using this last formula and (recall $C_n(z) = 0$ if $|z| < 2^{\theta(n)}$)

\[
\int_0^1 \int_0^\beta \tau^{-z}p_n(\tau) C_n(z)dzd\tau = \sum_{m_0=\theta(n)}^\infty \int_0^1 \int_{\beta_{m_0+1}}^{\beta_m} \tau^{-z}p_n(\tau) C_n(z)dzd\tau,
\]

one gets

\[
\left| \frac{1}{2\pi i} \int_0^1 \int_\beta \tau^{-z}p_n(\tau) C_n(z)dzd\tau \right| = O\left(\frac{\theta(n)2^{-\theta(n)-1}(n-1)+n(n-1)/2}{2^{2\theta(n)}} \max_{\tau \in [0,1]} |p_n(\tau)|\right).
\]

The lemma follows simplifying the $O$-symbol. \qed

Theorem A is a direct consequence of Lemmas 3 (a), 4, 5 and 6.

3. PROOFS OF THEOREM B AND OF COROLLARY

We need first the following lemma.

**Lemma 7.** Let $a_n, c_n, d_n$ be defined as in Theorem A. Then

i) $a_n, c_n \in \mathbb{Z}$.

ii) $d_n2^n\Pi_{k=\lfloor n/2 \rfloor}^n(2^k - 1) \in \mathbb{Z}$.

iii) For some constant $0 < c$, $\Pi_{k=\lfloor n/2 \rfloor}^n(2^k - 1) = O(2^{\frac{3}{8}n^2+cn})$.

**Proof.** The proof of (i-ii) is immediate using Lemma 3 (a), i.e. $2^n p_{n-1}(1) \in \mathbb{Z}$. Part (iii) is easy and is left to the reader. \qed

**Proof of Theorem B.** We apply Theorem A in its simplest form, i.e. we take $\theta(n) = n$. In this case the $O$-term of Theorem A is

\[
O\left(\frac{1}{2^{n^2/2-\ln n}}\right),
\]

where $c > 0$ is some positive constant. Next one has

\[
b'_n = a_n d_n + b_n.
\]

Thus Theorem A yields

\[
-\gamma a_n + b'_n + c_n + c_n \ln 2 = O\left(\frac{1}{2^{n^2/2-\ln n}}\right).
\]

Multiplying this by $g_n = 2^{n^2/2} \Pi_{k=\lfloor n/2 \rfloor}^n(2^k - 1)$ gives the desired result if one notices that by Lemma 7

\[
g_n = O(2^{(\frac{3}{8}+\rho)n^2+cn}),
\]

and that $a_n g_n, c_n g_n \in \mathbb{Z}$. This finishes the proof of Theorem B. \qed
Proof of Corollary. We prove the remark for the second sequence. Note that if \( \gamma \) were a rational number then \( \gamma n!a_n g_n \in \mathbb{Z} \) for \( n_0 < n \). Also \( n! = O(2^{c'n \ln n}) \) \( (0 < c') \) and therefore by Theorem B

\[
n!(-\gamma a_n g_n + b'_n g_n + e_n g_n + c_n g_n \ln 2) \to 0,
\]

and the assertion follows from this. For the first sequence the proof is similar and is left to the reader. \( \Box \)

References


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