BEST SIMULTANEOUS APPROXIMATION ON SMALL REGIONS BY RATIONAL FUNCTIONS

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Abstract. We study the behavior of best simultaneous \((l^q, L^p)\)-approximation by rational functions on an interval, when the measure tends to zero. In addition, we consider the case of polynomial approximation on a finite union of intervals. We also get an interpolation result.

1. Introduction

Let \(x_j \in \mathbb{R}, 1 \leq j \leq k, k \in \mathbb{N}\), and let \(B_j\) be pairwise disjoint closed intervals centered at \(x_j\) and radius \(\beta > 0\). Let \(n, m \in \mathbb{N} \cup \{0\}\) and we suppose that

\[n + m + 1 = kc + d, \quad c, d \in \mathbb{N} \cup \{0\}, \quad d < k.\]

We denote \(C_s(I), s \in \mathbb{N} \cup \{0\}\), the space of real functions defined on \(I := \bigcup_{j=1}^k B_j\), which are continuously differentiable up to order \(s\) on \(I\). For simplicity we write \(C(I)\) instead of \(C^0(I)\). We also denote \(\text{co}(I)\) the convex hull of \(I\). Let \(\Pi^n\) be the class of algebraic polynomials of degree at most \(n\), and \(\partial P\) the degree of \(P \in \Pi^n\).

We consider the set of rational functions

\[\mathcal{R}^n_{m} := \left\{ \frac{P}{Q} : P \in \Pi^n, \ Q \in \Pi^m, \ Q \neq 0 \right\}.\]

Clearly, we can assume \(\frac{P}{Q} \in \mathcal{R}^n_{m}\) with \(L^2\)-norm of \(Q\) equal to one on \(I\). Recall that \(\frac{P}{Q} \in \mathcal{R}^n_{m}\) is called normal if this expression is irreducible and either \(\partial P = n\) or \(\partial Q = m\), and the null function is called normal if \(m = 0\) (see [10]).

If \(h \in \mathcal{C}(I)\), we put

\[\|h\| := \left( \int_I |h(t)|^p \frac{dt}{|I|} \right)^{1/p}, \quad 1 \leq p < \infty,\]

where \(|I|\) is the Lebesgue measure of \(I\). If \(p = \infty\), as it is usual, \(\| \cdot \|\) will be the supreme norm. For each \(0 < \epsilon \leq 1\), we also put \(\|h\|_\epsilon = \|h^\epsilon\|\), where \(h^\epsilon(t) = h(\epsilon(t - x_j) + x_j), t \in B_j\).

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If $\chi_{B_j}$ is the characteristic function of the set $B_j$, we write $\|h\|_{B_j} = \|h\chi_{B_j}\|$. We denote $I_\epsilon = \bigcup_{j=1}^k [x_j - \epsilon, x_j + \epsilon]$.

Let $f_1, \ldots, f_l \in \mathcal{C}(I)$ and $1 \leq q < \infty$. The rational function $u \in \mathcal{R}_m^n$, $0 < \epsilon \leq 1$, is called a best simultaneous $(l^q, L^p)$-approximation $((l^q, L^p)$-b.s.a.) of $f_1, \ldots, f_l$ from $\mathcal{R}_m^n$ on $I_\epsilon$ if

$$\left( \sum_{i=1}^l \|f_i - u\|_\epsilon^q \right)^{1/q} = \inf_{u \in \mathcal{R}_m^n} \left( \sum_{i=1}^l \|f_i - u\|_\epsilon^q \right)^{1/q}. \quad (1)$$

For $q = \infty$, we need to consider in (1) the supreme norm on $\mathbb{R}^l$.

If a net $\{u_\epsilon\}$ has a limit in $\mathcal{R}_m^n$ as $\epsilon \to 0$, it is called a best simultaneous local $(l^q, L^p)$-approximation of $f_1, \ldots, f_l$ from $\mathcal{R}_m^n$ on $\{x_1, \ldots, x_k\}$ $((l^q, L^p)$-b.s.l.a.).

A pair $(P, Q) \in \Pi^n \times \Pi^m$ is a Padé approximant pair of $f$ on $\{x_1, \ldots, x_k\}$ if $Q \neq 0$ and

$$(Qf - P)(x) = o((x - x_j)^{\epsilon - 1}), \quad as \ x \to x_j, \ 1 \leq j \leq k.$$ If $\left( f - \frac{P}{Q} \right)(x) = o((x - x_j)^{\epsilon - 1})$, as $x \to x_j$, $1 \leq j \leq k$, then $\frac{P}{Q}$ is called a Padé rational approximant of $f$ on $\{x_1, \ldots, x_k\}$. This rational approximant may not exist. If $d = 0$ there is at most one, and we denote it by $Pa(f)$ when it exists.

In [6] the author studied properties of interpolation of best rational approximation to a single function with respect to an integral norm, which includes the $L^p$-norm, $1 \leq p < \infty$. In [7] the authors proved that the best approximation to $l^{-1} \sum_{j=1}^l f_j$ from an arbitrary class of functions, $S$, is identical with the $(l^2, L^2)$-b.s.a. of $f_1, \ldots, f_l$ from $S$. However it is known that the $(l^q, L^p)$-b.s.a., in general, does not match with the best approximation to the mean of the functions $f_1, \ldots, f_l$ when $S = \Pi^n$ (see [8]). The $(l^\infty, L^p)$-b.s.l.a. from $\Pi^n$ was studied in [4] and [5]. In [2], the authors showed that the $(l^q, L^p)$-b.s.l.a. to two functions is the average of their Taylor polynomials.

In this paper, we prove an interpolation property of any $(l^q, L^p)$-b.s.a. to two functions from $\mathcal{R}_m^n$. As a consequence, we prove the existence and characterization of the $(l^q, L^p)$-b.s.l.a. when $q > 1$ and $k = 1$. Analogous results over $(l^q, L^p)$-b.s.l.a. were obtained, for $m = 0$, in several intervals. All our theorems generalize previous results for a single function.

2. Preliminary results

Henceforward we suppose that $1 < p < \infty$ and $1 \leq q < \infty$, except in Lemma [4.3] and Theorem [4.4] where we assume $q > 1$. First, we establish an existence theorem for the $(l^q, L^p)$-b.s.a.

**Theorem 2.1.** Let $f_1, f_2 \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. Then there exists a $(l^q, L^p)$-b.s.a. of $f_1, f_2$ from $\mathcal{R}_m^n$ on $I_\epsilon$. 

Proof. Let \( \{ v_r = \frac{P_r}{Q_r} \in \mathcal{R}_m^n : r \in \mathbb{N} \} \) be such that
\[
\sum_{i=1}^{2} \| f_i - v_r \|^q \rightarrow \inf_{v \in \mathcal{R}_m^n} \sum_{i=1}^{2} \| f_i - v \|^q =: b \quad \text{as} \quad r \rightarrow \infty.
\]
It is easy to see that \( \{ \| v_r \|_\varepsilon : r \in \mathbb{N} \} \) is a bounded set. As the sequence \( \{ Q_r \}_{r \in \mathbb{N}} \) is uniformly bounded on compact sets, \( \{ \| P_r \|_\varepsilon : r \in \mathbb{N} \} \) is a bounded set. Now, following the same patterns of the proof of existence for best rational approximation to a single function (see [11, Theorem 2.1]), we can find a subsequence \( v_{r'} \), which converges to \( v \in \mathcal{R}_m^n \) verifying \( \sum_{i=1}^{2} \| f_i - v \|^q = b \), i.e., \( v \) is a \((l^q, L^p)\)-b.s.a. \( \Box \)

The following two lemmas can be proved analogously to [6, p. 88] and [11, p. 236], respectively.

**Lemma 2.2.** Let \( f_1, f_2 \in C(I) \) and \( 0 < \varepsilon < 1 \). Suppose that \( u_\varepsilon = \frac{P_\varepsilon}{Q_\varepsilon} \in \mathcal{R}_m^n \) is a \((l^q, L^p)\)-b.s.a. of \( f_1, f_2 \), from \( \mathcal{R}_m^n \) on \( I_\varepsilon \), and \( f_j \neq u_\varepsilon \) on \( I_\varepsilon \), \( 1 \leq j \leq 2 \). Then
\[
\sum_{j=1}^{2} \beta_j \left( \int_{I_\varepsilon} |f_j - u_\varepsilon|^{p-1} \text{sgn}(f_j - u_\varepsilon) \frac{P_\varepsilon Q - PQ_\varepsilon}{Q_\varepsilon} \right) \geq 0, \quad \frac{P_\varepsilon}{Q_\varepsilon} \in \mathcal{R}_m^n, \tag{2}
\]
where \( \beta_j = \beta_j(\varepsilon) := \frac{q}{p} \| f_j - u_\varepsilon \|_\varepsilon^{(\frac{q}{p} - 1)} \).  

**Remark 2.3.** If \( q \geq p \), the constraints \( f_j \neq u_\varepsilon \) on \( I_\varepsilon \), \( 1 \leq j \leq 2 \), are not necessary. Moreover, if \( q = p \) we observe that \( \beta_j = 1, 1 \leq j \leq 2 \).

**Lemma 2.4.** Let \( \gamma \in C(\text{co}(I)) \) be a strictly monotone function. If \( f \in C(I) \) and \( \int_I f \gamma^n = 0 \) for all \( n \in \mathbb{N} \cup \{0\} \), then \( f = 0 \).

**Lemma 2.5.** Let \( f_1, f_2 \in C(I) \) and \( 0 < \varepsilon < 1 \). Suppose that \( u_\varepsilon = \frac{P_\varepsilon}{Q_\varepsilon} \in \mathcal{R}_m^n \) is a \((l^q, L^p)\)-b.s.a. of \( f_1, f_2 \), from \( \mathcal{R}_m^n \) on \( I_\varepsilon \) and \( f_j \neq u_\varepsilon \) on \( I_\varepsilon \), \( 1 \leq j \leq 2 \). If \( u_\varepsilon \) is not normal then
\[
\sum_{j=1}^{2} \beta_j |f_j - u_\varepsilon|^{p-1} \text{sgn}(f_j - u_\varepsilon) = 0 \quad \text{on} \quad I_\varepsilon,
\]
where \( \beta_j \) was introduced in Lemma 2.2.

**Proof.** Suppose that \( u_\varepsilon \) is not normal. Let \( S = \{ S \in \Pi^1 : S(x) = x - a, a \in \mathbb{R} \setminus \text{co}(I) \} \). For \( \lambda \in \mathbb{R} \) and \( S \in S \), let \( P = P_\varepsilon S - \lambda \) and \( Q = Q_\varepsilon S \). Since \( u_\varepsilon = \frac{P_\varepsilon S}{Q_\varepsilon S} \) is a \((l^q, L^p)\)-b.s.a., by Lemma 2.2
\[
\sum_{j=1}^{2} \beta_j \left( \int_{I_\varepsilon} |f_j - u_\varepsilon|^{p-1} \text{sgn}(f_j - u_\varepsilon) \frac{\lambda}{Q_\varepsilon S} \right) \geq 0.
\]

Since \( \lambda \) is arbitrary, then
\[
\sum_{j=1}^{2} \beta_j \left( \int_{I_\varepsilon} |f_j - u_\varepsilon|^{p-1} \text{sgn}(f_j - u_\varepsilon) \frac{1}{Q_\varepsilon S} \right) = 0.
\]

Let \( h := \sum_{j=1}^{2} \beta_j |f_j - u_\varepsilon|^{p-1} \text{sgn}(f_j - u_\varepsilon) \frac{1}{Q_\varepsilon S} \in C(I) \). Then
\[
\int_{I_\varepsilon} h \frac{1}{S} = 0, \quad S \in S. \tag{3}
\]
Let \( \alpha < \min \text{co}(I) \) and \( \gamma(x) = \frac{a}{x-a}, \ a > 0 \). We choose a sufficiently small such that \( |\gamma(x)| < 1, \ x \in I \). For each \( \lambda \in [-1, 0] \) let \( S(x) = (x - \alpha) - \lambda a \). We observe that \( \sum_{n=0}^{\infty} [\lambda \gamma(x)]^n \) uniformly converges to \( \frac{1}{1-\lambda \gamma(x)} \) on \( I \). Since

\[
\int_{I_\epsilon} h(x) \frac{1}{S(x)} \, dx = \int_{I_\epsilon} \frac{h(x)}{(x - \alpha)(1 - \lambda \gamma(x))} \, dx
= \sum_{n=0}^{\infty} \lambda^n \int_{I_\epsilon} \frac{h(x)}{x - \alpha} \gamma^n(x) \, dx,
\]

from (3) we conclude that \( \int_{I_\epsilon} \frac{h(x)}{x-\alpha} \gamma^n(x) \, dx = 0, \ n \in \mathbb{N} \cup \{0\} \). As \( h \in C(I_\epsilon) \), using Lemma 2.4 for \( I_\epsilon \) instead of \( I \) we get the desired result. \( \Box \)

The following result was proved in [6, Theorem 2] for a single function.

**Theorem 2.6.** Let \( 0 < \epsilon \leq 1 \) and \( f_1, f_2 \in C(I) \). Let \( u_\epsilon \in \mathcal{R}_m^n \) be a non normal rational function. Then \( u_\epsilon \) is a \( (p, L^p) \)-b.s.a. of \( f_1 \) and \( f_2 \) from \( \mathcal{R}_m^n \) on \( I_\epsilon \) if and only if \( u_\epsilon = \frac{f_1 + f_2}{2} \) on \( I_\epsilon \).

**Proof.** By Remark 2.3, \( \beta_j = 1, \ j = 1, 2 \). Lemma 2.5 implies

\[|f_1 - u_\epsilon|^{p-1} \text{sgn}(f_1 - u_\epsilon) + |f_2 - u_\epsilon|^{p-1} \text{sgn}(f_2 - u_\epsilon) = 0 \] on \( I_\epsilon \).

If \( \text{sgn}(f_1 - u_\epsilon)(x) = - \text{sgn}(f_2 - u_\epsilon)(x) \), then \( u_\epsilon(x) = \frac{(f_1 + f_2)(x)}{2} \). Otherwise, \( u_\epsilon(x) = f_1(x) = f_2(x) = \frac{(f_1 + f_2)(x)}{2} \) on \( I_\epsilon \). Reciprocally, suppose \( u_\epsilon = \frac{f_1 + f_2}{2} \) on \( I_\epsilon \) and let \( u \in \mathcal{R}_m^n \). Then

\[
\|f_1 - u_\epsilon\|_p^p + \|f_2 - u_\epsilon\|_p^p = 2 \int_I \left( \frac{|f_1 - f_2|^p}{2} \right) \, dx
\leq 2 \int_I \left( \left( \frac{|f_1 - u|^p}{2} \right) + \left( \frac{|u - f_2|^p}{2} \right) \right) \, dx
\leq 2 \left( \int_I \|f_1 - u\|^p \, dx \right) + \left( \int_I \|f_2 - u\|^p \, dx \right)
= \|f_1 - u\|_p^p + \|f_2 - u\|_p^p.
\]

The proof is complete. \( \Box \)

### 3. An interpolation property

Next, we introduce some notation to prove an interpolation result. Let \( f_1, f_2 \in C(I) \) and \( 0 < \epsilon \leq 1 \). We write

\[y_i = y_i(\epsilon) := x_i + \epsilon \beta, \quad y^\epsilon_i = y^\epsilon_i(\epsilon) := x_{i+1} - \epsilon \beta, \quad 1 \leq i \leq k - 1.
\]

If \( g \in C(I_\epsilon) \), we denote

\[\mathcal{A}(g) = \{ i : g(y_i(\epsilon))g(y^\epsilon_i(\epsilon)) < 0, \ 1 \leq i \leq k - 1 \}\]

and \( k^*(g) \) the cardinal of \( \mathcal{A}(g) \). If \( k = 1 \), we put \( k^*(g) = 0 \).

Let $\tilde{f}_1, \tilde{f}_2 \in C(\text{co}(I))$ be extensions of $f_1$ and $f_2$, respectively. Now, we suppose that $\beta_j$, $1 \leq j \leq 2$, introduced in Lemma 2.2 is well defined. For a) $m = 0$ or b) $m \geq 1$, $k = 1$, the function
\[
\tilde{h}_\epsilon := \beta_1|f_1 - u_\epsilon|^{p-1} \text{sgn}(f_1 - u_\epsilon) + \beta_2|f_2 - u_\epsilon|^{p-1} \text{sgn}(f_2 - u_\epsilon)
\]
(4)
is well defined on co($I_\epsilon$). We write
\[
\alpha_j(\epsilon) = (\beta_j)^{1/p} \left( \sum_{i=1}^2 \beta_i^{1/p} \right)^{-1}.
\]
(5)

Now, we establish the main result of this section.

**Theorem 3.1.** Let $f_1, f_2 \in C(I)$ and $0 < \epsilon \leq 1$. Suppose that $u_\epsilon \in \mathcal{R}_m^n$ is a $(L^q, L^p)$-b.s.a. of $f_1$ and $f_2$ from $\mathcal{R}_m^n$ on $I_\epsilon$. If $f_j \neq u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$, and a) or b) holds, then $u_\epsilon$ interpolates to $\alpha_1(\epsilon)f_1 + \alpha_2(\epsilon)f_2$ in at least $n + m + 1$ different points of co($I_\epsilon$), where at least $n + m + 1 - k^*(\tilde{h}_\epsilon)$ of them belong to $I_\epsilon$.

**Proof.** Since $f_j \neq u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$, the function $\tilde{h}_\epsilon$ is defined. We consider two cases. First, suppose that $u_\epsilon$ is not normal, then Lemma 2.5 implies $\tilde{h}_\epsilon = 0$ on $I_\epsilon$. Now, we assume that $u_\epsilon := \frac{P_\epsilon}{Q_\epsilon}$ is normal. It is well known that $P_\epsilon \Pi^m + Q_\epsilon \Pi^n = \Pi^{n+m}$ (see [1] p. 240). Therefore by Lemma 2.2 we have
\[
\int_{I_\epsilon} \frac{\tilde{h}_\epsilon}{(Q_\epsilon)^2} v = 0, \quad v \in \Pi^{n+m}.
\]
(6)
Suppose that $\tilde{h}_\epsilon$ exactly changes of sign in $z_1, \ldots, z_s \in I_\epsilon$, with $s \leq n + m + 1 - k^*(\tilde{h}_\epsilon)$. We can choose $r_1, \ldots, r_{k^*(\tilde{h}_\epsilon)}$, with $r_i \in (y_i, y_i')$ such that $\tilde{h}_\epsilon(r_i) = 0$, $i \in A(\tilde{h}_\epsilon)$. Let $v := \eta \Pi_{i=1}^{s} (x - z_i) \Pi_{i \in A(\tilde{h}_\epsilon)} (x - r_i)$, $\eta := \pm 1$ be such that $v$ satisfies $\tilde{h}_\epsilon v \geq 0$ on $I_\epsilon$ and $\tilde{h}_\epsilon v > 0$ on a positive measure subset of $I_\epsilon$. This contradicts (6), so $s \geq n + m + 1 - k^*(\tilde{h}_\epsilon)$. In this way we have proved that $\tilde{h}_\epsilon$ has at least $n + m + 1$ different zeros in co($I_\epsilon$), where at least $n + m + 1 - k^*(\tilde{h}_\epsilon)$ of them belong to $I_\epsilon$.

Let $x \in \text{co}(I_\epsilon)$ be such that $\tilde{h}_\epsilon(x) = 0$, i.e.
\[
0 = \beta_1|f_1 - u_\epsilon|(x)^{p-1} \text{sgn}(f_1 - u_\epsilon)(x)
+ \beta_2|f_2 - u_\epsilon|(x)^{p-1} \text{sgn}(f_2 - u_\epsilon)(x).
\]
Now, the proof follows analogously to the first part in the proof of Theorem 2.6. \hfill \Box

We denote $l_j(\epsilon)$, $1 \leq j \leq k$, the cardinal of the set of points of $B_j$, where $u_\epsilon$ interpolates to the function $\alpha_1(\epsilon)f_1 + \alpha_2(\epsilon)f_2$, whenever $\alpha_j(\epsilon), 1 \leq j \leq 2$, are defined. The following corollary can be proved similarly to [5] Corollary 9.

**Corollary 3.2.** Under the same hypotheses of Theorem 3.1 there exists $j$, $1 \leq j \leq k$, such that $l_j(\epsilon) \geq c$.
4. Existence of $(l^q, L^p)$-b.s.l.a. from $\mathcal{R}_m^n$

First, in this section we obtain a general result about the asymptotic behavior of the error

$$\mathcal{E}_\epsilon := \|f_1 - u_\epsilon\|_q^q + \|f_2 - u_\epsilon\|_q^q.$$

**Theorem 4.1.** Let $f_1, f_2 \in \mathcal{C}(I), 0 < \epsilon \leq 1$, $u_\epsilon \in \mathcal{R}_m^n$ a $(l^q, L^p)$-b.s.a. of $f_1$ and $f_2$ from $\mathcal{R}_m^n$ on $I_\epsilon$. If there exists a Padé rational approximant of $\frac{f_1 + f_2}{2}$ on $\{x_1, \ldots, x_k\}$, then

$$\mathcal{E}_\epsilon^{1/q} = 2^{\frac{1}{q-2}}\|f_1 - f_2\|_\epsilon + o(\epsilon^{c-1}), \text{ as } \epsilon \to 0.$$

**Proof.** Let $R$ be a Padé rational approximant of $\frac{f_1 + f_2}{2}$ on $\{x_1, \ldots, x_k\}$. Consider the semi-norm on $\mathcal{C}(I) \times \mathcal{C}(I)$ defined by

$$\|(g_1, g_2)\|_\epsilon = (\|g_1\|_q^q + \|g_2\|_q^q)^{1/q}.$$

By the triangle inequality we have

$$\|f_1 - u_\epsilon\|_q^q + \|f_2 - u_\epsilon\|_q^q \leq \|(f_1 - R)\|_q^q + \|(f_2 - R)\|_q^q$$

$$= \left\| \left( \frac{f_1 - f_2}{2}, \frac{f_2 - f_1}{2} \right) + \left( \frac{f_1 + f_2}{2} - R, \frac{f_1 + f_2}{2} - R \right) \right\|_\epsilon^q$$

$$\leq 2^{1/q} \left\| \frac{f_1 - f_2}{2} \right\|_\epsilon + 2^{1/q} \left\| \frac{f_1 + f_2}{2} - R \right\|_\epsilon^q$$

$$\leq 2 \left( \|f_1 - f_2\|_\epsilon + o(\epsilon^{c-1}) \right)^q$$

$$= \frac{1}{2^{q-1}} \left( \|f_1 - f_2\|_\epsilon + o(\epsilon^{c-1}) \right)^q.$$

Since

$$(a + b)^q \leq 2^{q-1}(a^q + b^q), a, b \geq 0,$$

we get

$$\|f_1 - u_\epsilon\|_q^q + \|f_2 - u_\epsilon\|_q^q \geq \frac{1}{2^{q-1}} \|f_1 - f_2\|_\epsilon^2.$$  \hspace{1cm} (9)

From (7) and (9) we obtain the theorem. \hfill \Box

**Remark 4.2.** If $m = 0$ and $f_1, f_2 \in \mathcal{C}^c(I)$, with an analogous proof we have

$$\mathcal{E}_\epsilon^{1/q} = 2^{\frac{1}{q-2}}\|f_1 - f_2\|_\epsilon + O(\epsilon^c), \text{ as } \epsilon \to 0.$$

For $c > 0$ and $h \in \mathcal{C}^{c-1}(I)$, we consider the set

$$\mathcal{H}(h) = \{ P \in \Pi^n : P^{(i)}(x_j) = h^{(i)}(x_j), \ 0 \leq i \leq c - 1, \ 1 \leq j \leq k \}.$$

We define

$$A_j = \{ i : 0 \leq i \leq c - 1, \ f_1^{(i)}(x_j) \neq f_2^{(i)}(x_j) \}, \ 1 \leq j \leq k.$$

Let $m_j = \min A_j - 1$ if $A_j \neq \emptyset$, and $m_j = c - 1$ otherwise. Set

$$m = \min \{ m_j : 1 \leq j \leq k \}.$$  \hspace{1cm} (10)
For \( c = 0 \), we put \( \mathcal{H}(h) = \Pi^n \), and \( \overline{m} = -1 \). With these notations, we obtain the following lemma.

**Lemma 4.3.** Let \( q > 1 \) and assume \( c > 0 \), \( f_1, f_2 \in C^{c-1}(I) \) and \(-1 \leq \overline{m} \leq c - 2\). Under the same hypotheses of Theorem 4.1 if \( f_j \neq u_{\epsilon} \) on \( I_{\epsilon} \), \( 1 \leq j \leq 2 \), for small \( \epsilon \). Then \( \alpha_1(\epsilon) \) and \( \alpha_2(\epsilon) \) (see (3) above) are defined for small \( \epsilon \) and \( \lim_{\epsilon \to 0} \alpha_1(\epsilon) = \lim_{\epsilon \to 0} \alpha_2(\epsilon) = \frac{1}{2} \).

**Proof.** For simplicity all subnets \( \epsilon \to 0 \) will be denoted in the same way. Let \( g = \frac{1}{2}(f_1 + f_2) \), \( H = \frac{1}{2}(H_1 + H_2) \) with \( H_l \in \mathcal{H}(f_l) \), \( l = 1, 2 \), and \( u_{\epsilon} = \frac{P_{\epsilon}}{Q_{\epsilon}} \). Then \( (g - H)(x) = o((x - x_j)^{c-1}) \), as \( x \to x_j \), \( 1 \leq j \leq k \), and

\[
(f_1 - u_{\epsilon})(x) = \left( \frac{1}{2}(f_1 - f_2) + (g - H) + H - \frac{P_{\epsilon}}{Q_{\epsilon}} \right)(x).
\]

Hence

\[
\frac{Q_{\epsilon}(f_1 - u_{\epsilon})}{\|Q_{\epsilon}\|e^{m+1}}(x) = \frac{Q_{\epsilon}(x)}{\|Q_{\epsilon}\|e^{m+1}} \left( \frac{1}{2}(f_1 - f_2)(x) + (g - H(x)) \right)
\]

\[
+ \frac{Q_{\epsilon}(x)H(x) - P_{\epsilon}(x)}{\|Q_{\epsilon}\|e^{m+1}},
\]

for \( x \in B_j \), \( 1 \leq j \leq k \). By Theorem 4.1 and the definition of \( \overline{m} \) we obtain \( \|f_1 - u_{\epsilon}\|_c \leq \frac{\varepsilon^{1/q}}{\varepsilon^{m+1}} = O(1) \). Since \( Q_{\epsilon} \in \Pi^m \) on each \( B_j \), and \( \|\cdot\| \) can be also considered as a norm in \( (\Pi^m)^k \), the equivalence of norms in this space implies that there exists \( K > 0 \) such that \( \|Q_{\epsilon}^*\| := \max_{1 \leq j \leq k} \max_{B_j} |Q_{\epsilon}^*| \leq K \|Q_{\epsilon}\| \). As \( \overline{m} \leq c - 2 \), by (11) we get

\[
\left\| \frac{Q_{\epsilon}H - P_{\epsilon}}{\|Q_{\epsilon}\|} \right\|_c \leq \left\| \frac{Q_{\epsilon}^*}{\|Q_{\epsilon}^*\|} \right\|_c \left( \|f_1 - u_{\epsilon}\| + \sum_{j=1}^{k} \left( \frac{1}{2}(f_1 - f_2) + (g - H) \right) \chi_{B_j} \right)
\]

\[
\leq \left\| \frac{Q_{\epsilon}^*}{\|Q_{\epsilon}^*\|} \right\|_c \left( \varepsilon^{1/q} + \sum_{j=1}^{k} \left( \frac{1}{2}(f_1 - f_2) + (g - H) \right) \chi_{B_j} \right)
\]

\[
= O(\varepsilon^{m+1}).
\]

From (12) we have a subnet such that \( \frac{(Q_{\epsilon}H - P_{\epsilon})^s}{\|Q_{\epsilon}\|e^{m+1}} \to R \). Moreover, we can choose the subnet such that \( \frac{Q_{\epsilon}^*}{\|Q_{\epsilon}\|} \to S \). Here, \( R \) and \( S \) are polynomials on each \( B_j \). We denote

\[
\lambda(x) = \sum_{j=1}^{k} \frac{(f_1 - f_2 + 1)(m+1)(x_j)(x - x_j)^{m+1}}{(m+1)!} \chi_{B_j}(x) \quad \text{and} \quad T(x) = \frac{R(x)}{S(x)}.
\]

As \(-1 \leq \overline{m} \leq c - 2 \), \( \lambda \neq 0 \). Since \( \frac{(g - H)^s}{e^{m+1}} \to 0 \), from (11) we obtain

\[
\lim_{\epsilon \to 0} \frac{(f_1 - u_{\epsilon})^s}{e^{m+1}} = \frac{1}{2} \lambda + T
\]

on $I$ except possibly by the zeros of $S$. Similarly, we have
\[
\lim_{\epsilon \to 0} \frac{(f_2 - u_\epsilon)^\epsilon}{\epsilon^m + 1} = -\frac{1}{2} \lambda + T. \tag{14}
\]
By Fatou's Lemma, (13) and (14), there exists a subnet such that
\[
\left\| \frac{1}{2} \lambda + T \right\| \leq \lim_{\epsilon \to 0} \left\| f_1 - u_\epsilon \right\|_\epsilon \text{ and } \left\| \frac{1}{2} \lambda - T \right\| \leq \lim_{\epsilon \to 0} \left\| f_2 - u_\epsilon \right\|_\epsilon.
\]
Therefore, from (8) we have
\[
\|\lambda\|^q = \left\| \frac{1}{2} \lambda + T + \frac{1}{2} \lambda - T \right\| \leq \left( \left\| \frac{1}{2} \lambda + T \right\| + \left\| \frac{1}{2} \lambda - T \right\| \right)^q 
\]
\[
\leq 2^{q-1} \left( \left\| \frac{1}{2} \lambda + T \right\|^q + \left\| \frac{1}{2} \lambda - T \right\|^q \right) 
\]
\[
\leq 2^{q-1} \lim_{\epsilon \to 0} \frac{\|f_1 - u_\epsilon\|^q_\epsilon + \|f_2 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1} = \|\lambda\|^q,
\]
where the last equality holds by Theorem 4.1. So,
\[
\left\| \frac{1}{2} \lambda + T + \frac{1}{2} \lambda - T \right\| = \left\| \frac{1}{2} \lambda + T \right\| + \left\| \frac{1}{2} \lambda - T \right\|
\] (15)
and
\[
\frac{\left\| \frac{1}{2} \lambda + T \right\|^q_\epsilon + \left\| \frac{1}{2} \lambda - T \right\|^q_\epsilon}{2} = \left( \frac{\left\| \frac{1}{2} \lambda + T \right\| + \left\| \frac{1}{2} \lambda - T \right\|}{2} \right)^q. \tag{16}
\]
As $\| \cdot \|$ is strictly convex, from (15) there exists $a \geq 0$ such that
\[
\frac{1}{2} \lambda + T = a \left( \frac{1}{2} \lambda - T \right), \tag{17}
\]
i.e., $T = \frac{(a-1)\lambda}{2(1+a)}$. Also, as $x^q$ is strictly convex, from (16) we get
\[
\left\| \frac{1}{2} \lambda + T \right\| = \left\| \frac{1}{2} \lambda - T \right\|. \tag{18}
\]
If $\frac{1}{2} \lambda - T = 0$, then $\frac{1}{2} \lambda + T = 0$ and $\|\lambda\| = 0$, a contradiction. Therefore $\frac{1}{2} \lambda - T \neq 0$, so (17) and (18) imply $a = 1$. Therefore $T = 0$. Now, from (13) and (14), we have
\[
\lim_{\epsilon \to 0} \frac{(f_1 - u_\epsilon)^\epsilon}{\epsilon^m + 1} = \frac{\lambda}{2} \quad \text{and} \quad \lim_{\epsilon \to 0} \frac{(f_2 - u_\epsilon)^\epsilon}{\epsilon^m + 1} = -\frac{\lambda}{2}
\]
on $I$ except possibly by the zeros of $S$. Again, an application of Fatou's Lemma implies $\frac{\|\lambda\|^q}{2} \leq \lim_{\epsilon \to 0} \frac{\|f_1 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1}$ and $\frac{\|\lambda\|^q}{2} \leq \lim_{\epsilon \to 0} \frac{\|f_2 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1}$ for some subnet. Theorem 4.1 implies
\[
\lim_{\epsilon \to 0} \left( \frac{\|f_1 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1} + \frac{\|f_2 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1} \right) = \frac{\|\lambda\|^q}{2^{q-1}}.
\]
So,
\[
\frac{\|\lambda\|^q}{2} = \lim_{\epsilon \to 0} \frac{\|f_1 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1} = \lim_{\epsilon \to 0} \frac{\|f_2 - u_\epsilon\|^q_\epsilon}{\epsilon^m + 1}. \tag{19}
\]
Note that there exists $\epsilon_0 > 0$, such that for all $0 < \epsilon \leq \epsilon_0$, we have $\|f_j - u_\epsilon\|_\epsilon \neq 0, j = 1, 2$, because $\lambda \neq 0$. So, $f_j \neq u_\epsilon$ on $I_\epsilon, 1 \leq j \leq 2$, for
0 < \epsilon \leq \epsilon_0$, and $\alpha_1(\epsilon)$ and $\alpha_2(\epsilon)$ are defined for $0 < \epsilon \leq \epsilon_0$. Finally, from (5) and (19) we conclude that $\lim_{\epsilon \to 0} \alpha_1(\epsilon) = \lim_{\epsilon \to 0} \alpha_2(\epsilon) = \frac{1}{2}$. \hfill \square

Next, we prove the main result of this section, which extends [10] Theorem 1.

**Theorem 4.4.** Let $q > 1$ and assume $k = 1$. Let $f_1, f_2 \in C^{n+m}(I)$, $0 < \epsilon \leq 1$, and $u_\epsilon = \frac{P_\epsilon}{Q_\epsilon} \in R_m^n \cap (l^q, L^p)$-b.s.a. of $f_1$ and $f_2$ from $R_m^n$ on $I_\epsilon$. Suppose that there exists a subnet $\epsilon' \to 0$ such that $P_{\epsilon'} \to P_0$, $Q_{\epsilon'} \to Q_0$, and $(P_0, Q_0)$ is a Padé approximant pair of $\frac{f_1 + f_2}{2}$ on $\{x_1\}$. In addition, if the Padé approximant pair is unique, then $u_\epsilon$ converges pointwise to $\frac{P_0}{Q_0}$ as $\epsilon \to 0$, in a neighborhood of $x_1$ except possibly at $x_1$. Moreover, if $P_\epsilon \left( \frac{f_1 + f_2}{2} \right)$ is normal then $u_\epsilon$ uniformly converges to $P_\epsilon \left( \frac{f_1 + f_2}{2} \right)$, in a neighborhood of $x_1$.

**Proof.** Lemma 3.3 and Theorem 3.1 imply that for small $\epsilon$ there are $n + m + 1$ points in $co(I_\epsilon) = I_\epsilon$, say $z_0(\epsilon), \ldots, z_{n+m}(\epsilon)$, such that

$$P_\epsilon(z_i(\epsilon)) = Q_\epsilon(z_i(\epsilon))((\alpha_1(\epsilon)f_1(z_i(\epsilon)) + \alpha_2(\epsilon)f_2(z_i(\epsilon))) = 0 \leq i \leq n + m.$$ Consider $g_\epsilon = \alpha_1(\epsilon)f_1 + \alpha_2(\epsilon)f_2$. By the uniqueness of the interpolation polynomial of degree at most $n + m$, we get

$$P_\epsilon = H(z_0(\epsilon), \ldots, z_{n+m}(\epsilon))(Q_\epsilon g_\epsilon),$$

where the right-hand side denotes the interpolation polynomial of $Q_\epsilon g_\epsilon$ of degree $n + m$ on $\{z_0(\epsilon), \ldots, z_{n+m}(\epsilon)\}$. For a subnet $\epsilon' \to 0$, we have

$$Q_{\epsilon'} \to Q_0 \quad \text{and} \quad P_{\epsilon'} \to T_{n+m,x_1}(Q_0 g_\epsilon) =: P_0,$$

where $g$ is the limit of $g_{\epsilon'}$, and $T_{n+m,x_1}(h)$ represents the Taylor polynomial of $h$ of degree $n + m$ at $x_1$. First, we assume that $-1 \leq \overline{m} \leq c - 2$. By Lemma 4.3, $g = \frac{f_1 + f_2}{2}$ and

$$\left( Q_0 \left( \frac{f_1 + f_2}{2} - P_0 \right) \right)^{(i)}(x_1) = 0, \quad 0 \leq i \leq n + m.$$ Now, we suppose that $\overline{m} = c - 1$. Theorem 4.1 implies $\|f_1 - \frac{P_\epsilon}{Q_\epsilon}\|_\epsilon = o(\epsilon^{n+m})$. As a consequence $\|Q_\epsilon f_1 - P_\epsilon\|_\epsilon = o(\epsilon^{n+m})$, so

$$\|Q_\epsilon T_{n+m,x_1}(f_1) - P_\epsilon\|_\epsilon = o(\epsilon^{n+m}).$$

By definition of $\overline{m}$ we can replace $f_1$ by $\frac{f_1 + f_2}{2}$, and from a Pólya type inequality (see 3 Theorem 3) we have

$$\left( Q_0 \left( \frac{f_1 + f_2}{2} - P_0 \right) \right)^{(i)}(x_1) = 0, \quad 0 \leq i \leq n + m.$$ In any case, we conclude that $(P_0, Q_0)$ is a Padé approximant pair of $\frac{f_1 + f_2}{2}$ on $\{x_1\}$. On the other hand, if $P_\epsilon \left( \frac{f_1 + f_2}{2} \right)$ is normal, then $(P_0, Q_0)$ is the unique Padé approximant pair of $\frac{f_1 + f_2}{2}$ on $\{x_1\}$ and $Q_0(x_1) \neq 0$ (see [10] Lemma 3). Therefore
We denote $a)$ Let\(\epsilon\) uniformly converges to $\frac{f_1 + f_2}{2}$ on a neighborhood of $x_1$. \(\square\)

5. Existence of $({l}^q, L^p)$-b.s.l.a. from $\Pi^n$

Next, we prove a result about uniform boundedness of a net of best simultaneous approximations from $\Pi^n$.

**Theorem 5.1.** Let $f_1, f_2 \in C^n(I)$, $0 < \epsilon \leq 1$, and let $P_\epsilon \in \Pi^n$ be a $({l}^q, L^p)$-b.s.a. to $f_1$ and $f_2$ from $\Pi^n$ on $I_\epsilon$. Then the net $\{P_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \rightarrow 0$.

**Proof.** Without loss of generality we can assume that the extensions $\tilde{f}_1, \tilde{f}_2$ considered in page 61 belong to $C^n(\text{co}(I))$. By Theorem 3.1 there exists $z_0(\epsilon) < \cdots < z_n(\epsilon)$ in $\text{co}(I)$ such that $P_\epsilon = H_{z_0(\epsilon), \cdots, z_n(\epsilon)}(\gamma_1 f_1 + \gamma_2 f_2)$, where as before $H_{z_0(\epsilon), \cdots, z_n(\epsilon)}$ denotes the interpolation polynomial of $\gamma_1 f_1 + \gamma_2 f_2$ of degree $n$ on $\{z_0, \cdots, z_n\}$, $\gamma_1, \gamma_2 \geq 0$ and $\gamma_1 + \gamma_2 = 1$. Since the nets $\{z_0(\epsilon), \cdots, z_n(\epsilon)\}$ and $\{\gamma_1(\epsilon), \gamma_2(\epsilon)\}$ are bounded, we can find convergent subnets. Suppose that $\gamma_j(\epsilon') \rightarrow \gamma_j$, $j = 1, 2$, and $z_i(\epsilon') \rightarrow t_i$, $0 \leq i \leq n$, as $\epsilon' \rightarrow 0$. Clearly $t_0 \leq \cdots \leq t_n$. Using Newton’s divided difference formula and the continuity of the divided differences we get $P_{\epsilon'} \rightarrow H_{t_0, \cdots, t_n} (\gamma_1 \tilde{f}_1 + \gamma_2 \tilde{f}_2)$, as $\epsilon' \rightarrow 0$. Therefore the net $\{P_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \rightarrow 0$. \(\square\)

Now, we state results about the convergence of b.s.a. We consider a basis of $\Pi^n$, $\{u_{sv}\}_{0 \leq s \leq k}$ which satisfies

\[ u_{su}^{(i)}(x_j) = \delta_{(i,j)(s,v)}, \quad w_{su}^{(i)}(x_j) = 0, \quad 0 \leq i \leq c - 1, \quad 1 \leq j \leq k, \]

where $\delta$ is the Kronecker delta function.

In the next theorem we need to recall the number $m$ which was defined in [10].

**Theorem 5.2.** Assume $f_1, f_2 \in C^c(I)$, $0 < \epsilon \leq 1$. Let $P_\epsilon \in \Pi^n$ be a $({l}^q, L^p)$-b.s.a. to $f_1$ and $f_2$ from $\Pi^n$ on $I_\epsilon$, and let $A$ be the cluster point set of the net $\{P_\epsilon\}$ as $\epsilon \rightarrow 0$. Then:

a) $A$ is contained in $\mathcal{M}(f_1, f_2)$, the set of solutions of the following minimization problem:

\[
\min_{P_\epsilon \in \Pi^n} \left( \sum_{l=1}^{2} \sum_{j=1}^{k} \left( \frac{1}{|x_j|^{(m+1)}} \left| (f_l - P)^{(m+1)}(x_j) \right| \right)^{q/p} \right)^{p/q}
\]

with the constraints $P^{(i)}(x_j) = \frac{(f_1 + f_2)^{(i)}(x_j)}{2}$, $0 \leq i \leq m$, $1 \leq j \leq k$.

b) If $f_1, f_2 \in C^n(I)$, then $A \neq \emptyset$. In particular, if $\mathcal{M}(f_1, f_2)$ is unitary, there exists a unique $({l}^q, L^p)$-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$.

**Proof.** a) Let $P_0 \in A$. By definition of $A$, there is a net $\epsilon \downarrow 0$ such that $P_\epsilon \rightarrow P_0$. We denote $U_\epsilon = \frac{H_1 - P_0}{2}$ and $V_\epsilon = \frac{H_2 - P_0}{2}$, where $H_l \in \mathcal{H}(f_l)$, $l = 1, 2$. Clearly,

\[ \epsilon_\epsilon \geq \left( \frac{\|f_1 - P_\epsilon\|_2 + \|f_2 - P_\epsilon\|_2}{2} \right)^q. \]

Since \((H_l - f_l)(x) = O(\|(x - x_j)^c\|),\) as \(x \to x_j, l = 1, 2, 1 \leq j \leq k,\) we obtain
\[
\|U_\varepsilon\|_\varepsilon + \|V_\varepsilon\|_\varepsilon \leq \mathcal{E}_\varepsilon^{1/q} + O(\varepsilon).
\] (21)

By Remark 4.2,
\[
\mathcal{E}_\varepsilon^{1/q} = 2 \frac{1 - q}{\varepsilon^{m+1}} \left\| f_1 - f_2 \right\|_\varepsilon + O(1).
\] (22)

Expanding \((f_1 - f_2)^\varepsilon\) by its Taylor polynomial at \(x_j, 1 \leq j \leq k,\) up to order \(m,\) we have
\[
\lim_{\varepsilon \to 0} \left\| \frac{f_1 - f_2}{\varepsilon^{m+1}} \right\|_\varepsilon = \frac{1}{(m+1)!} \left( \sum_{i=1}^{k} \left| (f_1 - f_2)(\varepsilon^{m+1}) (x_j) \right|^p \left\| (t - x_j)^{m+1} \right\|_{B_j} \right)^{1/p} =: L \] (23)

From (22) and (23) we obtain that \(\mathcal{E}_\varepsilon^{1/q} \) is bounded as \(\varepsilon \to 0.\) So, (21) implies that \(\left\| \frac{U_\varepsilon}{\varepsilon^{m+1}} \right\|_{B_j} \) and \(\left\| V_\varepsilon \right\|_{B_j} \), \(1 \leq j \leq k,\) are bounded. Since \(\frac{U_\varepsilon}{\varepsilon^{m+1}}, \frac{V_\varepsilon}{\varepsilon^{m+1}} \in \Pi^m\) on \(B_j,\) then \(\left( \frac{U_\varepsilon}{\varepsilon^{m+1}} \right)^{(i)}(x_j) = (f_1 - f_0)^{(i)}(x_j) \varepsilon^{i-m-1}\) and \(\left( \frac{V_\varepsilon}{\varepsilon^{m+1}} \right)^{(i)}(x_j) = (f_2 - f_0)^{(i)}(x_j) \varepsilon^{i-m-1} \) are bounded for all \(0 \leq i \leq c - 1, 1 \leq j \leq k.\) Therefore there exists \(d_{ij}\) such that
\[
\lim_{\varepsilon \to 0} (f_l - P_\varepsilon)^{(i)}(x_j) \varepsilon^{i-m-1} = d_{ij}, \quad 0 \leq i \leq m, 1 \leq j \leq k, \quad l = 1, 2 \] (24)

for some subnet, that we again denote by \(\varepsilon.\) For \(t \in B_j\) we have
\[
\frac{(f_1 - P_\varepsilon)^\varepsilon(t)}{\varepsilon^{m+1}} = \sum_{i=0}^{m} \frac{(f_1 - P_\varepsilon)^{(i)}(x_j)}{i!} \varepsilon^{-(m+1)} (t - x_j)^i + \frac{(f_1 - P_\varepsilon)^{(m+1)}(\varepsilon(\xi_j(t) - x_j) + x_j)}{(m+1)!} (t - x_j)^{m+1},
\]
where \(\xi_j(t)\) belongs to the segment with ends \(t\) and \(x_j.\) From (24) we get
\[
\lim_{\varepsilon \to 0} \frac{(f_1 - P_\varepsilon)^\varepsilon(t)}{\varepsilon^{m+1}} = \sum_{i=0}^{m} \frac{d_{ij}}{i!} (t - x_j)^i + \frac{(f_1 - P_0)^{(m+1)}(x_j)}{(m+1)!} (t - x_j)^{m+1},
\]
uniformly on \(B_j.\) Therefore
\[
\lim_{\varepsilon \to 0} \left\| \frac{(f_1 - P_\varepsilon)^\varepsilon}{\varepsilon^{m+1}} \right\|_\varepsilon^p = \sum_{j=1}^{k} \left\| \frac{d_{ij}}{i!} (t - x_j)^i + \frac{(f_1 - P_0)^{(m+1)}(x_j)}{(m+1)!} (t - x_j)^{m+1} \right\|_{B_j}^p \] (25)

\[
\geq \sum_{j=1}^{k} \left( \frac{(f_1 - P_0)^{(m+1)}(x_j)}{(m+1)!} \right)^p J^p_j,
\]
where \( J_j = \inf_{Q \in \Pi^{i+1}} \| (t - x_j)^{i+1} - Q(t) \|_{B_j} \). Clearly (25) holds for \( f_2 \) instead of \( f_1 \).

From (24) we can assume \( P_0^{(i)}(x_j) = f_1^{(i)}(x_j) \) for \( 1 \leq j \leq k \), \( 0 \leq i \leq m \), so we can write

\[
P_0 = \sum_{v=1}^{k} \sum_{s=0}^{m} f_1^{(s)}(x_v) u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k} \sum_{s=m+1}^{c-1} \tau_{sv} u_{sv},
\]

for some real numbers \( \{ \bar{b}_e \}_{1 \leq e \leq d} \) and \( \{ \tau_{sv} \}_{0 \leq s \leq c-1} \). Given two sets of real numbers (independent of \( \epsilon \)), say \( \{ c_{sv} \}_{1 \leq s \leq k} \) and \( \{ b_{ve} \}_{1 \leq e \leq d} \), consider the following net of polynomials in \( \Pi^n \),

\[
R_\epsilon = \sum_{v=1}^{k} \sum_{s=0}^{m} (f_1^{(s)}(x_v) - c_{sv} \epsilon^{m+1-s}) u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k} \sum_{s=m+1}^{c-1} c_{sv} u_{sv}.
\]

We observe that \( R_\epsilon^{(i)}(x_j) = f_1^{(i)}(x_j) - c_{ij} \epsilon^{m+1-i}, \; 1 \leq j \leq k, \; 0 \leq i \leq m \).

Let \( h = \sum_{v=1}^{k} \sum_{s=0}^{m} f_1^{(s)}(x_v) u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k} \sum_{s=m+1}^{c-1} c_{sv} u_{sv} \). Expanding \( (f_1 - R_\epsilon)^e \) by its Taylor polynomial at \( x_j \) up to order \( m \), we obtain

\[
(f_1 - R_\epsilon)^e(t) = \sum_{i=0}^{c_{ij}} c_{ij} \frac{(t - x_j)^i}{i!} + \frac{(f_1 - h)^{(m+1)}(x_j - x_j)}{(m+1)!} (t - x_j)^m + \sum_{v=1}^{k} \sum_{s=0}^{m} c_{sv} \epsilon^{m+1-s} \frac{(t - x_j)^i}{i!} (x_v - x_j)^{m+1}, \quad t \in B_j,
\]

where \( \xi_j(t) \) belongs to the segment with ends \( t \) and \( x_j \). Since \( \lim_{\epsilon \to 0} \frac{(f_1 - R_\epsilon)^e(t)}{\epsilon^{m+1}} = \sum_{i=0}^{c_{ij}} c_{ij} \frac{(t - x_j)^i}{i!} + \frac{(f_1 - h)^{(m+1)}(x_j)}{(m+1)!} (t - x_j)^m + \sum_{v=1}^{k} \sum_{s=0}^{m} c_{sv} \epsilon^{m+1-s} \frac{(t - x_j)^i}{i!} (x_v - x_j)^{m+1}, \quad t \in B_j \), we have

\[
\lim_{\epsilon \to 0} \left\| \frac{(f_1 - R_\epsilon)^e}{\epsilon^{m+1}} \right\|_p = \sum_{j=1}^{k} \sum_{i=0}^{c_{ij}} \left( t - x_j \right)^i + \frac{(f_1 - h)^{(m+1)}(x_j)}{(m+1)!} (t - x_j)^m + \sum_{v=1}^{k} \sum_{s=0}^{m} c_{sv} \epsilon^{m+1-s} \frac{(t - x_j)^i}{i!} (x_v - x_j)^{m+1} \right\|_B_j.
\]

Let \( c_{ij}, \; 1 \leq j \leq k, \; 0 \leq i \leq m \), be such that \( \sum_{i=0}^{c_{ij}} \frac{c_{ij} \frac{(t - x_j)^i}{i!}}{(m+1)!} \) is the best approximation to \( \frac{(f_1 - h)^{(m+1)}(x_j)}{(m+1)!} (t - x_j)^m + \sum_{v=1}^{k} \sum_{s=0}^{m} c_{sv} \epsilon^{m+1-s} \frac{(t - x_j)^i}{i!} (x_v - x_j)^{m+1} \) with respect to \( \| . \|_{B_j} \). Then

\[
\lim_{\epsilon \to 0} \left\| \frac{(f_1 - R_\epsilon)^e}{\epsilon^{m+1}} \right\|_p = \sum_{j=1}^{k} \left| \frac{(f_1 - h)^{(m+1)}(x_j)}{(m+1)!} \right|^p J_j^p,
\]

and similarly we get

\[
\lim_{\epsilon \to 0} \left\| \frac{(f_2 - R_\epsilon)^e}{\epsilon^{m+1}} \right\|_p = \sum_{j=1}^{k} \left| \frac{(f_2 - h)^{(m+1)}(x_j)}{(m+1)!} \right|^p J_j^p.
\]
From (25)-(27) and the continuity of the function $|x|^{\frac{q}{p}} + |y|^{\frac{q}{p}}$, we have

$$\sum_{l=1}^{2} \left( \sum_{j=1}^{k} \left| \frac{(f_l - P_0)^{(m+1)}(x_j)}{(m+1)!} J_j^p \right|^{p} \right)^{q/p} \leq \liminf_{\epsilon \to 0} \frac{\mathcal{E}_\epsilon}{\epsilon^{(m+1)q}} \leq \sum_{l=1}^{2} \left( \sum_{j=1}^{k} \left| \frac{(f_l - h)^{(m+1)}(x_j)}{(m+1)!} J_j^p \right|^{p} \right)^{q/p},$$

(28)

for all $h = \sum_{s=0}^{\bar{m}} \sum_{v=1}^{d} j_l(s)(x_v)u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{d} c_{sv} u_{sv}$.

For all $1 \leq j \leq k$, $J_j = \inf_{Q \in \Pi^{m+1}} \left( |f_j - h|^{m+1} - Q(y)|^{p \frac{dy}{n}} \right)^{\frac{1}{p}} \neq 0$. Then $J_j$ does not depend on $j$. So, from (28) we obtain

$$\sum_{l=1}^{2} \left( \sum_{j=1}^{k} \left| (f_l - P_0)^{(m+1)}(x_j) \right|^{p} \right)^{q/p} \leq \sum_{l=1}^{2} \left( \sum_{j=1}^{k} \left| (f_l - h)^{(m+1)}(x_j) \right|^{p} \right)^{q/p}.$$

In addition, as $f_l^{(i)}(x_j) = f_l^{(i)}(x_j)$, $0 \leq i \leq m$, $1 \leq j \leq k$, then $P_0^{(i)}(x_j) = (f_{l_1/2})^{(i)}(x_j)$. The proof of a) is complete.

b) If $f_1, f_2 \in C^n(I)$, by Theorem 5.1 the net $\{P_{\epsilon}\}$ is uniformly bounded on compact sets, then there exists $P_0 \in A$. From a), $P_0 \in M(f_1, f_2)$. In particular, if $M(f_1, f_2)$ is unitary, there exists a unique ($l^p, L^p$)-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$.

The following theorem gives sufficient conditions for $M(f_1, f_2)$ to be a unitary set. Its proof is analogous to that of [5, Theorem 12].

**Theorem 5.3.** Let $f_1, f_2 \in C^{\alpha}(I)$ and $q > 1$. If either a) $\bar{m} = c - 2, d = 0$ or b) $\bar{m} = c - 1$, then $M(f_1, f_2)$ is a unitary set.

The next theorem shows that there always exists a unique ($l^2, L^2$)-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$.

**Theorem 5.4.** Let $f_1, f_2 \in C^{\alpha}(I)$. Then there exists a unique ($l^2, L^2$)-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$, and it is the best local approximation of $\frac{f_1 + f_2}{2}$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$ with respect to the norm $L^2$.

**Proof.** Let $0 < \epsilon \leq 1$ and let $\{P_{\epsilon}\}$ be a net of ($l^2, L^2$)-b.s.a. of $f_1$ and $f_2$ from $\Pi^n$ on $I_\epsilon$; then it is well known that $P_{\epsilon}$ is the best approximation to $\frac{f_1 + f_2}{2}$ with respect to the norm $L^2$ (see [12, Theorem 3]). Hence, we deduce that $\{P_{\epsilon}\}$ converges to the best local approximation of $\frac{f_1 + f_2}{2}$ (see [9, Theorem 4]).

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References


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