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GEOMETRIC PROPERTIES OF NEUTRAL SIGNATURE METRICS ON 4-DIMENSIONAL NILPOTENT LIE GROUPS

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Abstract. The classification of left invariant metrics of neutral signature on the 4-dimensional nilpotent Lie groups is presented. Their geometry is extensively studied with special emphasis on the holonomy groups and projective equivalence. Additionally, we focus our attention on the Walker metrics. They appear as the underlying structure of metrics on the nilpotent Lie groups with degenerate center. Finally, we give complete description of the isometry groups.

Introduction

The geometry of 4-dimensional Lie groups with a left invariant metric has been studied extensively through the years. For a comprehensive treatment and for references to the extensive literature on the subject of left invariant metrics on Lie groups in small dimensions one may refer to the book [2]. Initially motivated by works of Lauret, Cordero, and Parker, we became interested in the case of the nilpotent Lie groups. The first classification type results for small dimensions in the positive definite setting are due to Milnor [31] who classified 3-dimensional Lie groups with a left invariant positive definite metric. Riemannian nilmanifolds of dimension three and four were extensively studied by Lauret [26], while for dimension five we refer to [22]. The distinction between Riemannian and pseudo-Riemannian settings is obvious from the results of Rahmani [34]. It is shown there that on the 3-dimensional Heisenberg group H_3 there are three classes of non-equivalent Lorentz left invariant metrics, one of which is flat. In the positive definite case there is only one class and it is not flat. The classification of the Lorentz 3-dimensional nilpotent Lie groups was done by Cordero and Parker [10], while the classification in the 4-dimensional case can be found in [7].

A metric g of split $(2,2)$ signature is said to be a Walker metric if there exists a 2-dimensional null distribution which is parallel with respect to the Levi-Civita connection of g. This type of metrics has been introduced by Walker [36], who has

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shown that they have a (local) canonical form depending on three smooth functions. Walker metrics appear in a natural way being associated with tangent and cotangent bundles in various constructions (for more details on geometric properties of Walker metrics we refer to [8], [9] and references therein). Here they arose as the underlying structure of the metrics on the nilpotent Lie groups with degenerate center.

The other question of interest is a projective equivalence of left invariant metrics. The first examples of geodesically equivalent metrics are due to Lagrange [25]. Later, Beltrami [4] generalized this example to the metrics of constant negative curvature, and to the case of pseudo-Riemannian metrics of constant curvature. Though the examples of Lagrange and Beltrami are 2-dimensional, one can easily generalize them to every dimension and to every signature.

Intuitively, one might expect a link between projective relatedness and a holonomy type, since if $\nabla = \bar{\nabla}$, then (M, g) and $(M, \bar{g})$ have the same holonomy type. Indeed, holonomy theory can, to some extent, describe projective equivalence (see the work of Hall [19, 20, 37]).

Matveev gave an algorithm for obtaining a list of possible geodesically equivalent metrics to a given metric. It is interesting that the proposed algorithm does not depend on the signature of the metric. Manifolds of constant curvature allow geodesically equivalent metrics (see [25], [4], etc.). Sinjukov [35] showed that all geodesically equivalent metrics given on a symmetric space are affinely equivalent. For more comprehensive results in this area we refer to [24], [29], [30] and references therein.

Another striking difference between the Riemannian and pseudo-Riemannian set-up is apparent from the isometry group information. In particular, it can be significantly larger than in the Riemannian or pseudo-Riemannian case with non-degenerate center. It was shown that in the Riemannian case the group of isometric automorphisms coincides with the group of isometries of the 2-step nilpotent Lie group N fixing its identity element and the distribution

$$TN = \upsilon N \oplus \xi N. \tag{1}$$

These subbundles are obtained by the left translation of the splitting at the Lie algebra level $\mathfrak{n} = \upsilon \oplus \xi$, where $\mathfrak{n}$ is the Lie algebra of N, ξ denotes its center and υ the orthogonal complementary subspace of ξ. Further information regarding the positive definite case can be found in [14] and [23].

Lie groups acting by isometries on a fixed pseudo-Riemannian 2-step nilpotent Lie group endowed with a left invariant metric —the group of isometric automorphisms, the group of isometries preserving the splitting (1), and the full isometry group— were investigated by Cordero and Parker in [11] and later by del Barco and Ovando in [12]. In the same paper, del Barco and Ovando also considered bi-invariant metrics on the 2-step nilpotent Lie groups, showing that there are isometric automorphisms not preserving any kind of splitting.

Our paper is organized as follows. In Section 1 the classification of 4-dimensional left invariant metrics of neutral signature on the nilpotent Lie groups $H_3 \times \mathbb{R}$ and

G_4 is presented. This completes the full classification of metrics on the nilpotent Lie groups in dimensions three and four.

In Section 2 the geometry of the metrics is explored. The curvature tensor (Tables 1 and 2) and the holonomy groups (Table 3) were calculated. Also, we study a connection between holonomy groups and projective equivalence of metrics. It is interesting that if the center of the group is degenerate, the corresponding metric is of Walker type. We focus our attention on these metrics.

In Section 3 we find projective classes of the metrics (Theorems 3.1 and 3.2). Furthermore, we solve the problem of the projectively equivalent metrics on the 4-dimensional nilpotent Lie groups by showing that a left invariant metric g is either geometrically rigid or has projectively equivalent metrics that are also affinely equivalent (Theorem 3.3). Moreover, all affinely equivalent metrics are left invariant, while their signature may change.

Finally, Section 4 provides complete description of the isometry groups for both $H_3 \times \mathbb{R}$ and G_4.

1. Classification of non-equivalent inner products of neutral signature

Let (N, g) be a nilpotent Lie group with a left invariant metric g. If not stated otherwise, we assume that N is 4-dimensional and that the metric g is of neutral signature $(+, +, -, -)$.

Magnin [27] proved that, up to isomorphism, there are only two non-abelian nilpotent Lie algebras of dimension four: $\mathfrak{h}_3 \oplus \mathbb{R}$ and $\mathfrak{g}_4$, with corresponding simply connected Lie groups $H_3 \times \mathbb{R}$ and G_4.

With respect to the basis $\{x_1, x_2, x_3, x_4\}$, the Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}$ is defined by a non-zero commutator

$$[x_1, x_2] = x_3.$$

The Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}$ is 2-step nilpotent with a 2-dimensional center $Z(\mathfrak{h}_3 \oplus \mathbb{R}) = \mathcal{L}(x_4, x_3)$ and a 1-dimensional commutator subalgebra spanned by x_3.

The Lie algebra $\mathfrak{g}_4$, with respect to the basis $\{x_1, x_2, x_3, x_4\}$, is given by non-zero commutators

$$[x_1, x_2] = x_3, \qquad [x_1, x_3] = x_4.$$

The Lie algebra $\mathfrak{g}_4$ is 3-step nilpotent with a 1-dimensional center $Z(\mathfrak{g}_4) = \mathcal{L}(x_4)$ and a 2-dimensional commutator subalgebra

$$\mathfrak{g}_4' := [\mathfrak{g}_4, \mathfrak{g}_4] = \mathcal{L}(x_3, x_4).$$

Since the exponential map on any simply connected nilpotent Lie group N is a diffeomorphism, by the Hopf–Rinow theorem any positive definite metric on N is geodesically complete. However, Guediri [18] has found an example of a non-complete left invariant Lorentz metric on a 3-step nilpotent Lie group that corresponds to the Lie algebra $\mathfrak{g}_4$.

Denote by $Aut(\mathfrak{n})$ the group of automorphisms of the Lie algebra $\mathfrak{n}$ that is defined by

$$Aut(\mathfrak{n}) := \{F : \mathfrak{n} \to \mathfrak{n} \,|\, F \text{ linear, bijective, } \ [Fx, Fy] = F[x, y], \ \ x, y \in \mathfrak{n}\}.$$

Denote by $\mathcal{S}(\mathfrak{n})$ the set of non-equivalent neutral inner products (i.e. of signature $(+, +, -, -)$) of the Lie algebra $\mathfrak{n}$. Having a basis of the Lie algebra $\mathfrak{n}$ fixed, the set $\mathcal{S}(\mathfrak{n})$ is identified with symmetric matrices S of signature $(2, 2)$ modulo the following action of the automorphism group:

$$S \mapsto F^T S F, \quad F \in Aut(\mathfrak{n}). \tag{2}$$

Automorphisms of these Lie groups are described in the following two lemmas.

Lemma 1.1. [26] *The group $Aut(\mathfrak{g}_4)$ of automorphisms of the Lie algebra $\mathfrak{g}_4$, with respect to the basis $\{x_1, x_2, x_3, x_4\}$, consists of real matrices of the form*

$$Aut(\mathfrak{g}_4) = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & a_1 b_2 & 0 \\ a_4 & b_4 & a_1 b_3 & a_1^2 b_2 \end{pmatrix} \middle| \begin{array}{l} a_i \in \mathbb{R}, i = 1, \dots, 4, \\ b_j \in \mathbb{R}, j = 2, \dots, 4, \\ a_1, b_2 \neq 0 \end{array} \right\}.$$

Lemma 1.2. [26] *The group $Aut(\mathfrak{h}_3 \oplus \mathbb{R})$ of automorphisms of the Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}$, with respect to the basis $\{x_1, x_2, x_3, x_4\}$, consists of real matrices of the form*

$$Aut(\mathfrak{h}_3 \oplus \mathbb{R}) = \left\{ \begin{pmatrix} A & 0 & 0 \\ b_1 & \det A & \mu \\ b_2 & 0 & \lambda \end{pmatrix} \middle| \begin{array}{l} A \in GL(2, \mathbb{R}), \ b_1, b_2 \in \mathbb{R}^2, \\ \lambda, \mu \in \mathbb{R}, \ \lambda \neq 0 \end{array} \right\}.$$

It is interesting to notice that in the Riemannian case, up to equivalence (2), there is just one inner product for each Lie algebra (see [26]). Note that the situation in the Lorentz case is quite different — there are seven families of inner products in the case of the Lie algebra $\mathfrak{g}_4$ and six families in the case of $\mathfrak{h}_3 \oplus \mathbb{R}$ (see [7]).

Theorem 1.1. *The set $\mathcal{S}(\mathfrak{g}_4)$ of non-equivalent inner products of neutral signature on the Lie algebra $\mathfrak{g}_4$, with respect to the basis $\{x_1, x_2, x_3, x_4\}$, is represented by the following matrices:*

$$S_A = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix}, \quad \epsilon_1, \epsilon_2 \in \{-1, 1\},$$

$$S_1^{\pm\lambda} = \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \pm\lambda & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad S_2^{\pm\lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \mp 1 & 0 & 0 \\ 0 & 0 & \pm\lambda & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$S_3^{\pm\lambda} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}, \qquad S_4^{\pm\lambda} = \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix},$$

$$S_1^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad S_2^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \end{pmatrix},$$

with $\lambda > 0$. *Matrix* $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *has split* $(+,-)$ *signature if* $\epsilon_1 \neq \epsilon_2$, *and* $(+,+)$ *(respectively* $(-,-)$*) signature if* $\epsilon_1 = \epsilon_2 = \mp 1$.

Proof. The proof is entirely analogous to the case of Lorentz signature (see [7]).

Denote the center of $\mathfrak{g}_4$ by $Z = \mathcal{L}(x_4)$ and the commutator subalgebra by $\mathfrak{g}_4' = \mathcal{L}(x_3, x_4)$. Let us consider a signature of $\mathfrak{g}_4'$.

For the signatures $(+,+)$, $(-,-)$ and $(+,-)$, we get the inner products S_A (for an appropriate choice of ϵ_1, ϵ_2).

If $\mathfrak{g}_4'$ is partially degenerate, we distinguish two cases: when the center is null (inner products $S_1^{\pm\lambda}$ and $S_2^{\pm\lambda}$) and when it is not (inner products $S_3^{\pm\lambda}$ and $S_4^{\pm\lambda}$).

Let us examine more thoroughly the case when $\mathfrak{g}_4'$ is totally null. The matrix S representing the inner product reduces to

$$S = \begin{pmatrix} p & r & e & f \\ r & q & g & h \\ e & g & 0 & 0 \\ f & h & 0 & 0 \end{pmatrix}.$$

Since the action of $F \in Aut(\mathfrak{g}_4)$ cannot change the signature of the element h, after some calculation one can show that for $h = 0$ one gets the inner product S_1^0 and for $h \neq 0$ there exist two non-equivalent inner products represented by S_2^0. $\square$

Theorem 1.2. *The set* $\mathcal{S}(\mathfrak{h}_3 \oplus \mathbb{R})$ *of non-equivalent inner products of neutral signature on the Lie algebra* $\mathfrak{h}_3 \oplus \mathbb{R}$, *with respect to the basis* $\{x_1, x_2, x_3, x_4\}$, *is represented by the following matrices:*

$$S_\mu^{\pm} = \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 0 & 0 & \pm\mu & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \qquad S_\lambda^{\pm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \pm\lambda & 0 \\ 0 & 0 & 0 & \mp 1 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad S_{01}^{\pm} = \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix},$$

$$S_{02}^{\pm} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad S_0^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where $\lambda, \mu > 0$.

Proof. The center of the Lie algebra is $Z = \mathcal{L}(x_3, x_4)$ and the commutator subalgebra is $\mathcal{L}(x_3)$. Similar to the previous case, we investigate the signature of the center Z and obtain the classification. □

The inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ gives rise to a left invariant metric g on the corresponding Lie group. Now we find the coordinate description of those metrics defined by the inner product from Theorems 1.1 and 1.2.

If X_1, X_2, X_3, X_4 are the left invariant vector fields on G_4 defined by $x_1, x_2, x_3, x_4 \in \mathfrak{g}_4$, for global coordinates (x, y, z, w) on G_4 we have the relations

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + \frac{x^2}{2}\frac{\partial}{\partial w}, \quad X_3 = \frac{\partial}{\partial z} + x\frac{\partial}{\partial w}, \quad X_4 = \frac{\partial}{\partial w}.$$

Similarly, in global coordinates (x, y, z, w) on $H_3 \times \mathbb{R}$ we have the left invariant vector fields

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial w}.$$

Now, by a straightforward computation, we can prove the following two theorems.

Theorem 1.3. *Each left invariant metric of neutral signature on the Lie group* G_4*, up to an automorphism of* G_4*, is isometric to one of the following:*

$$\begin{aligned}
g_A &= \epsilon_1 dx^2 + \epsilon_2 dy^2 + a(xdy - dz)^2 - b(xdy - dz)(x^2dy - 2xdz + 2dw) \\
&\quad + \frac{c}{4}(x^2dy - 2xdz + 2dw)^2; \\
g_1^{\pm\lambda} &= \mp dx^2 + 2dydw + xdy(xdy - 2dz) \pm \lambda(xdy - dz)^2; \\
g_2^{\pm\lambda} &= \mp dy^2 + 2dxdw + xdx(xdy - 2dz) \pm \lambda(xdy - dz)^2; \\
g_3^{\pm\lambda} &= \mp dy^2 - 2dx(xdy - dz) \pm \frac{\lambda}{4}(x^2dy - 2xdz + 2dw)^2; \\
g_4^{\pm\lambda} &= \mp dx^2 - 2dy(xdy - dz) \pm \frac{\lambda}{4}(x^2dy - 2xdz + 2dw)^2; \\
g_1^0 &= x^2dxdy + 2dwdx + 2dydz - 2x(dy^2 + dxdz); \\
g_2^0 &= 2dxdz - 2xdxdy \pm dy(2dw - 2xdz + x^2dy).
\end{aligned}$$

Theorem 1.4. *Each left invariant metric of neutral signature on the Lie group* $H_3 \times \mathbb{R}$*, up to an automorphism of* $H_3 \times \mathbb{R}$*, is isometric to one of the following:*

$$\begin{aligned}
g_\mu^{\pm} &= \mp dx^2 \mp dy^2 \pm \mu(xdy - dz)^2 \pm dw^2; \\
g_\lambda^{\pm} &= dx^2 - dy^2 \pm \lambda(xdy - dz)^2 \mp dw^2; \\
g_1 &= dx^2 + 2dwdz - dy(dy + 2xdw);
\end{aligned}$$

$$\begin{aligned} g_{01}^{\pm} &= 2dy(dz - xdy) \pm (dw^2 - dx^2); \\ g_{02}^{\pm} &= 2dwdx \mp (dy + dz - xdy)(dz - (1+x)dy); \\ g_0^0 &= 2dwdx + 2dy(dz - xdy). \end{aligned}$$

Remark 1.1. *Note that metrics* $g_\mu^\pm$, $g_\lambda^\pm$ *and* $g_{01}^\pm$ *correspond to the direct product of the Lorentz metrics on* H_3 *(see* [34]*) with* $\mathbb{R}$.

All metrics on the Lie group $H_3 \times \mathbb{R}$ are geodesically complete [18]. Unfortunately, this is not the case with the Lie group G_4. For example, completeness of the metric g_A depends on various choices of parameters a, b and c.

2. Curvature and holonomy of metrics

Let g be a left invariant metric of neutral signature on a nilpotent Lie group of dimension four. To calculate a Levi-Civita connection ∇ of a metric g we use Koszul's formula

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) = X.g\left(Y,Z\right) - Y.g\left(Z,X\right) - Z.g\left(X,Y\right) \\ + g\left([X,Y],Z\right) - g\left([Y,Z],X\right) + g\left([Z,X],Y\right). \end{aligned}$$

Since the metric on the Lie group is left invariant, the components in the first row of Koszul's formula vanish when evaluated on left invariant vector fields.

With respect to the left invariant basis $\{X_1, X_2, X_3, X_4\}$ the metric g is represented by a symmetric matrix S in every point on the Lie group. Let $[v]$ denote the column of coordinates of the vector $v = \nabla_{X_i} X_j,\ i,j = 1,\dots,4$, with respect to the basis $\{X_1, X_2, X_3, X_4\}$. Then we have

$$[v] = S^{-1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T,$$

where

$$\alpha_k := g\left(v, X_k\right) = \frac{1}{2}\left(g\left([X_i,X_j],X_k\right) - g\left([X_j,X_k],X_i\right) + g\left([X_k,X_i],X_j\right)\right).$$

For the curvature tensor we use the formula

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

The Ricci curvature is defined as the trace of the operator

$$\rho(X,Y) = Tr(Z \mapsto R(Z,X)Y),$$

with components $\rho_{ij} = \rho(X_i, X_j)$. The scalar curvature is the trace of the Ricci operator

$$Scal = \sum_i \rho_i^i = \sum_{i,j} g^{ij}\rho_{ji},$$

where we have used the inverse $\{g^{ij}\} = S^{-1}$ of the metric to raise the index and obtain the Ricci operator.

Recall that, in the case of neutral signature in dimension four, the curvature tensor $R: \Lambda^2 T_pN \to \Lambda^2 T_pN$, considered as a mapping of two forms, can be decomposed as

$$R = \begin{pmatrix} W^+ & B \\ B* & W^- \end{pmatrix} + \frac{Scal}{12} I, \tag{3}$$

where W^+ and W^- are respectively the self-dual and the anti-self-dual part of the Weyl tensor W, and B is the Einstein part. The Weyl tensor $W = W^+ \oplus W^-$ is conformal invariant.

Remark 2.1. *From the next two tables it is obvious that none of the metrics is Einstein, since $\rho_{ij} \neq \frac{Scal}{4} g_{ij}$ or, equivalently, $B \neq 0$. Moreover, it is an easy exercise to show that the metric g_1 on the group $H_3 \times \mathbb{R}$ is conformally flat ($W = 0$), metrics g_2^0 on G_4 are self-dual ($W^- = 0$) and none of the others is anti-self-dual ($W^+ = 0$).*

Remark 2.2. *Milnor proved in* [31] *that in the Riemannian case if the Lie group G is solvable, then every left invariant metric on G is either flat, or has strictly negative scalar curvature. Note that in the pseudo-Riemannian setting this does not hold since we have found examples of metrics with strictly positive scalar curvature as well as non-flat metrics with zero scalar curvature.*

Remark 2.3. *The Lie group $H_3 \times \mathbb{R}$ can be realized as the group of matrices of the form*

$$\begin{pmatrix} 1 & \overline{z} & w \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad z, w \in \mathbb{C}.$$

Therefore, $H_3 \times \mathbb{R}$ carries invariant complex structures. On the other hand, no compact quotient of G_4 can have a complex structure (homogeneous or otherwise). Consequently, G_4 cannot have a left invariant complex structure. The proof of this fact can be found in [15].

Although none of our two algebras admits a hypercomplex structure, it was proven by Blažić and Vukmirović in [6] *that the algebra $\mathfrak{h}_3 \oplus \mathbb{R}$ admits an integrable para-hypercomplex structure if its center is totally null. The corresponding metric is flat and it is exactly our metric g_0^0.*

2.1. **Case of group G_4.** See Table 1.

Remark 2.4. *Note that for the metric g_A we denoted by $d = ac - b^2 = \det A$.*

Remark 2.5. *Metric $g_1^{-\lambda}$ is flat for $\lambda = 1$.*

2.2. **Case of group $H_3 \times \mathbb{R}$.** See Table 2.

2.3. **Holonomy.** Now we calculate holonomy algebras $\mathrm{hol}(g)$ of each metric g from Theorems 1.3 and 1.4. Note that, since the nilpotent Lie group is simply connected, the restricted holonomy group $Hol_0(N, g)$ coincides with the full holonomy group $Hol(N, g)$.

metric	curvature tensor R^l_{ijk} (non-vanishing components)		Ricci tensor ρ_{ij}	scalar curvature
g_A	$R^2_{112} = \frac{3a\epsilon_2}{4}$ $R^3_{112} = \frac{3bc}{4d}$ $R^4_{112} = \frac{d-3b^2}{4d}$ $R^1_{212} = -\frac{3a\epsilon_1}{4}$ $R^1_{312} = -b\epsilon_1$ $R^1_{412} = -\frac{c\epsilon_1}{4}$ $R^2_{114} = \frac{c\epsilon_2}{4}$ $R^3_{114} = -\frac{b\epsilon_2}{4}$ $R^4_{114} = -\frac{c^2}{4d}$ $R^1_{214} = -\frac{c\epsilon_1}{4}$ $R^1_{314} = \frac{b(ad\epsilon_2+c^2)\epsilon_1}{4d}$ $R^1_{414} = \frac{(b^2d\epsilon_2+c^3)\epsilon_1}{4d}$ $R^3_{234} = \frac{c\epsilon_1}{4}$ $R^4_{234} = -\frac{b\epsilon_1}{2}$ $R^2_{334} = \frac{(b^2-d)\epsilon_1\epsilon_2}{4}$ $R^3_{334} = \frac{bc^2\epsilon_1}{4d}$ $R^4_{334} = -\frac{ac^2\epsilon_1}{4d}$ $R^2_{434} = \frac{bc\epsilon_1\epsilon_2}{4}$ $R^3_{434} = \frac{c^3\epsilon_1}{4d}$ $R^4_{434} = -\frac{bc^2\epsilon_1}{4d}$	$R^2_{113} = b\epsilon_2$ $R^3_{113} = \frac{3c^2-ad\epsilon_2}{4d}$ $R^4_{113} = \frac{-bc}{d}$ $R^1_{213} = -b\epsilon_1$ $R^1_{313} = \frac{(a^2d\epsilon_2-c(3d-b^2))\epsilon_1}{4d}$ $R^1_{413} = \frac{b(ad\epsilon_2+c^2)\epsilon_1}{4d}$ $R^3_{223} = -\frac{a\epsilon_1}{4}$ $R^2_{323} = \frac{a^2\epsilon_1\epsilon_2}{4}$ $R^3_{323} = \frac{b(b^2-d)\epsilon_1}{4d}$ $R^4_{323} = \frac{a(b^2-d)\epsilon_1}{4d}$ $R^2_{423} = \frac{ab\epsilon_1\epsilon_2}{4}$ $R^3_{423} = \frac{c(b^2-d)\epsilon_1}{4d}$ $R^4_{423} = \frac{b(d-b^2)\epsilon_1}{4d}$ $R^3_{224} = -\frac{b\epsilon_1}{4}$ $R^2_{324} = \frac{ab}{4}\epsilon_1\epsilon_2$ $R^3_{324} = \frac{b^2c\epsilon_1}{4d}$ $R^4_{324} = -\frac{abc\epsilon_1}{4d}$ $R^2_{424} = \frac{b^2\epsilon_1\epsilon_2}{4}$ $R^4_{424} = \frac{bc^2\epsilon_1}{4d}$ $R^4_{424} = -\frac{b^2c\epsilon_1}{4d}$	$\rho_{11} = -\frac{ad\epsilon_2+c^2}{2d}$ $\rho_{22} = -\frac{a\epsilon_1}{2}$ $\rho_{23} = -\frac{b\epsilon_1}{2}$ $\rho_{33} = \frac{(a^2d\epsilon_2-c(d-b^2))\epsilon_1}{2d}$ $\rho_{34} = \frac{b(ad\epsilon_2+c^2)}{2d}$ $\rho_{44} = \frac{(b^2d\epsilon_2+c^3)\epsilon_1}{2d}$	$-\frac{(c^2\epsilon_2+ad)\epsilon_1}{2d}$
$g_1^{\pm\lambda}$	$R^4_{121} = \frac{(\lambda\pm1)(3\lambda\mp1)}{4\lambda}$ $R^1_{212} = \frac{(\lambda\pm1)(3\lambda\mp1)}{4\lambda}$	$R^3_{223} = \frac{(\lambda\pm1)^2}{4\lambda}$ $R^4_{323} = \frac{\mp(\lambda\pm1)^2}{4}$	$\rho_{22} = \frac{\lambda^2-1}{2\lambda}$	0
$g_2^{\pm\lambda}$	$R^2_{112} = -\frac{3\lambda}{4}$ $R^4_{212} = \mp\frac{3\lambda}{4}$	$R^3_{113} = \frac{\lambda}{4}$ $R^4_{313} = \mp\frac{\lambda^2}{4}$	$\rho_{11} = \frac{\lambda}{2}$	0
$g_3^{\pm\lambda}$	$R^4_{112} = \frac{1}{2}$ $R^3_{412} = \mp\frac{\lambda}{2}$ $R^1_{113} = \pm\frac{\lambda}{4}$ $R^3_{313} = \mp\frac{\lambda}{4}$	$R^2_{114} = -\frac{\lambda}{2}$ $R^3_{214} = \mp\frac{\lambda}{2}$ $R^4_{134} = \pm\frac{\lambda}{4}$ $R^3_{434} = -\frac{\lambda^2}{4}$	$\rho_{13} = \pm\frac{\lambda}{2}$ $\rho_{44} = -\frac{\lambda^2}{2}$	$\pm\frac{\lambda}{2}$
$g_4^{\pm\lambda}$	$R^2_{113} = \pm\frac{3\lambda}{4}$ $R^1_{313} = \frac{3\lambda}{4}$ $R^4_{223} = \mp\frac{1}{2}$ $R^3_{423} = \frac{\lambda}{2}$	$R^2_{224} = -\frac{\lambda}{2}$ $R^3_{324} = \frac{\lambda}{2}$ $R^4_{334} = \frac{\lambda}{4}$ $R^2_{434} = \mp\frac{\lambda^2}{4}$	$\rho_{24} = -\frac{\lambda}{2}$ $\rho_{33} = \frac{\lambda}{2}$	0
g_1^0	flat			0
g_2^0	$R^4_{112} = 1$ $R^1_{212} = -\frac{1}{4}$ $R^3_{212} = -1$	$R^4_{123} = -\frac{1}{4}$ $R^3_{223} = \frac{1}{4}$ $R^4_{312} = \frac{1}{4}$	$\rho_{22} = -\frac{1}{2}$	0

TABLE 1. Curvature of left invariant metrics on the Lie group G_4

metric	curvature tensor R^l_{ijk} (non-vanishing components)	Ricci tensor ρ_{ij}	scalar curvature
$g^{\pm}_{\mu}$	$R^2_{112} = -\frac{3\mu}{4}$ $R^1_{212} = \frac{3\mu}{4}$ $R^3_{113} = \frac{\mu}{4}$ $R^1_{313} = \frac{\mu^2}{4}$ $R^3_{223} = \frac{\mu}{4}$ $R^2_{323} = \frac{\mu^2}{4}$	$\rho_{11} = \frac{\mu}{2}$ $\rho_{22} = \frac{\mu}{2}$ $\rho_{33} = \frac{\mu^2}{2}$	$\mp\frac{\mu}{2}$
$g^{\pm}_{\lambda}$	$R^2_{112} = \mp\frac{3\lambda}{4}$ $R^1_{212} = \mp\frac{3\lambda}{4}$ $R^3_{113} = \pm\frac{\lambda}{4}$ $R^1_{313} = -\frac{\lambda^2}{4}$ $R^3_{223} = \mp\frac{\lambda}{4}$ $R^2_{323} = -\frac{\lambda^2}{4}$	$\rho_{11} = \pm\frac{\lambda}{2}$ $\rho_{22} = \mp\frac{\lambda}{2}$ $\rho_{33} = -\frac{\lambda^2}{2}$	$\pm\frac{\lambda}{2}$
g_1	$R^3_{114} = \frac{1}{4}$ $R^1_{414} = -\frac{1}{4}$ $R^3_{224} = -\frac{1}{4}$ $R^2_{424} = -\frac{1}{4}$	$\rho_{44} = -\frac{1}{2}$	0
$g^{\pm}_{01}$	flat		0
$g^{\pm}_{02}$	$R^2_{112} = -\frac{3}{4}$ $R^4_{212} = \pm\frac{3}{4}$ $R^3_{113} = \frac{1}{4}$ $R^4_{313} = \pm\frac{1}{4}$	$\rho_{11} = \frac{1}{2}$	0
g^0_0	flat		0

TABLE 2. Curvature of left invariant metrics on the Lie group $H_3 \times \mathbb{R}$

We know that the holonomy algebra is a subalgebra of an isometry algebra, i.e. $\mathrm{hol}(g) \leq o(2,2)$. According to the Ambrose–Singer theorem the algebra $\mathrm{hol}(g)$ is generated by the curvature operators $R(X_i, X_j)$ and their covariant derivatives of any order. Since the covariant derivatives and the curvature operators are known, we use the formula

$$(\nabla_{X_m} R(X_k, X_l))(X_j) = \nabla_{X_m}(R(X_k, X_l)(X_j)) - R(X_k, X_l)(\nabla_{X_m} X_j)$$

to calculate the derivative $\nabla_{X_m} R(X_k, X_l)$ of the curvature operator. Higher order derivatives are calculated using a similar formula.

The results are summarized in Table 3.

group	metric	hol(g)	basis	p.n.v.	dec.
G_4	g_A, $c \neq 0$	A_{32}		no	no
	g_A, $c = 0$	A_{32}		no	no
	$g_1^{+\lambda}$	A_{17}	$X_1 \wedge X_4, X_3 \wedge X_4$	X_4	no
	$g_1^{-\lambda}$, $\lambda \neq 1$	A_{17}	$X_1 \wedge X_4, X_3 \wedge X_4$	X_4	no
	g_1^{-1}	flat			
	$g_2^{\pm\lambda}$	A_{17}	$X_2 \wedge X_4, X_3 \wedge X_4$	X_4	no
	$g_3^{\pm\lambda}$	A_{32}		no	no
	$g_4^{\pm\lambda}$	A_{32}		no	no
	g_1^0	flat			
	g_2^0	A_{17}	$X_3 \wedge X_4, X_1 \wedge X_4$	X_4	no
$H_3 \times \mathbb{R}$	$g_\mu^\pm$	A_{22}	$X_1 \wedge X_2, X_1 \wedge X_3, X_2 \wedge X_3$	X_4	yes
	$g_\lambda^\pm$	A_{22}	$X_1 \wedge X_2, X_1 \wedge X_3, X_2 \wedge X_3$	X_4	yes
	g_1	A_{17}	$X_1 \wedge X_3, X_2 \wedge X_3$	X_3	no
	$g_{01}^\pm$	flat			
	$g_{02}^\pm$	A_{17}	$X_2 \wedge X_4, X_3 \wedge X_4$	X_4	no
	g_0^0	flat			

TABLE 3. Holonomy algebras of left invariant metrics on the Lie groups G_4 and $H_3 \times \mathbb{R}$. (p.n.v.: parallel null vector; dec.: decomposable.)

For the classification of holonomy algebras, we used the notation proposed by Ghanam and Thompson in [17]. Note that the results obtained in [17] were later improved by Galaev and Leistner in [16].

From the three algebras listed in the previous table only the full holonomy algebra A_{32} is irreducible and therefore the corresponding metrics are indecomposable. Let us examine the geometric structure for the other two algebras.

In the case of the algebra A_{17} we have an invariant 2-dimensional null distribution containing a parallel null vector field. Since the holonomy algebra A_{17} is 2-dimensional, according to [37], all projectively equivalent metrics are also affinely equivalent.

By the results of de Rham [13] and Wu [40], A_{22} corresponds to the product of an irreducible 3-dimensional Lorentzian metric and a 1-dimensional flat factor adjusted so as to obtain a neutral signature. It is easy to check that A_{22} is spanned by $X_1 \wedge X_2, X_1 \wedge X_3, X_2 \wedge X_3$, corresponding to the subalgebra $3(c)$ from [37]. In the same paper projective equivalence of the metric g of this type was investigated under some strict assumptions on the range of the curvature map $R:\ \Lambda^2 T_pN \to \Lambda^2 T_pN$ (at point p).

2.4. **Walker metrics.** We can observe that Walker metrics appear as the underlying structure of neutral signature metrics on the nilpotent Lie groups with the holonomy algebra A_{17}. It is a well known fact that metrics on the 2-step nilpotent Lie groups with degenerate center admit Walker metrics. Additionally, we find an example of this kind of metric with non-degenerate center. Interestingly, a metric on a 3-step nilpotent Lie group with degenerate center is not necessarily a Walker one. For example, a family of metrics g_A with $c = 0$ has degenerate center, but it is not Walker.

Let us recall that there exist local coordinates (x, y, z, w) such that a Walker metric, with respect to the frame $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}\}$, has the form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{4}$$

where a, b and c are smooth functions.

In [36] Walker described an algorithm for finding appropriate local coordinates.

Let $\mathcal{D}$ be a 2-dimensional null distribution. Since the space is 4-dimensional, we have $\mathcal{D} = \mathcal{D}^\perp$. There exist local coordinates (x_1, x_2, x_3, x_4) such that $\mathcal{D}$ is spanned by $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$. Since $\mathcal{D}$ is totally null, we have

$$g\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 0.$$

Now, we consider the vector fields $\{\xi_i\}$, $i \in \{1, 2\}$ defined by

$$g(\xi_i, X) = dx^{i+2}(X).$$

It follows that $\{\xi_i\}_{i=1}^2$ are orthogonal to any $X \in \mathcal{D}^\perp$ and hence they lie in $\mathcal{D}$. Moreover, they are linearly independent. Thus we can take

$$\frac{\partial}{\partial x} = \xi_1^j \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial y} = \xi_2^j \frac{\partial}{\partial x_j}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial w} = \frac{\partial}{\partial x_4}, \quad j \in \{1, \dots, 4\}$$

to be our new coordinate frame.

Furthermore, it can be proven that, if a Walker metric possesses parallel vector field, there exist local coordinates such that functions a, b and c do not depend on the variable x. Since the change of coordinates preserves the canonical form (4) of the metric g, it is easy to show that the coordinate transformations are given by

$$\begin{aligned} \widetilde{x} &= \frac{\partial z}{\partial \widetilde{z}}\, x + \frac{\partial z}{\partial \widetilde{w}}\, y + S^1(z, w), & \widetilde{z} &= \alpha(z, w), \\ \widetilde{y} &= \frac{\partial w}{\partial \widetilde{z}}\, x + \frac{\partial w}{\partial \widetilde{w}}\, y + S^2(z, w), & \widetilde{w} &= \beta(z, w), \end{aligned}$$

(see [3] for details).

Therefore, we can change a basis in such a way that the metric g has the simplest form. Corresponding functions a, b, c for Walker's form are given in Table 4.

group	metric	basis of null distribution	functions from Walker form
G_4	$g_1^{+\lambda}$	$X_1 - \frac{1}{\sqrt{\lambda}}X_3, X_4$	$a = a(y) = \frac{1}{\lambda+1}y^2$ $b = \lambda$ $c = c(y) = y$
	$g_1^{-\lambda}, \lambda \neq 1$	$X_1 - \frac{1}{\sqrt{\lambda}}X_3, X_4$	$a = a(y) = -\frac{1}{\lambda-1}y^2$ $b = -\lambda$ $c = c(y) = y$
	$g_2^{\pm\lambda}$	$X_2 + \frac{1}{\sqrt{\lambda}}X_3, X_4$	$a = a(y,z) = \pm\frac{1}{\lambda}(y+z)^2$ $b = \pm\lambda$ $c = c(y) = y$
	g_2^0	X_3, X_4	$a = a(w) = \pm w^2$ $b = 2$ $c = c(y) = y$
$H_3 \times \mathbb{R}$	g_1	$X_1 + X_2, X_3$	$a = 0$ $b = -1$ $c = c(y) = y$
	$g_{02}^{\pm}$	$X_2 + X_3, X_4$	$a = a(y) = \mp y^2$ $b = \mp 1$ $c = c(y) = y$

Table 4. Walker form of metrics

Remark 2.6. *Note that all of the Walker metrics from the Table 4 have zero scalar curvature. Hence, the components of the self-dual part of the Weyl tensor are given by*

$$W_{11}^{+} = W_{13}^{+} = W_{33}^{+} = a_{ww} + a_{yy}\frac{b^2}{4}, \qquad W_{12}^{+} = W_{22}^{+} = W_{23}^{+} = 0,$$

where indexes denote the partial derivatives. If the scalar curvature is zero, then the following holds (see [8]*):*

a) W^+ *vanishes if and only if* $W_{11}^+ = W_{12}^+ = 0$,
b) W^+ *is 2-step nilpotent operator if and only if* $W_{11}^+ \neq 0$, $W_{12}^+ = 0$,
c) W^+ *is 3-step nilpotent operator if and only if* $W_{12}^+ \neq 0$.

We can observe that all of our Walker metrics have 2-step nilpotent operator W^+*, except the metric* g_1 *which is previously shown to be conformally flat.*

More details on conformally flat Walker metrics in dimension four can be found in [1]*. Note that our metric* g_1 *is not considered in their study, since c is not constant.*

Remark 2.7. *If we consider the action of the curvature map R on $\Lambda^2 T_pN = \Lambda^+ \oplus \Lambda^-$ as in* (3)*, with the standard identification*

$$B = \begin{pmatrix} 0 & B \\ B* & 0 \end{pmatrix},$$

we conclude that our metric g_2^0 on the Lie group G_4 satisfies conditions $W^- = 0$, $B^2 \big| \Lambda^- = 0$ (here Λ^- is the bundle of anti-self-dual bivectors) and the scalar curvature is constant. Metrics with these properties are studied in [5]*.*

Lemma 2.1. *On the 4-dimensional nilpotent Lie group N, all left invariant Walker metrics are geodesically complete.*

Proof. All left invariant metrics on $H_3 \times \mathbb{R}$ are geodesically complete (see [18]), therefore we have to prove the statement only for the metrics on the Lie group G_4.

Let $\gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a geodesic curve on (G_4, g) with $\gamma(0) = (x_1^0, x_2^0, x_3^0, x_4^0)$ and $\dot\gamma(0) = (\dot x_1^0, \dot x_2^0, \dot x_3^0, \dot x_4^0)$.

The Lagrangian associated with the metric g has the form:

$$L(t) = b\dot x_4(t)^2 + \dot x_3(t)\left(2\dot x_1(t) + a\dot x_3(t)\right) + 2\dot x_4(t)\left(\dot x_2(t) + x_2(t)\dot x_3(t)\right).$$

Here $a = a(x_2, x_3, x_4)$ and b is constant. From the Euler–Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x_k}\right) - \frac{\partial L}{\partial x_k} = 0, \quad k = 1, \ldots, 4,$$

we get the following system of partial differential equations:

$$\begin{aligned} \ddot x_3 &= 0, \\ 2\dot x_3 \dot x_4 - 2\ddot x_4 + a_{x_2}\,\dot x_3^2 &= 0, \\ 2\left(x_2\ddot x_4 + \ddot x_1 + \dot x_2\dot x_4 + a_{x_4}\,\dot x_3\dot x_4\right) + \dot x_3\left(a_{x_3}\,\dot x_3 + 2a_{x_2}\,\dot x_2\right) &= 0, \\ -2\left(\dot x_2\dot x_3 + b\ddot x_4 + \ddot x_2\right) + a_{x_4}\,\dot x_3^2 &= 0. \end{aligned}$$

A standard but long calculation leads to the conclusion that all geodesics exist for every $t \in \mathbb{R}$. □

3. Projective equivalence

In this section we look for metrics on the nilpotent Lie groups $H_3 \times \mathbb{R}$ and G_4 that are geodesically equivalent to the left invariant metrics from Theorems 1.3 and 1.4.

We say that the metrics g and $\bar g$ are *geodesically equivalent* if every geodesic of g is a (reparameterized) geodesic of $\bar g$. We say that they are *affinely equivalent* if their Levi-Civita connections coincide. We call a metric g *geodesically rigid* if every metric $\bar g$, geodesically equivalent to g, is proportional to g. Weyl [39] proved that two conformally and geodesically equivalent metrics are proportional with a constant coefficient of proportionality.

The two connections $\nabla = \{\Gamma^i_{jk}\}$ and $\bar\nabla = \{\bar\Gamma^i_{jk}\}$ have the same unparameterized geodesics, if and only if their difference is a pure trace: there exists a $(0,1)$-tensor ϕ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_k\phi_j + \delta^i_j\phi_k. \tag{5}$$

For a parameterized geodesic $\gamma(\tau)$ of $\bar{\nabla}$, the curve $\gamma(\tau(t))$ is a parameterized geodesic of ∇ if and only if the parameter transformation $\tau(t)$ satisfies the following ODE:

$$\phi_\alpha \dot{\gamma}^\alpha = \frac{1}{2}\frac{d}{dt}\left(\log\left|\frac{d\tau}{dt}\right|\right),$$

where the velocity vector of γ with respect to the parameter t is denoted by $\dot{\gamma}$.

Suppose that ∇ and $\bar{\nabla}$, related by (5), are the Levi-Civita connections of the metrics g and $\bar{g}$, respectively. Contracting (5) with respect to i and j, we obtain $\bar{\Gamma}^\alpha_{\alpha i} = \Gamma^\alpha_{\alpha i} + (n+1)\phi_i$. From the other side, for the Levi-Civita connection ∇ of a metric g, we have $\Gamma^\alpha_{\alpha k} = \frac{1}{2}\frac{\partial \log|\det(g)|}{\partial x_k}$. Thus,

$$\phi_i = \frac{1}{2(n+1)}\frac{\partial}{\partial x_i}\log\left|\frac{\det(\bar{g})}{\det(g)}\right| = \phi_{,i}$$

for the function $\phi : M \longrightarrow \mathbb{R}$ given by

$$\phi := \frac{1}{2(n+1)}\log\left|\frac{\det(\bar{g})}{\det(g)}\right|. \tag{6}$$

The formula (5) implies that g and $\bar{g}$ are geodesically equivalent if and only if for the function ϕ, given by (6), the following holds:

$$\bar{g}_{ij;k} - 2\bar{g}_{ij}\phi_k - \bar{g}_{ik}\phi_j - \bar{g}_{jk}\phi_i = 0, \tag{7}$$

where "semi-colon" denotes the covariant derivative with respect to the connection ∇.

Consider the projective Weyl tensor

$$W^i_{jkl} := R^i_{jkl} - \frac{1}{n-1}\left(\delta^i_l R_{jk} - \delta^i_k R_{jl}\right),$$

where R^i_{jkl} and R_{jk} are components of the curvature and the Ricci tensor of manifold (M^n, g). Weyl has shown [38] that the projective Weyl tensor does not depend on the choice of a connection within the projective class. Note that the converse is not true (see [21]).

In order to find the projective class of our 4-dimensional metrics, we use the formula

$$-2g^{a(i}W^{j)}_{akl} = g^{ab}W^{(i}_{ab[l}\delta^{j)}_{k]}, \tag{8}$$

proposed by Matveev in [28], where the brackets "[ ]" denote the skew-symmetrization without division, and the brackets "( )" denote the symmetrization without division.

Every metric g geodesically equivalent to $\bar{g}$ has the same projective Weyl tensor as $\bar{g}$. We view the equation (8) as the system of homogeneous linear equations on the components of g; every metric g geodesically equivalent to $\bar{g}$ satisfies this system of equations (8).

Notice that this system is quite complicated already in dimension four, thus it is less useful in higher dimensions.

General algorithm:

In order to find all metrics $\bar{g}$ that are projectively equivalent to g we first look for the solutions of the homogeneous system of the equations (8). The condition (8) is necessary but not sufficient, so we call such metrics *candidates*. Then, we check if each candidate satisfies the condition (7) which is sufficient for projective equivalence. If all candidates satisfying condition (7) are proportional to g, by a result of Weyl [39] the metric g is geodesically rigid. If the function ϕ satisfying relations (7) is constant, then $\bar{g}$ is affinely equivalent to g.

Remark 3.1. *Although the proposed algorithm was presented in* [28] *considering only the Lorentzian metrics, it is also valid for the metrics of arbitrary signature. Note that, by computing the candidates, we find all the possible pairs of geodesically equivalent metrics, regardless of their signature.*

Applying the proposed algorithm, in a similar manner as in the case of Lorentz signature (see [7]) we can prove the following two theorems.

Theorem 3.1. *Let g and $\bar{g}$ be geodesically equivalent metrics on the Lie group G_4 and let g be non-flat, left invariant. Then the following two possibilities hold:*

a) *If g is of type $g_1^{\pm\lambda}$, $g_2^{\pm\lambda}$, or g_2^0, then g and $\bar{g}$ are affinely equivalent. The family of metrics $\bar{g}$ is 2-dimensional. Every metric $\bar{g}$ is left invariant and of neutral signature.*

b) *If g is of type g_A, $g_3^{\pm\lambda}$ or $g_4^{\pm\lambda}$, then it is geodesically rigid, i.e. $\bar{g} = cg$, $c \in \mathbb{R}$.*

Note: In Table 5 we denote by $c_1, c_2 \in \mathbb{R}, c_1 \neq 0$ constants and by $p, q, r, s \in C^\infty(G_4)$ functions on the Lie group.

Theorem 3.2. *Let g and $\bar{g}$ be geodesically equivalent metrics on the Lie group $H_3 \times \mathbb{R}$, with g non-flat, left invariant. Then $\bar{g}$ is affinely equivalent to g and left invariant. The family of metrics $\bar{g}$ is 2-dimensional.*

Note: In Table 6, we denote by $c_1, c_2, c_3 \in \mathbb{R}, c_1,\ c_3 \neq 0$ constants, and by $p, q \in C^\infty(H_3 \times \mathbb{R})$ functions on the Lie group.

Remark 3.2. *Note that if a metric g is flat and g and $\bar{g}$ are geodesically equivalent, then their Levi-Civita connections coincide (moreover, they are equal to zero). Therefore, flat metrics were excluded from our calculations.*

Our results, together with the known facts for the Riemannian and Lorentz case, give rise to the following, more general statement:

Theorem 3.3. *Let g be a left invariant metric on the 4-dimensional nilpotent Lie group. If g and $\bar{g}$ are geodesically equivalent, then they are either affinely equivalent or g is geodesically rigid. The metric $\bar{g}$ is also left invariant, but not necessarily the same signature as g.*

4. Isometry groups

Let us denote by $I(N)$ the isometry group of N. Set $O(N) = Aut(N) \cap I(N)$ and let $I^{aut}(N) = O(N) \ltimes N$, where N acts by the left translations. Furthermore,

original metric (g)	candidate metrics	equivalent metric
$g_A = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix}$ $\epsilon_1, \epsilon_2 \in \{-1, 1\}$	$p \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix}$	$c_1 \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix}$
$g_1^{+\lambda} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$p \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -pq & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \lambda \neq \frac{1}{3}$ $\begin{pmatrix} q & -pqr & 0 & 0 \\ -pqr & p^2(qr^2 - s) & 0 & p \\ 0 & 0 & \frac{1}{3}p & 0 \\ 0 & p & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & c_1 \\ 0 & 0 & \lambda c_1 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix}$
$g_1^{-\lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ case $\lambda \neq 1$	$p \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -pq & 0 & 1 \\ 0 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & c_1 \\ 0 & 0 & -\lambda c_1 & 0 \\ 0 & c_1 & 0 & 0 \end{pmatrix}$
$g_2^{\pm\lambda} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \mp 1 & 0 & 0 \\ 0 & 0 & \pm\lambda & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$p \begin{pmatrix} -pq & 0 & 0 & 1 \\ 0 & \mp 1 & 0 & 0 \\ 0 & 0 & \pm\lambda & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} c_2 & 0 & 0 & c_1 \\ 0 & \mp c_1 & 0 & 0 \\ 0 & 0 & \pm\lambda c_1 & 0 \\ c_1 & 0 & 0 & 0 \end{pmatrix}$
$g_3^{\pm\lambda} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}$	$p \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}$	$c_1 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}$
$g_4^{\pm\lambda} = \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}$	$p \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}$	$c_1 \begin{pmatrix} \mp 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \pm\lambda \end{pmatrix}$
$g_2^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pm 1 \\ 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \end{pmatrix}$	$p \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -pq & 0 & \pm 1 \\ 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & c_1 & 0 \\ 0 & c_2 & 0 & \pm c_1 \\ c_1 & 0 & 0 & 0 \\ 0 & \pm c_1 & 0 & 0 \end{pmatrix}$

TABLE 5. Projectively equivalent metrics on the Lie group G_4

original metric (g)	candidate metrics	equivalent metric
$g_\mu^\pm = \begin{pmatrix} \mp1 & 0 & 0 & 0 \\ 0 & \mp1 & 0 & 0 \\ 0 & 0 & \pm\mu & 0 \\ 0 & 0 & 0 & \pm1 \end{pmatrix}$	$\begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & \mu p & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$	$\begin{pmatrix} -c_1 & 0 & 0 & 0 \\ 0 & -c_1 & 0 & 0 \\ 0 & 0 & \mu c_1 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix}$
$g_\lambda^\pm = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \pm\lambda & 0 \\ 0 & 0 & 0 & \mp1 \end{pmatrix}$	$\begin{pmatrix} \pm p & 0 & 0 & 0 \\ 0 & \mp p & 0 & 0 \\ 0 & 0 & \lambda p & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$	$\begin{pmatrix} \pm c_1 & 0 & 0 & 0 \\ 0 & \mp c_1 & 0 & 0 \\ 0 & 0 & \lambda c_1 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix}$
$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & p & -p^2q \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & c_1 & c_2 \end{pmatrix}$
$g_{02}^\pm = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -p^2q & 0 & 0 & p \\ 0 & \pm p & 0 & 0 \\ 0 & 0 & \mp p & 0 \\ p & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} c_2 & 0 & 0 & c_1 \\ 0 & \pm c_1 & 0 & 0 \\ 0 & 0 & \mp c_1 & 0 \\ c_1 & 0 & 0 & 0 \end{pmatrix}$

Table 6. Projectively equivalent metrics on the Lie group $H_3 \times \mathbb{R}$

let $\widetilde{O}(N)$ be the subgroup of $I(N)$ that fixes the identity element of N, then the following holds:

$$I(N) = \widetilde{O}(N) \cdot N. \tag{9}$$

Let M be a pseudo-Riemannian manifold and let g be the corresponding pseudo-Riemannian metric. A vector field X on M is called a Killing vector field if

$$g(\nabla_V X, W) + g(V, \nabla_W X) = 0 \tag{10}$$

holds for any vector fields W and V on M.

All left invariant pseudo-Riemannian metrics on $H_3 \times \mathbb{R}$ are geodesically complete (see [18]), thus the set of all Killing fields is the Lie algebra of the full isometry group. Also, the 1-parameter groups of isometries constituting the flow of any Killing field are global. Therefore isometries produced by integration of Killing fields on $H_3 \times \mathbb{R}$ are global. It is important to notice that not all metrics on the Lie group G_4 are complete.

Theorem 4.1. *a) If the center of the Lie group G_4 is non-degenerate, then $\widetilde{O}(G_4)$ is discrete and the isometry group $I(G_4)$ of the corresponding metric is given by*

$$I(G_4) \simeq \operatorname{diag}(\epsilon, \bar\epsilon, \epsilon\bar\epsilon, \bar\epsilon) \ltimes G_4, \quad \epsilon, \bar\epsilon \in \{-1, 1\}.$$

b) In the case of the metrics $g_\mu^\pm$ and $g_\lambda^\pm$ on the Lie group $H_3 \times \mathbb{R}$ the isometry group is given by $I(H_3 \times \mathbb{R}) = \widetilde{O}(H_3 \times \mathbb{R}) \ltimes (H_3 \times \mathbb{R})$, where

$$\widetilde{O}(H_3 \times \mathbb{R}) = \left\{ \left(\begin{array}{cccc} \epsilon\cos t & -\epsilon\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \bar\epsilon \end{array} \right) \middle| \begin{array}{l} t \in \mathbb{R}, \\ \epsilon, \bar\epsilon \in \{-1, 1\} \end{array} \right\}, \text{ for } g_\mu^\pm,$$

$$\widetilde{O}(H_3 \times \mathbb{R}) = \left\{ \left(\begin{array}{cccc} \epsilon\cosh t & \epsilon\sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \bar\epsilon \end{array} \right) \middle| \begin{array}{l} t \in \mathbb{R}, \\ \epsilon, \bar\epsilon \in \{-1, 1\} \end{array} \right\}, \text{ for } g_\lambda^\pm.$$

c) In the case of the metric g_1 on the group $H_3 \times \mathbb{R}$, the isometry group is given by $\widetilde{O}(H_3 \times \mathbb{R}) \cdot (H_3 \times \mathbb{R})$, where $\widetilde{O}(H_3 \times \mathbb{R})$ is a 3-dimensional Lie group with the corresponding Lie algebra defined by the structure equations

$$[\xi^1, \xi^2] = \xi^2, \quad [\xi^1, \xi^3] = 0, \quad [\xi^2, \xi^3] = 0. \tag{11}$$

Proof. (a) After solving the Killing equation (10), we observe that the algebra of isometries is generated only by the left translations. The group of isometries fixing the identity coincides with the group of isometric automorphisms, therefore the proposition holds.

(b) A direct calculation shows that the algebra of Killing vector fields is 5-dimensional. Also, $H_3 \times \mathbb{R}$ is a normal subgroup of $I(H_3 \times \mathbb{R})$, thus in (9) we have the semi-direct product.

(c) Since $\nabla R \equiv 0$, the metric g_1 makes the simply connected group $H_3 \times \mathbb{R}$ a symmetric space. The group of isometries fixing the identity is identified with the group of linear isomorphisms of $\mathfrak{h}_3 \oplus \mathbb{R}$ preserving the curvature tensor. This shows that the isotropy of the identity inside $I(H_3 \times \mathbb{R})$ has dimension three, so $I(H_3 \times \mathbb{R})$ has at least seven linearly independent Killing vector fields.

Also, note that g_1 is the only Walker metric corresponding to non-degenerate center. We can change basis in such a way that the metric g_1 has the form presented in Table 4.

By solving the Killing equation (10), we obtain exactly seven Killing vectors $\{\xi^1, \dots, \xi^7\}$ given by:

$$\xi^1 = -\frac{w^2}{2}\frac{\partial}{\partial x} + (w - y)\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \quad \xi^4 = -e^{-z}\frac{\partial}{\partial y},$$

$$\xi^2 = -w\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad \xi^5 = -e^z y\frac{\partial}{\partial x} + \frac{1}{2}e^z\frac{\partial}{\partial y} + e^z\frac{\partial}{\partial w},$$

$$\xi^3 = \frac{\partial}{\partial z}, \qquad \xi^6 = \frac{\partial}{\partial x},$$

$$\xi^7 = \frac{\partial}{\partial w}.$$

One can check that $\{\xi^1, \xi^2, \xi^3\}$ span the Lie algebra with the structure equations (11). The remaining four vectors generate a Lie algebra defined by non-zero commutator $[\xi^4, \xi^5] = \xi^6$, i.e. the Lie algebra $\mathfrak{h}_3 \oplus \mathbb{R}$.

The following relations are satisfied:

$$\begin{aligned}
&[\xi^1,\xi^4] = -\xi^4, && [\xi^2,\xi^4] = 0, && [\xi^3,\xi^4] = -\xi^4,\\
&[\xi^1,\xi^5] = \xi^5, && [\xi^2,\xi^5] = 0, && [\xi^3,\xi^5] = \xi^5,\\
&[\xi^1,\xi^6] = 0, && [\xi^2,\xi^6] = 0, && [\xi^3,\xi^6] = 0,\\
&[\xi^1,\xi^7] = -\xi^2 - \xi^7, && [\xi^2,\xi^7] = \xi^6, && [\xi^3,\xi^7] = 0.
\end{aligned}$$

Since $[\xi^1, \xi^7] = -\xi^2 - \xi^7 \notin \mathcal{L}(\xi^4, \xi^5, \xi^6, \xi^7)$, $H_3 \times \mathbb{R}$ is not a normal subgroup of $I(H_3 \times \mathbb{R})$.

On the other hand, an algebra generated by $\{\xi^2, \xi^4, \xi^5, \xi^6, \xi^7\}$ is a nilpotent Lie algebra corresponding to the algebra L_5^4 from the classification of Morozov [32] or $\mathfrak{g}_{5,1}$ from Magnin's classification [27]. Therefore, the algebra of isometries is isomorphic to $\mathbb{R}^2 \ltimes \mathfrak{g}_{5,1}$. □

Remark 4.1. *If we consider the metric g_A on the Lie group G_4 when $c = 0$, i.e. in the case of degenerate center, but a non-degenerate commutator subalgebra, we get that the isometry group is the same as in a non-degenerate case.*

Example 1. Metrics $g_{01}^{\pm}$ and g_0^0 on the Lie group $H_3 \times \mathbb{R}$ and metrics g_1^0 and $g_1^{-\lambda}$ (for $\lambda = 1$) on the Lie group G_4 are flat. Thus, the isometry group is $O(2,2) \ltimes N$. ◇

From the above considerations, we conclude that if the center is degenerate and the holonomy algebra is A_{17}, the corresponding metrics are Walker metrics of the specific form given in the Table 4. Since all Walker metrics on a 4-dimensional nilpotent Lie group N are geodesically complete (see Lemma 2.1), the Lie algebra of the isometry group coincides with the Lie algebra of the Killing vector fields.

Example 2. First, let us consider the metric $g_{02}^{\pm}$ on the Lie group $H_3 \times \mathbb{R}$. The algebra of Killing vector fields is 6-dimensional and given by

$$\begin{aligned}
\xi^1 &= \left(\mp 6w - 3yz^2\right)\frac{\partial}{\partial x} \mp 3z(z-2)\frac{\partial}{\partial y} + z^3\frac{\partial}{\partial w}, && \xi^4 = \frac{\partial}{\partial w},\\
\xi^2 &= -2yz\frac{\partial}{\partial x} \mp 2(z-1)\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial w}, && \xi^5 = \frac{\partial}{\partial z},\\
\xi^3 &= -y\frac{\partial}{\partial x} \mp \frac{\partial}{\partial y} + z\frac{\partial}{\partial w}, && \xi^6 = \frac{\partial}{\partial x}.
\end{aligned}$$

The subalgebra spanned by vectors $\{\xi^3, \xi^4, \xi^5, \xi^6\}$ is defined by the non-zero commutator $[\xi^5, \xi^3] = \xi^4$, thus it is isomorphic to $\mathfrak{h}_3 \oplus \mathbb{R}$.

Two remaining vectors ξ^1 and ξ^2 span an algebra isomorphic to $\mathbb{R}^2$ and they satisfy the following relations:

$$\begin{aligned}
&[\xi^1,\xi^3] = 0, && [\xi^1,\xi^4] = \pm 6\xi^6, && [\xi^1,\xi^5] = -3\xi^2, && [\xi^1,\xi^6] = 0,\\
&[\xi^2,\xi^3] = \mp 2\xi^6, && [\xi^2,\xi^4] = 0, && [\xi^2,\xi^5] = -2\xi^3, && [\xi^2,\xi^6] = 0.
\end{aligned}$$

The algebra of isometries is nilpotent, containing the maximal abelian ideal of the third order. Thus, it corresponds to the algebra of type 21 from the classification of Morozov [32] (see also [27]). ◇

Things get more complicated if we consider Walker metrics on the Lie group G_4.

Example 3. First, let us consider the metric g_2^0. The algebra of Killing vector fields is spanned by

$$\begin{aligned} \xi^1 &= \frac{\partial}{\partial z}, & \xi^4 &= \mp(z+1)w\frac{\partial}{\partial x} \pm z\frac{\partial}{\partial y} + \frac{\partial}{\partial w}, \\ \xi^2 &= \frac{\partial}{\partial x}, & \xi^5 &= -e^{-z}\frac{\partial}{\partial y}, \\ \xi^3 &= -w\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, & \xi^6 &= -e^{z}(y \pm w)\frac{\partial}{\partial x} - \left(1 \mp \frac{1}{2}\right)e^{z}\frac{\partial}{\partial y} + e^{z}\frac{\partial}{\partial w}. \end{aligned}$$

It is an easy calculation to show that the algebra of isometries is solvable and it contains the maximal nilpotent ideal generated by vectors $\{\xi^2, \xi^3, \xi^4, \xi^5, \xi^6\}$. This ideal is isomorphic to $\mathfrak{g}_{5,4}$, therefore we observe that the algebra of isometries is isomorphic to $\mathfrak{g}_{6,84}$ (from the the classification of Mubarakzjanov [33]). ◇

Example 4. In the case of the metric $g_2^{\pm\lambda}$, the Killing vectors have the form

$$\begin{aligned} \xi^1 &= \left(\mp 6\lambda w - 3yz^2 + \frac{3}{4}z^4 - 2z^3\right)\frac{\partial}{\partial x} \mp 3\lambda z(z-2)\frac{\partial}{\partial y} + z^3\frac{\partial}{\partial w}, \\ \xi^2 &= \frac{1}{3}z\left(-6y + 2z^2 - 3z\right)\frac{\partial}{\partial x} \mp 2\lambda(z-1)\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial w}, \\ \xi^3 &= \left(-y + \frac{1}{2}z^2\right)\frac{\partial}{\partial x} \mp \lambda\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}, \\ \xi^4 &= \frac{\partial}{\partial w}, \quad \xi^5 = w\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \xi^6 = \frac{\partial}{\partial x}. \end{aligned}$$

Thus the algebra of isometries is 6-dimensional and the subalgebra spanned by $\{\xi^3, \xi^4, \xi^5, \xi^6\}$ is 3-step nilpotent.

After an appropriate change of basis vectors, we can see that the algebra of isometries is exactly the algebra from Example 2. ◇

Example 5. Finally, we discuss the metrics $g_1^{+\lambda}$ and $g_1^{-\lambda}$ (for $\lambda \neq 1$).

For convenience, set $\mu = \frac{1}{1\pm\lambda}$. Note that $0 < \mu < 1$ for every $\lambda > 0$ in the case of the metric $g_1^{+\lambda}$, while in the case of $g_1^{-\lambda}$ two possibilities may occur: $\mu > 1$ for $0 < \lambda < 1$ and $\mu < 0$ for $\lambda > 1$.

After a long but straightforward calculation, we obtain 6-dimensional algebras of Killing vector fields:

$$\begin{aligned} \xi^3 &= (w-y)\frac{\partial}{\partial x} - \frac{1}{\mu}\frac{\partial}{\partial y} + z\frac{\partial}{\partial w}, & \xi^5 &= \frac{\partial}{\partial w}, \\ \xi^4 &= \frac{\partial}{\partial x}, & \xi^6 &= \frac{\partial}{\partial z}, \end{aligned}$$

where

$$\xi^1 = e^{z\sqrt{\mu}} \left(-y\sqrt{\mu}\frac{\partial}{\partial x} + \frac{\mu - \sqrt{\mu}}{\mu}\frac{\partial}{\partial y} + \frac{\partial}{\partial w} \right),$$
$$\xi^2 = e^{-z\sqrt{\mu}} \left(y\sqrt{\mu}\frac{\partial}{\partial x} + \frac{\mu + \sqrt{\mu}}{\mu}\frac{\partial}{\partial y} + \frac{\partial}{\partial w} \right),$$

when $\mu > 0$, and

$$\begin{aligned}\xi^1 &= \mu y \sin(z\sqrt{-\mu})\frac{\partial}{\partial x} + (\sin(z\sqrt{-\mu}) - \sqrt{-\mu}\cos(z\sqrt{-\mu}))\frac{\partial}{\partial y} \\ &\quad - \sqrt{-\mu}\cos(z\sqrt{-\mu})\frac{\partial}{\partial w}, \\ \xi^2 &= -\mu y \cos(z\sqrt{-\mu})\frac{\partial}{\partial x} - (\cos(z\sqrt{-\mu}) + \sqrt{-\mu}\sin(z\sqrt{-\mu}))\frac{\partial}{\partial y} \\ &\quad - \sqrt{-\mu}\sin(z\sqrt{-\mu})\frac{\partial}{\partial w},\end{aligned}$$

when $\mu < 0$.

In both cases $\{\xi^3, \xi^4, \xi^5, \xi^6\}$ generate the algebra of left translations and $\{\xi^1, \xi^2\}$ form an algebra isomorphic to $\mathbb{R}^2$.

The algebra of Killing vector fields for $g_1^{+\lambda}$ and $g_1^{-\lambda}$ (for $\lambda \neq 1$) is solvable, containing the maximal nilpotent ideal of dimension five and it is isomorphic to the algebra $\mathfrak{g}_{6,84}$ (see [33]). ◇

The preceding examples and Remark 4.1 directly imply:

Lemma 4.1. *(a) If the center of the Lie group $H_3 \times \mathbb{R}$ is degenerate and $H_3 \times \mathbb{R}$ is non-flat, then the corresponding algebra of isometries is 6-dimensional and isomorphic to the algebra listed in the table below.*

(b) If the Lie group G_4 is non-flat with degenerate center, then the corresponding algebra of isometries is either 4-dimensional, generated by the left translations (in the case of g_A, $c = 0$) or it is 6-dimensional and isomorphic to one of the algebras presented in Table 7.

Corollary 4.1. *For any left invariant metric the following inequality holds:*

$$\dim I(H_3 \times \mathbb{R}) > \dim(H_3 \times \mathbb{R}).$$

This does not hold for the Lie group G_4.

Corollary 4.2. *If the center of a 4-dimensional nilpotent Lie group N is degenerate and N is non-flat, then $\dim \widetilde{O}(N) \leq 2$. The equality holds for all metrics except the metric g_A (for $c = 0$) on the Lie group G_4, when $\widetilde{O}(N)$ is discrete.*

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group	metric	algebra of isometries	nilpotent/ solvable
G_4	$g_1^{+\lambda}$ $g_1^{-\lambda}$ $(\lambda \neq 1)$	$[e_2, e_4] = e_1, [e_3, e_5] = e_1,$ $[e_2, e_6] = e_2, [e_4, e_6] = -e_4, [e_5, e_6] = e_3$	solvable
	$g_2^{\pm\lambda}$	$[e_1, e_2] = e_3, [e_1, e_5] = e_6,$ $[e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_3, e_4] = e_6$	nilpotent
	g_2^0	$[e_2, e_4] = e_1, [e_3, e_5] = e_1,$ $[e_2, e_6] = e_2, [e_4, e_6] = -e_4, [e_5, e_6] = e_3$	solvable
$H_3 \times \mathbb{R}$	$g_{02}^{\pm}$	$[e_1, e_2] = e_3, [e_1, e_5] = e_6,$ $[e_2, e_3] = e_4, [e_2, e_4] = e_5, [e_3, e_4] = e_6$	nilpotent

Table 7. Isometry algebras for Walker metrics

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