ANOTHER PROOF OF CHARACTERIZATION OF BMO VIA BANACH FUNCTION SPACES

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Abstract. Our aim is to give a characterization of the BMO norm via Banach function spaces based on the Rubio de Francia algorithm. Our proof is different from the one by Ho [Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces, Anal. Math. 38 (2012), 173–185].

1. Introduction

The BMO space $\text{BMO}(\mathbb{R}^n)$ is known as the dual space of the Hardy space, and plays an important role in real analysis. The space $\text{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions $b$ satisfying that the semi-norm

$$
\|b\|_{\text{BMO}} := \sup_{Q: \text{cube}} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx
$$

is finite, where, for each cube $Q \subset \mathbb{R}^n$, $|Q|$ is its Lebesgue measure and $b_Q$ is the mean value of the function $b$ on $Q$, namely

$$
b_Q := \frac{1}{|Q|} \int_Q b(y) \, dy.
$$

The definition above tells us that every function $b \in \text{BMO}(\mathbb{R}^n)$ has bounded mean oscillation. The semi-norm $\|b\|_{\text{BMO}}$ is called the BMO norm. If $b \in \text{BMO}(\mathbb{R}^n)$, then there exist positive constants $C_1$ and $C_2$ such that for all cubes $Q$ and $\lambda > 0$,

$$
|\{x \in Q : |b(x) - b_Q| > \lambda\}| \leq C_1 |Q| \exp \left( - \frac{C_2 \lambda}{\|b\|_{\text{BMO}}} \right). \tag{1.1}
$$

The inequality (1.1) is proved by John and Nirenberg in [11] and implies that the BMO norm $\|b\|_{\text{BMO}}$ is equivalent to

$$
\|b\|_{\text{BMO},L_p} := \sup_{Q: \text{cube}} \frac{1}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}} \|b - b_Q\chi_Q\|_{L^p(\mathbb{R}^n)} \tag{1.2}
$$

for any constant $1 \leq p < \infty$, where $\chi_Q$ is the characteristic function for $Q$.

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The author has considered generalization of the equivalent BMO norm and proved the following:

1. (The author, [7]) By using a bounded measurable function \( p(\cdot) : \mathbb{R}^n \to [1, \infty) \) we generalize the semi-norm \((1.2)\) to

\[
\|b\|_{\text{BMO}_{L^p(\cdot)}} := \sup_{Q \in \text{cube}} \frac{1}{|Q|} \| \chi_Q \|_{L^p(\cdot)(\mathbb{R}^n)} \| (b - b_Q) \chi_Q \|_{L^p(\cdot)(\mathbb{R}^n)},
\]

where the variable Lebesgue norm \( \| f \|_{L^p(\cdot)(\mathbb{R}^n)} \) is defined by

\[
\| f \|_{L^p(\cdot)(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\lambda} \, dx \leq 1 \right\}.
\]

The generalized Lebesgue space \( L^p(\cdot)(\mathbb{R}^n) \) consists of all measurable functions \( f \) such that the norm \( \| f \|_{L^p(\cdot)(\mathbb{R}^n)} \) is finite. If \( p(\cdot) \) satisfies \( 1 < \inf p(x) \) and the Hardy–Littlewood maximal operator \( M \), defined by

\[
Mf(x) := \sup_{Q \ni x, Q \in \text{cube}} \frac{1}{|Q|} \int_Q |f(y)| \, dy,
\]

is bounded on \( L^p(\cdot)(\mathbb{R}^n) \), then the generalized BMO norm \( \| b \|_{\text{BMO}_{L^p(\cdot)}} \) is equivalent to the classical one \( \| b \|_{\text{BMO}} \).

2. (The author and Sawano, [9]) If a bounded measurable function \( p(\cdot) : \mathbb{R}^n \to [1, \infty) \) satisfies \( 1 \leq \inf p(x) \) and the log-Hölder conditions:

\[
|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)} \quad \text{for } x, y \in \mathbb{R}^n, |x - y| \leq 1/2,
\]

\[
|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)} \quad \text{for } x \in \mathbb{R}^n,
\]

for some constants \( C \) and \( p_\infty \) independent of \( x, y \), then \( \| b \|_{\text{BMO}_{L^p(\cdot)}} \) is equivalent to \( \| b \|_{\text{BMO}} \).

3. (The author, Sawano and Tsutsui, [10]) If a variable exponent \( p(\cdot) : \mathbb{R}^n \to [1, \infty) \) is bounded and \( M \) is of weak type \( (p(\cdot), p(\cdot)) \), that is, there exists a constant \( C > 0 \) such that for all \( f \in L^p(\cdot)(\mathbb{R}^n) \) and all \( \lambda > 0 \),

\[
\| \chi_{\{Mf > \lambda\}} \|_{L^p(\cdot)(\mathbb{R}^n)} \leq C \lambda^{-1} \| f \|_{L^p(\cdot)(\mathbb{R}^n)},
\]

then \( \| b \|_{\text{BMO}_{L^p(\cdot)}} \) is equivalent to \( \| b \|_{\text{BMO}} \).

On the other hand, Ho [6] has proved an interesting characterization in the context of general function spaces including Lebesgue spaces. Given a Banach function space \( X \) equipped with a norm \( \| \cdot \|_X \), we define a generalized BMO norm

\[
\| b \|_{\text{BMO}_X} := \sup_{Q \in \text{cube}} \frac{1}{\| \chi_Q \|_X} \| (b - b_Q) \chi_Q \|_X.
\]

If \( M \) is bounded on the associate space \( X' \), then \( \| b \|_{\text{BMO}_X} \) is equivalent to \( \| b \|_{\text{BMO}} \).

We remark that Ho's result [6] has included the authors' one [7, 9]. The statements in [7, 9] are deeply depending on Diening’s work [3] on variable exponent analysis. On the other hand, Ho’s proof is self-contained and obtained as a by-product of the new results about atomic decomposition introduced in [6].
In the present paper we give a new proof of the characterization of the BMO norm initially proved by Ho. We use the Rubio de Francia algorithm [13, 14, 15].

2. Preliminaries

2.1. Banach function spaces. We present the definition of Banach function space and some fundamental properties of it based on Bennett and Sharpley [1]. For further informations on the theory of Banach function space including the proof of Lemma 2.1 below we refer to that book.

Definition 2.1. Let $\mathcal{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$, and $X$ be a linear subspace of $\mathcal{M}$.

1. The space $X$ is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathcal{M} \rightarrow [0, \infty]$ satisfying the following properties: Let $f, g, f_j \in \mathcal{M}$ ($j = 1, 2, \ldots$), then
   (a) $f \in X$ holds if and only if $\|f\|_X < \infty$.
   (b) Norm property:
      (i) Positivity: $\|f\|_X \geq 0$.
      (ii) Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.
      (iii) Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds for all complex numbers $\lambda$.
      (iv) Triangle inequality: $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.
   (c) Symmetry: $\|f\|_X = \|f\|_X$.
   (d) Lattice property: If $0 \leq g(x) \leq f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leq \|f\|_X$.
   (e) Fatou property: If $0 \leq f_j(x) \leq f_{j+1}(x)$ for all $j$ and $f_j(x) \rightarrow f(x)$ as $j \rightarrow \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim_{j \rightarrow \infty} \|f_j\|_X = \|f\|_X$.
   (f) For every measurable set $F \subset \mathbb{R}^n$ such that $|F| < \infty$, $\|\chi_F\|_X$ is finite. Additionally there exists a constant $C_F > 0$ depending only on $F$ such that for all $h \in X$,
      $$\int_F |h(x)| \, dx \leq C_F \|h\|_X.$$

2. Suppose that $X$ is a Banach function space equipped with a norm $\|\cdot\|_X$. The associate space $X'$ is defined by

$$X' := \{ f \in \mathcal{M} : \|f\|_{X'} < \infty \},$$

where

$$\|f\|_{X'} := \sup_g \left\{ \int_{\mathbb{R}^n} f(x)g(x) \, dx : \|g\|_X \leq 1 \right\}.$$

Lemma 2.1. Let $X$ be a Banach function space. Then the following hold:

1. (The Lorentz–Luxemburg theorem.) $(X')' = X$ holds; in particular, the norms $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.
(2) (The generalized Hölder inequality.) If \( f \in X \) and \( g \in X' \), then we have
\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq \|f\|_X \|g\|_{X'}.
\]

The usual Lebesgue space \( L^p(\mathbb{R}^n) \) with constant exponent \( 1 \leq p \leq \infty \) is an example of Banach function spaces. Kováčik and Rákosník [12] have proved that the generalized Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) with variable exponent \( p(\cdot) \) is a Banach function space and the associate space is \( L^{p'(\cdot)}(\mathbb{R}^n) \) with norm equivalence, where \( p'(\cdot) \) is the conjugate exponent given by \( \frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1 \).

If we assume some conditions for boundedness of the Hardy–Littlewood maximal operator \( M \) on \( X \), then the norm \( \| \cdot \|_X \) has properties similar to the Muckenhoupt weights.

**Lemma 2.2.** Let \( X \) be a Banach function space. Suppose that the Hardy–Littlewood maximal operator \( M \) is weakly bounded on \( X \), that is, there exists a positive constant \( C \) such that
\[
\| \chi_{\{Mf>\lambda\}} \|_X \leq C \lambda^{-1} \|f\|_X
\]
is true for all \( f \in X \) and \( \lambda > 0 \). Then we have
\[
\sup_{Q: \text{cube}} \frac{1}{|Q|} \|\chi_Q \|_X \|\chi_Q \|_{X'} < \infty.
\]

**Proof.** The proof is found in the author’s papers [8, Lemmas 2.4 and 2.5] and [10, Lemmas G’ and H]. For the reader’s convenience we give the self-contained proof. Take a cube \( Q \) and a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). Suppose that \( |f|_Q > 0 \). Because \( |f|_Q \chi_Q(x) \leq M(f \chi_Q)(x) \) holds for almost every \( x \in \mathbb{R}^n \), we obtain \( M(f \chi_Q)(x) > \lambda \) for almost every \( x \in Q \), where \( \lambda := |f|_Q/2 \). Hence by assumption (2.1) we get
\[
|f|_Q \chi_Q \leq |f|_Q \chi_{\{M(f \chi_Q) > \lambda\}} \leq |f|_Q \cdot C \lambda^{-1} \|f \chi_Q\|_X
\leq 2C \|f \chi_Q\|_X.
\]
Therefore we have
\[
\frac{1}{|Q|} \|\chi_Q \|_X \|\chi_Q\|_{X'} = \frac{1}{|Q|} \|\chi_Q \|_X \cdot \sup_g \left\{ \int_{\mathbb{R}^n} \left| g(x) \chi_Q(x) \right| \, dx : g \in X, \|g\|_X \leq 1 \right\}
= \sup_g \{ |g|_Q \chi_Q \|_X : g \in X, \|g\|_X \leq 1 \}
\leq \sup_g \{ 2C |g \chi_Q|_X : g \in X, \|g\|_X \leq 1 \}
\leq 2C.
\]

**Remark 2.1.** If \( M \) is bounded on \( X \), that is, there exists a positive constant \( C \) such that
\[
\|Mf\|_X \leq C \|f\|_X
\]
holds for all \( f \in X \), then we can easily check that (2.1) holds. On the other hand, if \( M \) is bounded on the associate space \( X' \), then Lemma 2.1 shows that (2.2) is true.
2.2. Muckenhoupt weights.

**Definition 2.2.** Let $w$ be a locally integrable and positive function on $\mathbb{R}^n$. The function $w$ is said to be a Muckenhoupt $A_1$ weight if there exists a positive constant $C_1$ such that $Mw(x) \leq C_1 w(x)$ holds for almost every $x \in \mathbb{R}^n$. The set $A_1$ consists of all Muckenhoupt $A_1$ weights. For every $w \in A_1$, the finite value

$$[w]_{A_1} := \sup_{Q: \text{cube}} \left\{ \frac{1}{|Q|} \int_Q w(x) \, dx \cdot \|w^{-1}\|_{L^\infty(Q)} \right\}$$

is said to be a Muckenhoupt $A_1$ constant.

We remark that if $w \in A_1$, then

$$\frac{1}{|Q|} \int_Q w(x) \, dx \leq [w]_{A_1} \inf_{x \in Q} w(x)$$

for all cubes $Q$. We will use a classical result on the Muckenhoupt weights.

**Lemma 2.3** (Chapter 7 in [4], Chapter 9 in [5]). Let $w \in A_1$. We write $w(Q) := \int_Q w(x) \, dx$ for a cube $Q$. Then the reverse Hölder inequality holds, that is, there exist positive constants $q > 1$ and $C$ depending on $n$ and $[w]_{A_1}$ such that for all cubes $Q$,

$$\left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right)^{1/q} \leq C |Q|^{-1} w(Q).$$

3. The main result

Ho [6, Theorem 2.3] has proved the following characterization of the BMO norm via Banach function spaces.

**Theorem 3.1.** Let $X$ be a Banach function space. If the Hardy–Littlewood maximal operator $M$ is bounded on the associate space $X'$, then there exist positive constants $C_1 \leq C_2$ such that for all $b \in \text{BMO}({\mathbb{R}^n})$,

$$C_1 \|b\|_{\text{BMO}} \leq \sup_{Q: \text{cube}} \frac{1}{\|\chi_Q\|_X} \|(b - b_Q)\chi_Q\|_X \leq C_2 \|b\|_{\text{BMO}}. \tag{3.1}$$

Ho’s proof is based on the theory of Hardy spaces. We will give another proof of Theorem 3.1 by virtue of the Rubio de Francia algorithm.

**Proof.** Below we take $b \in \text{BMO}({\mathbb{R}^n})$ arbitrarily and write $f := b - b_Q$ for any cube $Q$.

We first prove the left hand side inequality in (3.1). Using Lemma 2.2 and Remark 2.1 we get for all cubes $Q$:

$$\frac{1}{|Q|} \int_Q |f(x)| \, dx \leq \frac{1}{|Q|} \|f\chi_Q\|_X \|\chi_Q\|_X \leq C \cdot \frac{1}{\|\chi_Q\|_X} \|f\chi_Q\|_X,$$

where $C > 0$ is a constant independent of $b$ and $Q$. This shows the desired estimate.
Next we prove the right hand side inequality in (3.1). Our idea is based on [2, Proof of Lemma 3.3]. Take \( g \in X' \) with \( \|g\|_{X'} \leq 1 \). Let \( B := \|M\|_{X' \to X'} \) and define a function
\[
Rg(x) := \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2B)^k}, \quad (g \in X'),
\]
where
\[
M^k g := \begin{cases} 
|g| & (k = 0), \\
Mg & (k = 1), \\
M(M^k - 1)g & (k \geq 2).
\end{cases}
\]
For every \( g \in X \), the function \( Rg \) satisfies the following properties:
1. \( |g(x)| \leq Rg(x) \) for almost every \( x \in \mathbb{R}^n \).
2. \( \|Rg\|_{X'} \leq 2\|g\|_{X'} \), namely the operator \( R \) is bounded on \( X' \).
3. \( M(Rg)(x) \leq 2BRg(x) \), that is, \( Rg \) is a Muckenhoupt \( A_1 \) weight.

By Lemma 2.3, there exist positive constants \( q > 1 \) and \( C \) independent of \( g \) such that for all cubes \( Q \),
\[
\left( \frac{1}{|Q|} \int_Q Rg(x)^q \, dx \right)^{1/q} \leq C \frac{|Q|}{|Q|} Rg(Q).
\]
By virtue of the generalized Hölder inequality, we obtain
\[
\| (Rg)\chi_Q \|_{L^q(\mathbb{R}^n)} = |Q|^{1/q} \left( \frac{1}{|Q|} \int_Q Rg(x)^q \, dx \right)^{1/q} \\
\leq |Q|^{1/q} \cdot \frac{C}{|Q|} Rg(Q) \\
\leq C \cdot |Q|^{-1/(q')} \|Rg\|_{X'} \|\chi_Q\|_X \\
\leq C \cdot |Q|^{-1/(q')} \|\chi_Q\|_X.
\]
Then we have
\[
\int_Q |f(x)g(x)| \, dx \leq \int_Q |f(x)| Rg(x) \, dx \\
\leq \|f\chi_Q\|_{L^{q'}(\mathbb{R}^n)} \| (Rg)\chi_Q \|_{L^q(\mathbb{R}^n)} \\
\leq C \left( \frac{1}{|Q|} \int_Q |f(x)|^{q'} \, dx \right)^{1/(q')} \|\chi_Q\|_X \\
\leq C \|b\|_{BMO} \|\chi_Q\|_X.
\]
By Lemma 2.1 we get
\[
\|(b - b_Q)\chi_Q \|_X = \|f\chi_Q\|_X \\
\leq C \sup_g \left\{ \int_Q |f(x)g(x)| \, dx : g \in X', \|g\|_{X'} \leq 1 \right\} \\
\leq C \|b\|_{BMO} \|\chi_Q\|_X.
\]
Consequently we have proved the theorem.
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