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MONADIC WAJSBERG HOOPS

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Abstract. Wajsberg hoops are the $\{\odot, \to, 1\}$-subreducts (hoop-subreducts) of Wajsberg algebras, which are term equivalent to MV-algebras and are the algebraic models of Łukasiewicz infinite-valued logic. Monadic MV-algebras were introduced by Rutledge [Ph.D. thesis, Cornell University, 1959] as an algebraic model for the monadic predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. In this paper we study the class of $\{\odot, \to, \forall, 1\}$-subreducts (monadic hoop-subreducts) of monadic MV-algebras. We prove that this class, denoted by $\mathcal{MWH}$, is an equational class and we give the identities that define it. An algebra in $\mathcal{MWH}$ is called a monadic Wajsberg hoop. We characterize the subdirectly irreducible members in $\mathcal{MWH}$ and the congruences by monadic filters. We prove that $\mathcal{MWH}$ is generated by its finite members. Then, we introduce the notion of width of a monadic Wajsberg hoop and study some of the subvarieties of monadic Wajsberg hoops of finite width k. Finally, we describe a monadic Wajsberg hoop as a monadic maximal filter within a certain monadic MV-algebra such that the quotient is the two element chain.

1. Introduction

Hoops are a particular class $\mathcal{H}$ of algebraic structures which were introduced in an unpublished manuscript by Büchi and Owens in the mid-seventies and later investigated in [4] and [13]. They are partially ordered commutative residuated integral monoids satisfying a further divisibility condition. Bosbach showed that $\mathcal{H}$ is a variety.

The variety of hoops includes two classes of algebras that are closely related to familiar algebras of logic: the variety of Brouwerian semilattices, defined relative to $\mathcal{H}$ by the identity $x \odot x \approx x$, and the variety of Wajsberg hoops, defined relative to $\mathcal{H}$ by the axiom $(x \to y) \to y \approx (y \to x) \to x$. It is known that Brouwerian semilattices are the $\{\wedge, \to, 1\}$-subreducts of Heyting algebras, which are the algebraic models of intuitionistic propositional logic, and Wajsberg hoops are the $\{\odot, \to, 1\}$-subreducts (hoop-subreducts) of Wajsberg algebras, which are term equivalent to MV-algebras and are the algebraic models of Łukasiewicz infinite-valued logic.

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Monadic MV-algebras (MMV-algebras for short) were introduced and studied by Rutledge in [15] as an algebraic model for the monadic predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. He gave MMV-algebras the name of monadic Chang algebras. Rutledge followed Halmos' study of monadic boolean algebras and represented each subdirectly irreducible MMV-algebra as a subalgebra of a functional MMV-algebra. From this representation, he proved the completeness of the monadic predicate calculus. As the MMV-algebras form the algebraic semantic of the monadic predicate infinite-valued calculus of Łukasiewicz, then the subvarieties of the variety $\mathcal{MMV}$ of MMV-algebras are in one-to-one correspondence with the intermediate logics.

In this paper we study the class of all monadic hoop-subreducts of MMV-algebras, that is, the class of all $\{\odot, \to, \forall, 1\}$-subreducts of MMV-algebras. In §3 we prove that this class is an equational class, which we denote by $\mathcal{MWH}$, and we give the identities that define it. An algebra in $\mathcal{MWH}$ is called a *monadic Wajsberg hoop*. We characterize the congruences of each monadic Wajsberg hoop $\mathbf{H}$ by means of monadic filters. More precisely, we establish an order isomorphism from the lattice $\mathbf{Con}_{\mathcal{MWH}}(\mathbf{H})$ of congruences of $\mathbf{H}$ and the lattice $\mathcal{F}_M(\mathbf{H})$ of monadic filters of $\mathbf{H}$, both ordered by inclusion. Moreover, we prove that the lattice $\mathcal{F}_M(\mathbf{H})$ is isomorphic to the lattice $\mathcal{F}(\forall\mathbf{H})$ of filters of the Wajsberg hoop $\forall\mathbf{H}$. From this, we characterize the subdirectly irreducible members of $\mathcal{MWH}$. Since $\mathcal{MWH}$ is exactly the class of all monadic hoop-subreducts of MMV-algebras, we obtain a closed relationship between varieties of monadic Wajsberg hoops and varieties of MMV-algebras. As an application, we prove that $\mathcal{MWH}$ is generated by the set $\{\mathbf{C}_m^k : m, k \in \mathbb{N}\}$, where $\mathbf{C}_m^k$ is the monadic Wajsberg hoop-reduct of the MMV-algebra $\mathbf{S}_m^k$.

In §4 we introduce the notion of width of a monadic Wajsberg hoop. We characterize a subdirectly irreducible algebra of finite width k as a subalgebra of a functional monadic Wajsberg hoop. Finally, we study some of the subvarieties of algebras of a finite width k, including cancellative subvarieties.

In [1] the authors construct, given a Wajsberg hoop $\mathbf{H}$, an MV-algebra $\mathbf{MV}(\mathbf{H})$ such that the underlying set H of $\mathbf{H}$ is a maximal filter of $\mathbf{MV}(\mathbf{H})$ and the quotient $\mathbf{MV}(\mathbf{H})/H$ is the two element chain. In §5, we see that if the given Wajsberg hoop $\mathbf{H}$ is a monadic Wajsberg hoop, then we can enrich $\mathbf{MV}(\mathbf{H})$ with an existencial operator and a universal operator making $\mathbf{MV}(\mathbf{H})$ an MMV-algebra, obtaining from here a description of a monadic Wajsberg hoop as a monadic maximal filter within an MMV-algebra.

2. Preliminaries

In this section we recall some definitions and collect the properties of MMV-algebras and Wajsberg hoops needed in the rest of the paper.

An *MV-algebra* is an algebra $\mathbf{A} = \langle A; \oplus, \neg, 0\rangle$ of type $(2, 1, 0)$ satisfying the following identities:

(MV1) $x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$,
(MV2) $x \oplus y \approx y \oplus x$,
(MV3) $x \oplus 0 \approx x$,
(MV4) $\neg\neg x \approx x$,
(MV5) $x \oplus \neg 0 \approx \neg 0$,
(MV6) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

MV-algebras were introduced by C. C. Chang in [6] as algebraic models for Łukasiewicz infinite-valued logic. On each MV-algebra $\mathbf{A}$ we define the constant 1 and the operations $\odot$ and $\rightarrow$ as follows: $1 := \neg 0$, $x \odot y := \neg(\neg x \oplus \neg y)$, and $x \rightarrow y := \neg x \oplus y$. For any two elements a and b of $\mathbf{A}$, we define $a \leq b$ if and only if $a \rightarrow b = 1$. It follows that $\leq$ is a partial order, which is called the natural order of $\mathbf{A}$. The natural order determines a lattice structure in A. Specifically the join $a \vee b$ and the meet $a \wedge b$ of a and b are given by $a \vee b = (a \rightarrow b) \rightarrow b$ and $a \wedge b = a \odot (a \rightarrow b)$.

The real interval $[0,1]$ enriched with the operations $a \oplus b = \min\{1, a+b\}$ and $\neg a = 1 - a$, is an MV-algebra denoted by $[\mathbf{0},\mathbf{1}]$. Chang proved in [7] that this algebra generates the variety $\mathcal{MV}$ of MV-algebras. Let $\mathbb{N}$ be the set of positive integer numbers. For every $n \in \mathbb{N}$, let $S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ and denote by $\mathbf{S}_n = \langle S_n; \oplus, \neg, 0\rangle$ the finite MV-subalgebra of $[\mathbf{0},\mathbf{1}]$ with $n+1$ elements.

Mundici defined a functor Γ between MV-algebras and (abelian) ℓ-groups with strong unit, and proved that Γ is a categorical equivalence ([14]). For every abelian ℓ-group $\mathbf{G}$, the functor Γ equips the unit interval $[0,u]$ with the operations $x \oplus y = u \wedge (x+y)$, $\neg x = u - x$, and $1 = u$. The resulting structure $\langle [0,u]; \oplus, \neg, 0\rangle$ is an MV-algebra. Set $\mathbf{S}_{n,\omega} = \Gamma(\mathbb{Z} \times \mathbb{Z}, (n,0))$, where $\mathbb{Z}$ is the totally ordered additive group of integers and $\mathbb{Z} \times \mathbb{Z}$ is the lexicographic product of $\mathbb{Z}$ by itself. Let us observe that $\mathbf{S}_n$ is isomorphic to $\Gamma(\mathbb{Z}, n)$, and we write $\mathbf{S}_n \cong \Gamma(\mathbb{Z}, n)$.

A *hoop* is an algebra $\mathbf{H} = \langle H; \odot, \rightarrow, 1\rangle$ of type $(2,2,0)$ such that $\langle H; \odot, 1\rangle$ is a commutative monoid and the following identities are satisfied:

(H1) $x \rightarrow x \approx 1$,
(H2) $x \odot (x \rightarrow y) \approx y \odot (y \rightarrow x)$,
(H3) $x \rightarrow (y \rightarrow z) \approx x \odot y \rightarrow z$.

The first systematic study of the structural properties of hoops appeared in Ferreirim's thesis [13]. We also refer to [4] for a study of hoops and for further references.

If $\mathbf{H} = \langle H; \odot, \rightarrow, 1\rangle$ is a hoop then $\langle H; \odot, 1\rangle$ is a naturally ordered residuated commutative monoid where the order is defined by $a \leq b$ iff $a \rightarrow b = 1$, and residuation means that $a \odot b \leq c$ iff $a \leq b \rightarrow c$. Any hoop is a meet semilattice and the meet is term-definable as $a \wedge b = a \odot (a \rightarrow b)$. In the following lemma we collect some properties of hoops.

Lemma 2.1. *Let $\mathbf{H} = \langle H; \odot, \rightarrow, 1\rangle$ be a hoop. For every $a, b, c \in H$, the following holds:*

(H4) $1 \rightarrow a = a$,
(H5) $a \rightarrow 1 = 1$,
(H6) $a \rightarrow b \leq (c \rightarrow a) \rightarrow (c \rightarrow b)$,
(H7) $a \leq b \rightarrow a$,
(H8) $a \leq (a \rightarrow b) \rightarrow b$,
(H9) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$,

(H10) $a \to b \leq (b \to c) \to (a \to c)$,
(H11) $a \leq b$ *implies* $b \to c \leq a \to c$ *and* $c \to a \leq c \to b$.

The variety $\mathcal{WH}$ of Wajsberg hoops is defined relative to the variety of hoops by the axiom $(T)\,(x \to y) \to y \approx (y \to x) \to x$. The underlying ordering of a Wajsberg hoop is a distributive lattice ordering and the join is term-definable by $a \vee b = (a \to b) \to b$. Since any hoop satisfies the identity $(x \to y) \to (y \to x) \approx y \to x$ ([5]), any Wajsberg hoop satisfies $(x \to y) \vee (y \to x) \approx 1$. Then 1 is join-irreducible and, in consequence, any subdirectly irreducible Wajsberg hoop is totally ordered.

A hoop $\mathbf{H} = \langle H; \odot, \to, 1\rangle$ is cancellative if $\langle H; \odot, 1\rangle$ is cancellative as a monoid. Cancellative hoops form a variety, axiomatized relative to hoops by the identity $x \approx y \to (x \odot y)$ ([4]). It can be shown that cancellative hoops coincide with negative cones of abelian ℓ-groups in the following sense. If $\mathbf{G}$ is an abelian ℓ-group and G^- is its negative cone, then the structure $\langle G^-; \odot, \to, 1\rangle$ is a cancellative hoop, where $a \odot b := a + b$, $a \to b := (b - a) \wedge 0$, and $1 := 0^G$. Conversely, if $\mathbf{H}$ is a cancellative hoop, then $\mathbf{H}$ can be identified with the negative cone of an abelian ℓ-group ([4]). Any cancellative hoop is a Wajsberg hoop and cancellative hoops are axiomatized (relative to Wajsberg hoops) by the identity $x \to (x \odot x) \approx x$. If $\mathbf{H}$ is a nontrivial totally ordered cancellative Wajsberg hoop, then $\mathcal{V}(\mathbf{H})$ (the variety generated by $\mathbf{H}$) is equal to $\mathcal{V}(\mathbf{C}_\omega)$, where $\mathbf{C}_\omega$ is the cancellative hoop given by the negative cone of $\mathbb{Z}$.

A hoop is bounded if it has a bottom element with respect to the order. A Wajsberg algebra is a bounded Wajsberg hoop in the enriched language $(\odot, \to, 1, 0)$, where 0 is the constant for the bottom element. Wajsberg algebras and MV-algebras are term-wise equivalent. Indeed, if $\langle A; \odot, \to, 1, 0\rangle$ is a Wajsberg algebra and we define $\neg a := a \to 0$ and $a \oplus b := \neg a \to b$, then $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra. Conversely, if $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra and we set $1 := \neg 0$, $a \to b := \neg a \oplus b$, and $a \odot b := \neg(\neg a \oplus \neg b)$, then $\langle A; \odot, \to, 1, 0\rangle$ is a Wajsberg algebra.

In any Wajsberg hoop $\mathbf{H} = \langle H; \odot, \to, 1\rangle$ we can define the operation

$$a \oplus b = (a \to (a \odot b)) \to b.$$

If $\mathbf{H}$ has a least element, then it is easily seen that $\oplus$ is the usual Łukasiewicz sum, while if $\mathbf{H}$ is a cancellative hoop then $a \oplus b = 1$ for all $a, b \in H$ ([2]). Moreover, the identity $x \oplus x \approx 1$ axiomatizes the variety of cancellative hoops relative to the variety of Wajsberg hoops.

Congruences in MV-algebras and Wajsberg hoops are completely determined by filters. A filter of a Wajsberg hoop or MV-algebra $\mathbf{A}$ is a nonempty subset F of A such that $a \in F$ and $a \leq b$ implies $b \in F$, and $a, b \in F$ implies $a \odot b \in F$. Note that a filter is a subhoop. Filters form an algebraic lattice that is isomorphic with the congruence lattice of $\mathbf{A}$. If $\mathbf{A}$ is a bounded hoop, then its congruences do not depend on the fact that we look at $\mathbf{A}$ as an MV-algebra or a Wajsberg hoop.

We define $a^1 = a$ and $a^{n+1} = a \odot a^n$, for each $n \in \mathbb{N}$.

Let $\mathbf{H}$ be a nontrivial bounded Wajsberg hoop, with bottom element 0. For every $a \in H$, let the order of a, written $\operatorname{ord} a$, be the least $n \in \mathbb{N}$ such that $a^n = 0$, provided that such an n exists. If $a^n \neq 0$ for every n, then we set $\operatorname{ord} a = \omega$. We define the order of $\mathbf{H}$ by

$$\operatorname{ord}(\mathbf{H}) = \sup\{\operatorname{ord} a : a \in H - \{1\}\}.$$

If this supremum does not exist then we say that the order of $\mathbf{H}$ is infinite and we write $\operatorname{ord}(\mathbf{H}) = \omega$. Let the radical of $\mathbf{H}$, written $\operatorname{Rad}(\mathbf{H})$, be the intersection of all maximal filters of $\mathbf{H}$. If $\mathbf{H}$ is totally ordered then $\operatorname{Rad}(\mathbf{H}) = \{a \in H : \operatorname{ord} a = \omega\}$. It is easily seen that $\operatorname{Rad}(\mathbf{H})$ is a proper filter of $\mathbf{H}$ and a cancellative subhoop of $\mathbf{H}$. We define the rank of $\mathbf{H}$ by

$$\operatorname{rank}(\mathbf{H}) = \operatorname{ord}(\mathbf{H}/\operatorname{Rad}(\mathbf{H})).$$

There is an important relation between varieties of MV-algebras and varieties of Wajsberg hoops. If $\mathcal{K}$ is a class of MV-algebras, let $\mathcal{S}^h(\mathcal{K})$ be the class of hoop-subreducts of algebras in $\mathcal{K}$. It follows that if $\mathcal{V}$ is a variety of MV-algebras then $\mathcal{S}^h(\mathcal{V})$ is a variety of Wajsberg hoops. In particular, the variety $\mathcal{WH}$ is the class of hoop-subreducts of $\mathcal{MV}$ ([4]).

An algebra $\mathbf{A} = \langle A; \oplus, \neg, \exists, 0\rangle$ of type $(2, 1, 1, 0)$ is called a *monadic MV-algebra* (an MMV-algebra for short) if $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra and $\exists$ satisfies the following identities:

(MMV1) $x \leq \exists x$,
(MMV2) $\exists(x \vee y) \approx \exists x \vee \exists y$,
(MMV3) $\exists\neg\exists x \approx \neg\exists x$,
(MMV4) $\exists(\exists x \oplus \exists y) \approx \exists x \oplus \exists y$,
(MMV5) $\exists(x \odot x) \approx \exists x \odot \exists x$,
(MMV6) $\exists(x \oplus x) \approx \exists x \oplus \exists x$.

MMV-algebras were defined by Rutledge in [15]. Let $\mathbf{A}$ be an MMV-algebra and let us define $\forall a := \neg\exists\neg a$, for every $a \in A$. Clearly, the following identities dual to (MMV1)–(MMV6) are satisfied in every MMV-algebra:

(MMV7) $\forall x \leq x$,
(MMV8) $\forall(x \wedge y) \approx \forall x \wedge \forall y$,
(MMV9) $\forall\neg\forall x \approx \neg\forall x$,
(MMV10) $\forall(\forall x \odot \forall y) \approx \forall x \odot \forall y$,
(MMV11) $\forall(x \odot x) \approx \forall x \odot \forall x$,
(MMV12) $\forall(x \oplus x) \approx \forall x \oplus \forall x$.

For our purposes, it is more convenient to consider the operator $\forall$ instead of $\exists$. So, from now on, we consider an algebra $\mathbf{A} = \langle A; \oplus, \neg, \forall, 0\rangle$ as an MMV-algebra if $\forall$ satisfies the identities (MMV7)–(MMV12). We often write $\langle A; \forall\rangle$ for short.

The next lemma collects some basic properties of MMV-algebras.

Lemma 2.2 ([15])**.** *Let $\mathbf{A} \in \mathcal{MMV}$ and $a, b \in A$. Then the following properties hold:*

(MMV13) $\forall 0 = 0$,
(MMV14) $\forall 1 = 1$,
(MMV15) $\forall\forall a = \forall a$,
(MMV16) $\forall(\forall a \oplus \forall b) = \forall a \oplus \forall b$,
(MMV17) $\forall(\forall a \to \forall b) = \forall a \to \forall b$,
(MMV18) $\forall(a \to b) \leq \forall a \to \forall b$,
(MMV19) $\forall(a \vee \forall b) = \forall a \vee \forall b$,
(MMV20) $\forall\exists a = \exists a$, $\exists\forall a = \forall a$.

Let $\forall A$ be the set $\{a \in A : a = \forall a\} = \{a \in A : a = \exists a\}$. It follows that $\forall\mathbf{A} = \langle \forall A; \oplus, \neg, 0\rangle$ is an MV-subalgebra of the MV-reduct of $\mathbf{A}$.

Let $\mathbf{A}$ be an MV-algebra and $X = \{1, \ldots, k\}$ be a finite set. Let us consider the product algebra $\mathbf{A}^X$, where $\oplus$, $\neg$ and 0 are defined pointwise. If we define

$\forall_\wedge(\langle a_1,\dots,a_k\rangle) = \langle a_1\wedge\cdots\wedge a_k,\dots,a_1\wedge\cdots\wedge a_k\rangle$, then $\mathbf{A}^X = \langle A^X;\oplus,\neg,\forall_\wedge,0\rangle$ is an MMV-algebra. We denote this algebra simply by $\mathbf{A}^k$.

Let us consider the identity

$$(\alpha^k)\quad \bigvee_{1\le i<j\le k+1}\left(\forall(x_i\vee x_j)\to\bigvee_{s=1}^{k+1}\forall x_s\right)\approx 1,$$

if $k\ge 2$, and

$$(\alpha^1)\ x\approx\forall x.$$

The width of an MMV-algebra $\mathbf{A}$, denoted by width$(\mathbf{A})$, is the least integer k such that (α^k) holds in $\mathbf{A}$. If k does not exist, then we say that the width of $\mathbf{A}$ is infinite and we write width$(\mathbf{A})=\omega$. This definition is motived by the following result.

Proposition 2.3 ([10])**.** *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra that satisfies* (α^k)*. Then* $\mathbf{A}$ *is isomorphic to a subalgebra of* $(\forall\mathbf{A})^k$*.*

If $\mathbf{A}$ is a subdirectly irreducible MMV-algebra and width$(\mathbf{A})=k<\omega$, then the algebra of complemented elements $\mathbf{B}(\mathbf{A})$ is isomorphic to a subalgebra of the simple monadic boolean algebra $\mathbf{2}^k=\langle 2^k;\vee,\wedge,',\forall,\langle 0,\dots,0\rangle,\langle 1,\dots,1\rangle\rangle$, where $2=\{0,1\}$ and $\forall\colon 2^k\to 2^k$ is defined by $\forall a=\begin{cases}\langle 0,\dots,0\rangle & \text{if } a<\langle 1,\dots,1\rangle\\ \langle 1,\dots,1\rangle & \text{if } a=\langle 1,\dots,1\rangle\end{cases}$. It is well known that there is a correspondence between the family of all subalgebras of $\mathbf{2}^k$ and the partitions of the set of coatoms of $\mathbf{2}^k$. In addition, the partitions of this set are in a natural correspondence with the partitions of the set $\{1,\dots,k\}$. So we have a one-to-one onto correspondence between the set of subalgebras of $\mathbf{2}^k$ and the set of all partitions of $\{1,\dots,k\}$. Then we can associate to $\mathbf{A}$ the partition $\mathbf{P}=\{P_1,P_2,\dots,P_s\}$ of $\{1,\dots,k\}$ determined by the subalgebra $\mathbf{2}^s\cong\mathbf{B}(\mathbf{A})$ of $\mathbf{B}(\mathbf{S}_n^k)$.

Let

$$(\beta_n^s)\quad\left(\bigwedge_{1\le i<j\le s+1}\forall(2x_i^{n+1}\vee 2x_j^{n+1})\right)\to\bigvee_{j=1}^{s+1}\forall(2x_j^{n+1})\approx 1.$$

Let $\mathbf{A}$ be a subdirectly irreducible MMV-algebra such that rank$(\forall\mathbf{A})=n$, ord$(\forall\mathbf{A})=\omega$, and width$(\mathbf{A})=k$. Then, the identity (β_n^s) holds in $\mathbf{A}$ if and only if $\mathbf{B}(\mathbf{A})$ is isomorphic to a subalgebra of $\mathbf{2}^s$. The *boolean width of* $\mathbf{A}$, denoted by bwidth$(\mathbf{A})$, is the least integer s such that (β_n^s) holds in $\mathbf{A}$ ([10]).

Let $\mathbf{A}$ be a subdirectly irreducible MMV-algebra such that width$(\mathbf{A})=k$. The following results are proven in [10]:

- If ord$(\forall\mathbf{A})=m$, then $\mathcal{V}(\mathbf{A})=\mathcal{V}(\mathbf{S}_m^k)$.
- If rank$(\forall\mathbf{A})=n$, ord$(\forall\mathbf{A})=\omega$, bwidth$(\mathbf{A})=k$, then $\mathcal{V}(\mathbf{A})=\mathcal{V}(\mathbf{S}_{n,\omega}^k)$.
- If rank$(\forall\mathbf{A})=n$, ord$(\forall\mathbf{A})=\omega$, bwidth$(\mathbf{A})=1$, then $\mathcal{V}(\mathbf{A})=\mathcal{V}(\mathbf{S}_{n,\omega}^{k,1})$, where $\mathbf{S}_{n,\omega}^{k,1}$ is the MMV-subalgebra of $\mathbf{S}_{n,\omega}^k$ generated by the constant elements $\forall(\mathbf{S}_{n,\omega}^k)$ and the radical Rad$(\mathbf{S}_{n,\omega}^k)$ of $\mathbf{S}_{n,\omega}^k$.

- If $\mathrm{rank}(\forall\mathbf{A}) = n$, $\mathrm{ord}(\forall\mathbf{A}) = \omega$, $\mathrm{bwidth}(\mathbf{A}) = s$ with $1 < s < k$, then there exists a partition $\mathbf{P} = \{P_1, \ldots, P_s\}$ of the set $\{1, \ldots, k\}$ associated with $\mathbf{A}$ such that the following identity is satisfied in $\mathbf{A}$:

$$\left[\left((\bigwedge_{i=1}^{s} 2x_i^{n+1}) \leftrightarrow 0 \right) \wedge \left(\bigwedge_{1\leq i<j\leq s} ((2x_i^{n+1} \vee 2x_j^{n+1}) \leftrightarrow 1) \right) \wedge \right.$$
$$\left. \left(\bigwedge_{i=1}^{s} (\forall 2x_i^{n+1} \leftrightarrow 0) \right) \wedge \left(\bigwedge_{i=1}^{s} (\exists 2x_i^{n+1} \leftrightarrow 1) \right) \right] \qquad (f_n^{k,\mathbf{P}})$$
$$\to \left[\bigvee_{\sigma\in\mathbb{P}(\{1,\ldots,s\})} \left(\bigwedge_{i=1}^{s} \alpha^{p_{\sigma(i)}}_{\forall_{2x^{n+1}_{\sigma(i)}}} (z_1^{\sigma(i)}, \ldots, z^{\sigma(i)}_{p_{\sigma(i)}+1}) \right) \right] \approx 1,$$

where the cardinal of each subset P_i is denoted by p_i, the operation $\leftrightarrow$ is defined by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$, $\mathbb{P}(\{1, \ldots, s\})$ is the set of all permutations of the set $\{1, \ldots, s\}$, and

$$\alpha^{p_{\sigma(i)}}_{\forall_{a2x^{n+1}_{\sigma(i)}}} (z_1^{\sigma(i)}, \ldots, z^{\sigma(i)}_{p_{\sigma(i)}+1})$$

is an abbreviation of

$$\left(\forall \left(\bigvee_{r=1}^{p_{\sigma(i)}+1} \bigwedge Z_r^- \right) \vee 2x^{n+1}_{\sigma(i)} \right) \to \bigvee_{j=1}^{p_{\sigma(i)}+1} \left(\forall z_j^{\sigma(i)} \vee 2x^{n+1}_{\sigma(i)} \right),$$

where $Z = \{z_1^{\sigma(i)}, \ldots, z^{\sigma(i)}_{p_{\sigma(i)}+1}\}$ and $Z_r^- = Z - \{z_r^{\sigma(i)}\}$. Moreover, $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}^{k,P}_{n,\omega})$, where

$$\mathbf{S}^{k,\mathbf{P}}_{n,\omega} = \left\{ a \in \mathbf{S}^k_{n,\omega} : a(i)/\mathrm{Rad}(\mathbf{S}_{n,\omega}) = a(j)/\mathrm{Rad}(\mathbf{S}_{n,\omega}) \text{ if } i, j \in P_t \text{ for some } t \right\}.$$

3. Monadic Wajsberg hoops

In this section we define the variety $\mathcal{MWH}$ of monadic Wajsberg hoops. We prove general properties of $\mathcal{MWH}$ and characterize the congruences of each monadic Wajsberg hoop $\mathbf{H}$ by means of monadic filters. More precisely, we establish an order isomorphism from the lattice $\mathbf{Con}_{\mathcal{MWH}}(\mathbf{H})$ of congruences of $\mathbf{H}$ and the lattice $\mathcal{F}_M(\mathbf{H})$ of monadic filters of $\mathbf{H}$, both sets ordered by inclusion. Moreover, we prove that the lattice $\mathcal{F}_M(\mathbf{H})$ is isomorphic to the lattice $\mathcal{F}(\forall\mathbf{H})$ of filters of the Wajsberg hoop $\forall\mathbf{H}$. From this, we characterize the subdirectly irreducible members of $\mathcal{MWH}$. We also prove that $\mathcal{MWH}$ is exactly the class of all monadic hoop-subreducts of MMV-algebras, obtaining a closed relationship between varieties of monadic Wajsberg hoops and varieties of MMV-algebras. As a first application, we prove that $\mathcal{MWH}$ is generated by the set $\{\mathbf{C}_m^k : m, k \in \mathbb{N}\}$, where $\mathbf{C}_m^k$ is the monadic Wajsberg hoop-reduct of the MMV-algebra $\mathbf{S}_m^k$.

Definition 3.1. An algebra $\mathbf{H} = \langle H; \odot, \to, \forall, 1\rangle$ of type (2,2,1,0) is a *monadic Wajsberg hoop* if $\langle H; \odot, \to, 1\rangle$ is a Wajsberg hoop, and the following identities are satisfied:

(MH1) $\forall 1 \approx 1$,
(MH2) $\forall x \to x \approx 1$,
(MH3) $\forall((x \to \forall y) \to \forall y) \approx (\forall x \to \forall y) \to \forall y$,
(MH4) $\forall(x \to y) \to (\forall x \to \forall y) \approx 1$,
(MH5) $\forall(\forall x \to \forall y) \approx \forall x \to \forall y$,
(MH6) $\forall(x \odot x) \approx \forall x \odot \forall x$,
(MH7) $\forall((x \to \forall y) \to x) \approx (\forall x \to \forall y) \to \forall x$,
(MH8) $\forall(x \wedge y) \approx \forall x \wedge \forall y$,
(MH9) $\forall(\forall x \odot \forall y) \approx \forall x \odot \forall y$.

Let us observe that (MH3) can be written as $\forall(x \vee \forall y) \approx \forall x \vee \forall y$.

Lemma 3.2. *Let* $\mathbf{H}$ *be a monadic Wajsberg hoop, and let* $a, b \in H$. *Then we have the following.*

(MH10) $\forall a = \forall\forall a$.
(MH11) *If* $a \leq b$ *then* $\forall a \leq \forall b$.
(MH12) $\forall(\forall a \oplus \forall b) = \forall a \oplus \forall b$.
(MH13) $\forall(a \oplus a) = \forall a \oplus \forall a$.
(MH14) *If* $\forall H = \{\forall a : a \in H\}$ *then* $\forall\mathbf{H} = \langle \forall H; \odot, \to, \forall, 1\rangle$ *is a subalgebra of* $\mathbf{H}$.

Proof. From (MH2) and (MH3) we have

$$\forall a = \forall\forall a \vee \forall a = \forall(\forall a \vee \forall a) = \forall\forall a.$$

If $a \leq b$, and taking into account (MH4) and (MH1), we have

$$1 = \forall(a \to b) \to (\forall a \to \forall b) = \forall 1 \to (\forall a \to \forall b) = 1 \to (\forall a \to \forall b) = \forall a \to \forall b.$$

So, $\forall a \leq \forall b$.

From (MH9) and (MH5), we write

$$\begin{aligned}\forall(\forall a \oplus \forall b) &= \forall((\forall a \to (\forall a \odot \forall b)) \to \forall b) = \forall((\forall a \to \forall(\forall a \odot \forall b)) \to \forall b)\\ &= \forall(\forall(\forall a \to \forall(\forall a \odot \forall b)) \to \forall b) = \forall(\forall a \to \forall(\forall a \odot \forall b)) \to \forall b\\ &= (\forall a \to \forall(\forall a \odot \forall b)) \to \forall b = (\forall a \to (\forall a \odot \forall b)) \to \forall b = (\forall a \oplus \forall b).\end{aligned}$$

Let us prove now (MH13). From $\forall a \leq a$, we have that $\forall a \oplus \forall a \leq a \oplus a$. Taking into account (MH12) and (MH11), we obtain $\forall a \oplus \forall a = \forall(\forall a \oplus \forall a) \leq \forall(a \oplus a)$. Since $\forall(a \odot a) \leq a \odot a$, then $(a \to (a \odot a)) \to a \leq (a \to \forall(a \odot a)) \to a$. That is, $a \oplus a \leq (a \to \forall(a \odot a)) \to a$. Consequently, from (MH7) and (MH6), we have $\forall(a \oplus a) \leq \forall((a \to \forall(a \odot a)) \to a) = (\forall a \to \forall(a \odot a)) \to \forall a = (\forall a \to (\forall a \odot \forall a)) \to \forall a = \forall a \oplus \forall a$. So, (MH13) holds.

Finally, we obtain that $\forall\mathbf{H} = \langle \forall H; \odot, \to, \forall, 1\rangle$ is a subalgebra of $\mathbf{H}$ from (MH1), (MH5), (MH9) and (MH10). □

Let $\mathbf{H}$ be a Wajsberg hoop and $\mathbf{H}^k$ the Wajsberg hoop direct product of $\mathbf{H}$, where k is a positive integer number. We define $\forall_\wedge \colon H^k \to H^k$ by $\forall_\wedge(\langle a_1, \ldots, a_k\rangle) = \langle c, \ldots, c\rangle$, where $c = \bigwedge_{i=1}^k a_i$. The next result is straightforward.

Lemma 3.3. *If* $\mathbf{H}$ *is a Wajsberg hoop then* $\mathbf{H}^k = \langle H^k; \odot, \to, \forall_\wedge, 1\rangle$ *is a monadic Wajsberg hoop.*

A filter F of a monadic Wajsberg hoop $\mathbf{H}$ is a *monadic filter* of $\mathbf{H}$ if $\forall a \in F$, for each $a \in F$. Let us observe that if F is a monadic filter of $\mathbf{H}$, then F is closed by all the operations of $\mathbf{H}$. In consequence, every monadic filter is a subuniverse of $\mathbf{H}$.

Theorem 3.4. *Let* $\mathbf{H} \in \mathcal{MWH}$. *The correspondence* $\mathbf{Con}_{\mathcal{MWH}}(\mathbf{H}) \to \mathcal{F}_M(\mathbf{H})$ *defined by* $\theta \mapsto 1/\theta$, *where* $1/\theta = \{a \in H : (a,1) \in \theta\}$, *is an order isomorphism whose inverse is given by* $F \mapsto \theta_F$, *where*

$$\theta_F = \{(a,b) \in H \times H : (a \to b) \odot (b \to a) \in F\}\,.$$

Proof. Let $\theta \in Con_{\mathcal{MWH}}(\mathbf{H})$. We only need to prove that $1/\theta$ is a monadic filter. For that, let $a \in 1/\theta$, that is, $(a,1) \in \theta$. Then, $(\forall a, \forall 1) = (\forall a, 1) \in 1/\theta$.

Let $F \in \mathcal{F}_M(\mathbf{H})$ and $(a,b) \in H \times H$ such that $(a \to b) \odot (b \to a) \in F$. Then, $a \to b \in F$ and $b \to a \in F$. Since F is monadic, we have that $\forall(a \to b) \in F$ and $\forall(b \to a) \in F$. Taking into account that F is increasing and from (MH4), we obtain that $\forall a \to \forall b \in F$ and $\forall b \to \forall a \in F$. Thus, $(\forall a \to \forall b) \odot (\forall b \to \forall a) \in F$ and $\theta_F \in Con_{\mathcal{MWH}}(\mathbf{H})$.

The order isomorphism $\theta \mapsto 1/\theta$ is now a straigthforward computation. □

Given $\mathbf{H} \in \mathcal{MWH}$ and a non-empty set $X \subseteq H$, let $\mathrm{FMg}(X)$ denote the monadic filter generated by X in $\mathbf{H}$. Then,

$$\mathrm{FMg}(X) = \{b \in H : \forall a_1 \odot \forall a_2 \odot \cdots \odot \forall a_n \leq b \text{ for some } a_1, \ldots, a_n \in X\}\,.$$

In particular, if $X = \{a\}$ then $\mathrm{FMg}(\{a\}) = \{b \in H : (\forall a)^n \leq b, \text{ for some } n \in \mathbb{N}\}$, and it is denoted simply by $\mathrm{FMg}(a)$. Let us observe also that $\mathrm{FMg}(X) = \mathrm{Fg}(\forall X)$.

Theorem 3.5. *Let* $\mathbf{H} \in \mathcal{MWH}$. *The correspondence* $\mathcal{F}_M(\mathbf{H}) \to \mathcal{F}(\forall \mathbf{H})$ *defined by* $F \mapsto F \cap \forall H$ *is an order isomorphism whose inverse is given by* $M \mapsto \mathrm{FMg}(M)$.

Proof. If $F \in \mathcal{F}_M(\mathbf{H})$, then it is easy to see that $F \cap \forall H \in \mathcal{F}(\forall \mathbf{H})$. Let us prove that $\mathrm{FMg}(F \cap \forall H) = F$. From $F \cap \forall H \subseteq F$, we have that $\mathrm{FMg}(F \cap \forall H) \subseteq F$. Let us consider now $f \in F$. Then, $\forall f \in F \cap \forall H$ and consequently $\forall f \in \mathrm{FMg}(F \cap \forall H)$. Taking into account that $\mathrm{FMg}(F \cap \forall H)$ is increasing in H and $\forall f \leq f$, we obtain that $f \in \mathrm{FMg}(F \cap \forall H)$.

Let $M \in \mathcal{F}(\forall \mathbf{H})$. Let us see that $\mathrm{FMg}(M) \cap \forall H = M$. Clearly, $M \subseteq \mathrm{FMg}(M) \cap \forall H$. Let $z \in \mathrm{FMg}(M) \cap \forall H$. Then there are $a_1, a_2, \ldots, a_n \in M$ such that $a_1 \odot a_2 \odot \cdots \odot a_n = \forall a_1 \odot \forall a_2 \odot \cdots \odot \forall a_n \leq z$. Since M is a filter of $\forall \mathbf{H}$, we have that $a_1 \odot a_2 \odot \cdots \odot a_n \in M$. From $z \in \forall H$ and M increasing in $\forall H$, we obtain that $z \in M$. □

Corollary 3.6. *If* $\mathbf{H} \in \mathcal{MWH}$ *then*

$$\mathbf{Con}_{\mathcal{MWH}}(\mathbf{H}) \cong \mathcal{F}_M(\mathbf{H}) \cong \mathcal{F}(\forall \mathbf{H}) \cong \mathbf{Con}_{\mathcal{WH}}(\forall \mathbf{H}).$$

As an immediate consequence, we have the following characterization of the subdirectly irreducible members of the variety.

Corollary 3.7. *Let* $\mathbf{H} \in \mathcal{MWH}$. *Then,* $\mathbf{H}$ *is subdirectly irreducible (simple) if and only if* $\forall \mathbf{H}$ *is a subdirectly irreducible (simple) Wajsberg hoop.*

Since subdirectly irreducible algebras in the variety $\mathcal{WH}$ are totally ordered, the following result is also immediate.

Lemma 3.8. *If* $\mathbf{H}$ *is a subdirectly irreducible monadic Wajsberg hoop then* $\forall\mathbf{H}$ *is totally ordered.*

Then we clearly have the following result.

Proposition 3.9. *Every monadic Wajsberg hoop is a subdirect product of a family of monadic Wajsberg hoops* $\{\mathbf{H}_i\}$, $i \in I$, *such that* $\forall\mathbf{H}_i$ *is totally ordered.*

We use the order isomorphism between congruences and monadic filters to prove that $\mathcal{MWH}$ has the congruence extension property.

Proposition 3.10. *The variety of all monadic Wajsberg hoops has the congruence extension property.*

Proof. Let $\mathbf{A}$ be a monadic Wajsberg hoop and $\mathbf{B}$ a subalgebra of $\mathbf{A}$. Let F be a monadic filter of B and let us consider $F' = \mathrm{FMg}(F)$, the monadic filter generated by F in $\mathbf{A}$. It is clear that $F \subseteq F' \cap B$. Let $b \in F' \cap B$. Then there exist $a_1, \ldots, a_n \in F$ such that $\forall a_1 \odot \cdots \odot \forall a_n \leq b$. Since F is a monadic filter of $\mathbf{B}$, then $\forall a_i \in F$ for each i. So, $\forall a_1 \odot \cdots \odot \forall a_n \in F$ since F is closed under multiplication. In addition, from $b \in B$ and F upward closed we have that $b \in F$. Hence, $F = F' \cap B$. □

It is known that a totally ordered Wajsberg hoop is either bounded or cancellative. It is bounded and cancellative if and only if it is trivial (see [2, Proposition 2.1]).

Proposition 3.11. *If* $\mathbf{H}$ *is a monadic Wajsberg hoop such that* $\forall\mathbf{H}$ *is totally ordered, then* $\mathbf{H}$ *is bounded or cancellative. If it is both, then it is trivial.*

Proof. We know that $\forall\mathbf{H}$ is bounded or $\forall\mathbf{H}$ is cancellative. If $\forall\mathbf{H}$ is bounded then $\mathbf{H}$ is bounded. Indeed, if there exists a bottom element $\forall b$ in $\forall\mathbf{H}$, then $\forall b \leq \forall a \leq a$, for all $a \in H$. If $\forall\mathbf{H}$ is cancellative then $\forall a \oplus \forall a = 1$, for all $a \in H$. Then, from (MH2), $a \oplus a \geq \forall a \oplus \forall a = 1$. So, $a \oplus a = 1$ for all $a \in H$. Thus, $\mathbf{H}$ is cancellative.

If $\forall\mathbf{H}$ is bounded and cancellative, then $\forall H = \{1\}$. This implies that $H = \{1\}$, taking (MH2) into account. □

The following characterization will be needed in §4.

Corollary 3.12. *If* $\mathbf{H}$ *is a subdirectly irreducible monadic Wajsberg hoop, then* $\mathbf{H}$ *is bounded or cancellative.*

Proof. It follows from Lemma 3.8 and the last proposition. □

In the next lemma we prove that in every bounded monadic Wajsberg hoop we can define an MMV-algebra structure.

Lemma 3.13. *Let* $\mathbf{H} = \langle H; \odot, \rightarrow, \forall, 1\rangle$ *be a bounded monadic Wajsberg hoop with first element 0. Let us define the operations*

$$\neg a := a \rightarrow 0 \qquad \textit{and} \qquad a \oplus b := \neg a \rightarrow b,$$

for $a, b \in H$. *Then* $\langle H; \oplus, \neg, \forall, 0\rangle$ *is an MMV-algebra.*

Proof. It is known that $\langle H; \oplus, \neg, 0\rangle$ is an MV-algebra. So, it is enough to see that $\langle H; \oplus, \neg, \forall, 0\rangle$ satisfies the MMV-axioms. Axioms (MMV7), (MMV8), (MMV10), (MMV11) and (MMV12) are immediate from (MH2), (MH8), (MH9), (MH6), and (MH13) respectively. Clearly, if 0 is the bottom element then $\forall 0 = 0$. Now, from (MH5), we have that $\forall(\neg\forall a) = \forall(\forall a \to 0) = \forall a \to 0 = \neg\forall a$, for all $a \in H$. Then, (MMV9) holds. □

Lemma 3.14. *The monadic hoop-reduct, that is, the* $\{\odot, \to, \forall, 1\}$*-reduct of an MMV-algebra, is a monadic Wajsberg hoop.*

Proof. The identities (MH1), (MH2), (MH3), (MH4), (MH5), (MH6), (MH8), and (MH9) are immediate from (MMV14), (MMV7), (MMV19), (MMV18), (MMV17), (MMV11), (MMV8), and (MMV10), respectively. The identity (MH7) is proved in [11, Lemma 3.1]. □

In the following, we prove that the variety of monadic Wajsberg hoops is exactly the class of all monadic hoop-subreducts of MMV-algebras.

Let $\mathbf{H}$ be a monadic Wajsberg hoop and let $a \in H$. Let $[a) = \{b \in H : a \leq b\}$ and $(a] = \{b \in H : b \leq a\}$. For each $x, y \in [a)$, we define

$$x \odot_a y := (x \odot y) \vee a.$$

It is easy to see that $\langle [a); \odot_a, \to, 1\rangle$ is a bounded Wajsberg hoop. Let us observe that $[a)$ is closed under $\to$, since $x \leq y \to x$.

Lemma 3.15. *Let* $\mathbf{H} = \langle H; \odot, \to, \forall, 1\rangle \in \mathcal{MWH}$ *and* $a \in \forall H$. *Then,* $[\mathbf{a}) = \langle [a); \odot_a, \to, \forall, 1\rangle$ *is a bounded monadic Wajsberg hoop.*

Proof. From the remarks above it is enough to prove the MWH-axioms. Identities (MH1), (MH2), (MH3), (MH4), (MH5), and (MH7) are immediate.

Let $x, y \in [a)$. Let us prove first that (MH6) holds. Indeed, $\forall(x \odot_a x) = \forall((x \odot x) \vee a) = \forall(x \odot x) \vee a = (\forall x \odot \forall x) \vee a = \forall x \odot_a \forall x$.

For (MH8), we write $\forall(x \wedge_a y) = \forall(x \odot_a (x \to y)) = \forall((x \odot (x \to y)) \vee a) = \forall(x \odot (x \to y)) \vee a = (\forall x \wedge \forall y) \vee a = (\forall x \odot (\forall x \to \forall y)) \vee a = \forall x \odot_a (\forall x \to \forall y) = \forall x \wedge_a \forall y$.

Finally, $\forall(\forall x \odot_a \forall y) = \forall((\forall x \odot \forall y) \vee a) = \forall(\forall x \odot \forall y) \vee a = \forall x \odot_a \forall y$. So, (MH9) is satisfied.

Hence, $[\mathbf{a})$ is a bounded monadic Wajsberg hoop. □

Theorem 3.16. *If* $\mathbf{H}$ *is a monadic Wajsberg hoop then* $\mathbf{H} \in \mathrm{ISP_U}(\{[\forall \mathbf{z}) : z \in H\})$.

Proof. Let us consider the family $\{(a] : a \in H\}$. Since $(a \wedge b] = (a] \cap (b]$, the family $\{(a] : a \in H\}$ has the finite intersection property. Thus, there exists an ultrafilter F in the boolean algebra $\mathbf{Su}(\mathbf{H})$ of subsets of H, containing all the members of the family. Let $\psi \colon \mathbf{H} \to \left(\prod_{z\in H}[\forall \mathbf{z})\right)/F$ be defined by

$$\psi(a) = (a \vee \forall z)_{z\in H}/F.$$

So, given $a, b \in H$,

$$\psi(a) = \psi(b) \text{ if and only if } \{z \in H : \forall z \vee a = \forall z \vee b\} \in F.$$

Let us see that $\psi(\forall a) = \forall(\psi(a))$, $\psi(a \odot b) = \psi(a) \odot \psi(b)$, $\psi(a \to b) = \psi(a) \to \psi(b)$ and that ψ is injective.

For each $a \in H$, $\forall z \vee \forall a = \forall(a \vee \forall z)$. Then, $\forall(\psi(a)) = \forall\left((a \vee \forall z)_{z\in H}/F\right)$ $= \forall\left((a \vee \forall z)_{z\in H}\right)/F = \left(\forall(a \vee \forall z)\right)_{z\in H}/F = \left(\forall a \vee \forall z\right)_{z\in H}/F = \psi(\forall a)$.

If $z \leq a \odot b$ then $\forall z \vee (a \odot b) = a \odot b = (\forall z \vee a) \odot (\forall z \vee b)$. Then, $(a \odot b] \subseteq \{z \in H : \forall z \vee (a \odot b) = (\forall z \vee a) \odot (\forall z \vee b)\}$. Since $(a \odot b] \in F$, then $\{z \in H : \forall z \vee (a \odot b) = (\forall z \vee a) \odot (\forall z \vee b)\} \in F$. Thus, $\psi(a \odot b) = \psi(a) \odot \psi(b)$.

Note that $\psi(a \to b) = \psi(a) \to \psi(b)$ if and only if

$$\{z \in H : \forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\} \in F.$$

Let $c \in H$ be such that $c \leq a$ and $c \leq b$. Let us see that

$$(c] \subseteq \{z \in H : \forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\}.$$

Indeed, if $z \in (c]$ then $\forall z \leq z \leq c \leq a, b$. So, $\forall z \leq a \to b$ and consequently $\forall z \vee (a \to b) = a \to b = (\forall z \vee a) \to (\forall z \vee b)$. Since $(c] \in F$, then $\{z \in H : \forall z \vee (a \to b) = (\forall z \vee a) \to (\forall z \vee b)\} \in F$.

Finally, let us consider $a, b \in H$ such that $\psi(a) = \psi(b)$. Since $(a] \in F$, then $(a] \cap \{z \in H : \forall z \vee a = \forall z \vee b\} \in F$. In particular, this intersection is not empty. Let $w \in (a] \cap \{z \in H : \forall z \vee a = \forall z \vee b\}$. Then, $a = \forall w \vee a = \forall w \vee b$ and $b \leq a$. Analogously, considering $(b] \in F$ we obtain that $a \leq b$. So we prove that $a = b$. □

Corollary 3.17. $\mathbf{C}^k_\omega \in \mathrm{ISP_U}\left(\{\mathbf{C}^k_m : m \in \mathbb{N}\}\right)$.

Proof. If $z \in C^k_\omega$ then $[\forall \mathbf{z})$ is isomorphic to $\mathbf{C}^k_m$, for some $m \in \mathbb{N}$. □

Since in every bounded monadic Wajsberg hoop we can define an MMV-algebra structure (Lemma 3.13), it follows from Theorem 3.16 that every monadic Wajsberg hoop can be embeddable in an MMV-algebra.

Corollary 3.18. *If* $\mathbf{H}$ *is a subdirectly irreducible monadic Wajsberg hoop then* $\mathbf{H}$ *can be embeddable in an MMV-algebra* $\mathbf{A}$ *such that* $\forall\mathbf{A}$ *is totally ordered.*

Proof. Let $\mathbf{H}$ be a subdirectly irreducible monadic Wajsberg hoop. From Theorem 3.16, we know that $\mathbf{H}$ can be embeddable in an MMV-algebra $\mathbf{A}$. From Lemma 3.8 we also know that $\forall\mathbf{H}$ is totally ordered and consequently, for each $z \in H$, we have that $\forall([\forall\mathbf{z}))$ is totally ordered. Since the property of being totally ordered is preserved under ultraproducts, then $\forall\mathbf{A}$ is totally ordered too. □

If $\mathcal{K}$ is a class of MMV-algebras, let $\mathcal{S}^{hm}(\mathcal{K})$ be the class of all monadic hoop-subreducts of the algebras in $\mathcal{K}$, that is, monadic Wajsberg hoops that are monadic Wajsberg subhoops of some algebra in $\mathcal{K}$.

Proposition 3.19. *If* $\mathcal{V}$ *is a variety of MMV-algebras, then* $\mathcal{S}^{hm}(\mathcal{V})$ *is a variety of monadic Wajsberg hoops. In particular,* $\mathcal{MWH}$ *is the class of all monadic hoop-subreducts of MMV-algebras.*

Proof. Clearly $\mathcal{S}^{hm}(\mathcal{V})$ is closed under subalgebras and direct products. In addition, since $\mathcal{MWH}$ has the congruence extension property (Proposition 3.10), it follows that $\mathcal{S}^{hm}(\mathcal{V})$ is closed under homomorphic images.

From Lemma 3.14, Lemma 3.15 and Theorem 3.16, we have that $\mathcal{S}^{hm}(\mathcal{MMV}) = \mathcal{MWH}$. □

Corollary 3.20. *Let* $\mathbf{A}$ *be an MMV-algebra and* $\mathbf{H}$ *its monadic hoop-reduct. Then* $\mathcal{V}(\mathbf{H}) = \mathcal{S}^{hm}(\mathcal{V}(\mathbf{A}))$.

The variety of MMV-algebras is generated by the finite algebras $\mathbf{S}_m^k$ ([10]). Then the following result is a consequence of Proposition 3.19.

Corollary 3.21. *The variety* $\mathcal{MWH}$ *is generated by the set* $\{\mathbf{C}_m^k : m, k \in \mathbb{N}\}$.

Let us observe that in an MMV-algebra we can express the operation $\exists$ by means of $\to$ and $\forall$ in the following way:

$$\exists x \approx \forall(x \to \forall x) \to \forall x.$$

To prove this it is enough to see that $\exists_\vee a = \forall_\wedge(a \to \forall_\wedge a) \to \forall_\wedge a$, for each $a \in [0,1]^k$, since the variety of MMV-algebras is generated by the MMV-algebras $\langle [0,1]^k; \exists_\vee \rangle$. Let $a = \langle a_1, \ldots, a_k \rangle \in [0,1]^k$, and let us consider that $\min_s\{a_s\} = a_i$ and $\max_s\{a_s\} = a_j$. Then $a_j \to a_i \leq a_s \to a_i$, for all $1 \leq s \leq k$. Then $\forall_\wedge(a \to \forall_\wedge a) \to \forall_\wedge a = \langle (a_j \to a_i) \to a_i, \ldots, (a_j \to a_i) \to a_i = \langle a_j \vee a_i, \ldots, a_j \vee a_i \rangle = \langle a_j, \ldots, a_j \rangle = \exists_\vee a$.

Taking the above expression of $\exists$ into account, in any monadic Wajsberg hoop $\mathbf{H} = \langle H; \odot, \to, \forall, 1 \rangle$ we can define

$$\exists a = \forall(a \to \forall a) \to \forall a.$$

In addition, since the variety $\mathcal{MWH}$ is exactly the class of monadic hoop-subreducts of the variety $\mathcal{MMV}$, then every monadic Wajsberg hoop satisfies any MMV-identity in which $\neg$ and 0 are not involved. In particular, identities (MMV1), (MMV2), (MMV4), (MMV5), (MMV6), and (MMV20) are satisfied in every monadic Wajsberg hoop. The following identity will also be necessary for §5.

Lemma 3.22. *Let* $\mathbf{H} = \langle H; \odot, \to, \forall, 1 \rangle$ *be a monadic Wajsberg hoop, and let* $a, b \in H$. *Then,*

(MMV21) $\exists(b \to a \odot b) = \forall b \to \exists a \odot \forall b.$

Proof. Let us prove that (MMV21) holds in every MMV-algebra. For that, let $\mathbf{A}$ be an MMV-algebra and let $a, b \in A$. Then $b \to a \odot b = \neg b \oplus (a \odot b) = \neg b \oplus (\neg(\neg a \oplus \neg b)) = \neg(b \odot (\neg a \oplus \neg b)) = \neg(b \odot (b \to \neg a)) = \neg(b \wedge \neg a)$. Thus, $\exists(b \to a \odot b) = \exists(\neg(b \wedge \neg a)) = \neg\forall(b \wedge \neg a) = \neg(\forall b \wedge \forall \neg a) = \neg(\forall b \wedge \neg\exists a) = \forall b \to \exists a \odot \forall b$.

Since $\mathcal{MWH}$ is the class of all monadic hoop-subreducts of $\mathcal{MMV}$, (MMV21) holds in every monadic Wajsberg hoop. □

4. Varieties of monadic Wajsberg hoops of width k

In this section we define the width of a monadic Wajsberg hoop. We characterize a subdirectly irreducible algebra of finite width k as a subalgebra of a functional monadic Wajsberg hoop. Finally, we describe some of the subvarieties of finite width.

Let us consider the Wajsberg hoop $\mathbf{V}^X$ of all functions from a nonempty set X to a Wajsberg hoop $\mathbf{V}$, where the operations $\odot$, $\to$, and 1 are defined pointwise. If for $p \in V^X$ there exists the infimum of the set $\{p(y) : y \in X\}$, denoted by $\inf\{p(y) : y \in X\}$, then we can define the constant function $\forall_\wedge(p) \in V^X$ as

$$\forall_\wedge(p)(x) = \inf\{p(y) : y \in X\},$$

for every $x \in X$.

A *functional monadic Wajsberg hoop* $\mathbf{H}$ is a monadic Wajsberg hoop whose hoop-reduct is a Wajsberg subhoop of $\mathbf{V}^X$ and such that the universal operator is the function $\forall_\wedge$. Observe that $\mathbf{H}$ satisfies that

(1) for each $p \in H$, $\inf\{p(y) : y \in X\}$ exists in V;
(2) for each $p \in H$, the constant function $\forall_\wedge(p)$ is in H.

The proof of the next result follows completely analogously to [10, Proposition 4.5]. Because of this, we do not include it in this paper.

Proposition 4.1. *If* $\mathbf{H}$ *is a subalgebra of a functional monadic Wajsberg hoop* $\langle V^k; \forall_\wedge\rangle$ *such that* $\mathbf{V}$ *and* $\forall\mathbf{H}$ *are totally ordered, then* $\mathbf{H}$ *is isomorphic to a subalgebra of* $\langle(\forall\mathbf{H})^k; \forall_\wedge\rangle$.

Let us consider the identity

$$(\alpha^k) \quad \bigvee_{1\le i<j\le k+1} \left(\forall(x_i \vee x_j) \to \bigvee_{s=1}^{k+1} \forall x_s\right) \approx 1,$$

if $k \geq 2$, and

$$(\alpha^1)\ x \approx \forall x.$$

We define the *width* of a monadic Wajsberg hoop $\mathbf{H}$, denoted by $\mathrm{width}(\mathbf{H})$, as the least integer k such that (α^k) holds in $\mathbf{H}$. If k does not exist, then we say that the width of $\mathbf{H}$ is infinite and we write $\mathrm{width}(\mathbf{H}) = \omega$.

Theorem 4.2. *If* $\mathbf{H}$ *is a subdirectly irreducible monadic Wajsberg hoop that satisfies* (α^k) *then* $\mathbf{H}$ *is isomorphic to a subalgebra of* $\langle(\forall\mathbf{H})^k; \forall_\wedge\rangle$.

Proof. From Corollary 3.18, we have that $\mathbf{H}$ is a monadic hoop-subreduct of an MMV-algebra $\mathbf{A}$ such that $\forall\mathbf{A}$ is totally ordered. If $\mathbf{H}$ satisfies (α^k), then $[\forall\mathbf{z})$ also satisfies (α^k), for each $z \in H$. Then, by the construction of $\mathbf{A}$, it follows that (α^k) holds in $\mathbf{A}$ too.

Then we know that $\mathbf{A}$ is isomorphic to a subalgebra of the functional MMV-algebra $\langle(\forall\mathbf{A})^k; \forall_\wedge\rangle$ (see [10, Corollary 4.6]). From Lemma 3.14, the monadic hoop-reduct of $\langle(\forall\mathbf{A})^k; \forall_\wedge\rangle$ is a functional monadic Wajsberg hoop. Finally, the result follows from Proposition 4.1. □

Remark 4.3. Let us observe that if $\mathbf{H}$ is a subdirectly irreducible monadic Wajsberg hoop and $\mathrm{width}(\mathbf{H}) = k > 1$ then the representation given in Theorem 4.2 has exactly k factors. Indeed, let us suppose that there is some i_0, $1 \leq i_0 \leq k$, such that $a(i_0) = 1$ for all $a \in H$. For simplicity, we are here identifying $\mathbf{H}$ with the corresponding subalgebra of $\langle(\forall\mathbf{H})^k; \forall_\wedge\rangle$. We will show next that $\mathbf{H}$ satisfies (α^{k-1}) and this contradicts that $\mathrm{width}(\mathbf{H}) = k$. Let $A = \{a_1, a_2, \ldots, a_k\} \subseteq (\forall H)^k$

and let $A_i^- = A \setminus \{a_i\}$ for each $i \in \{1, \ldots, k\}$. Let us denote by $\bigwedge A$ and $\bigwedge A_i^-$ the infimum of all the elements of A and A_i^- respectively. Since $\forall\mathbf{H}$ is a totally ordered Wajsberg hoop, A has k elements and $a_i(i_0) = 1$ for all i, then there is some j such that $\bigwedge A = \bigwedge A_j^-$. Thus, $\bigwedge A_j^- \leq a_j$. In addition, if $i \neq j$ then $\bigwedge A_i^- \leq a_j$, and from this we have that $\bigvee_{i=1}^k \bigwedge A_i^- \leq a_j$. Then $\forall\left(\bigvee_{i=1}^k \bigwedge A_i^-\right) \leq \forall a_j$ and $\forall\left(\bigvee_{i=1}^k \bigwedge A_i^-\right) \to \forall a_j = 1$. In consequence, $\bigvee_{j=1}^k \left(\forall\left(\bigvee_{i=1}^k \bigwedge A_i^-\right) \to \forall a_j\right) = \forall\left(\bigvee_{i=1}^k \bigwedge A_i^-\right) \to \bigvee_{j=1}^k \forall a_j = \left(\bigwedge_{1\leq i<j\leq k} \forall(a_i \vee a_j)\right) \to \bigvee_{j=1}^k \forall a_j = 1$. This implies that (α^{k-1}) is satisfied (Cf. [10, Theorem 4.13]).

In what follows, we describe some subvarieties of monadic Wajsberg hoops of finite width. We begin by studying cancellative subvarieties.

Proposition 4.4. *Let* $\mathbf{H}$ *be a subdirectly irreducible monadic Wajsberg hoop such that* $\operatorname{width}(\mathbf{H}) = k$. *If* $\mathbf{H}$ *is cancellative, then* $\mathbf{H}$ *has a subalgebra isomorphic to* $\mathbf{C}_\omega^k$.

Proof. If $k = 1$, then $\mathbf{H}$ is a cancellative totally ordered Wajsberg hoop. In this case, the result is a well-known fact.

Let us suppose now that $k > 1$. From Theorem 4.2, we have that $\mathbf{H}$ is isomorphic to a subalgebra of $\langle (\forall\mathbf{H})^k; \forall_\wedge\rangle$. For simplicity, we identify $\mathbf{H}$ with this subalgebra. Since $\operatorname{width}(\mathbf{H}) = k$, there are k elements $a_j \in H \subseteq (\forall H)^k$, satisfying $a_j(i) = 1$ if $i \neq j$, and $a_j(j) = c_j$, where $c_j \in \forall H \setminus \{1\}$ and $1 \leq j \leq k$ (see Remark 4.3). Since $\forall\mathbf{H}$ is totally ordered, we can assume without loss of generality that $c_1 \leq \cdots \leq c_j \leq \cdots \leq c_k$. Let $b_j = \left(\bigodot_{s\in\{1,\ldots,k\}\setminus\{j\}} a_s\right) \to \forall a_k$. It is easy to see that $b_j(i) = 1$ if $i \neq j$, and $b_j(j) = c_k$. It is now straightforward to prove that the subalgebra of $\mathbf{H}$ generated by the set $\{b_j : 1 \leq j \leq k\}$ is isomorphic to $\mathbf{C}_\omega^k$. □

Lemma 4.5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two Wajsberg hoops such that* $\mathbf{A} \in \mathcal{V}_{\mathcal{WH}}(\mathbf{B})$. *Then the monadic Wajsberg hoop* $\langle \mathbf{A}^k; \forall_\wedge\rangle \in \mathcal{V}_{\mathcal{MWH}}(\langle \mathbf{B}^k; \forall_\wedge\rangle)$.

Proof. Let $\mathbf{A} \in \mathcal{V}_{\mathcal{WH}}(\mathbf{B}) = \mathrm{HSP}_{\mathcal{WH}}(\mathbf{B})$. Then there is $\mathbf{W} \in \mathrm{SP}(\mathbf{B})$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{W}$. Let $h\colon \mathbf{W} \to \mathbf{A}$ be the hoop-epimorphism from $\mathbf{W}$ onto $\mathbf{A}$. Then h induces naturally a monadic hoop-epimorphism $\bar{h}\colon \langle W^k; \forall_\wedge\rangle \to \langle A^k; \forall_\wedge\rangle$ defined by $\bar{h}(\langle w_1, \ldots, w_k\rangle) = \langle h(w_1), \ldots, h(w_k)\rangle$. On the other hand, $\mathbf{W}$ is a subalgebra of a direct product $\prod_{i\in I} \mathbf{B}$ of $\mathbf{B}$. It is straightforward to see that $\langle W^k; \forall_\wedge\rangle$ is a monadic Wajsberg subhoop of $\langle \left(\prod_{i\in I} \mathbf{B}\right)^k; \forall_\wedge\rangle$. Moreover, $\varphi\colon \left(\prod_{i\in I} \mathbf{B}\right)^k \to \prod_{i\in I} \mathbf{B}^k$ defined by $\varphi\langle (a_i^1)_{i\in I}, \ldots, (a_i^k)_{i\in I}\rangle = (\langle a_i^1, \ldots, a_i^k\rangle)_{i\in I}$ is an isomorphism. So $\mathbf{A}^k \in \mathrm{HSP}(\mathbf{B}^k)$, and this means that $\mathbf{A}^k \in \mathcal{V}_{\mathcal{MWH}}(\mathbf{B}^k)$. □

Theorem 4.6. *If* $\mathbf{H}$ *is a subdirectly irreducible* cancellative *monadic Wajsberg hoop such that* $\operatorname{width}(\mathbf{H}) = k$, *then* $\mathcal{V}(\mathbf{H}) = \mathcal{V}(\mathbf{C}_\omega^k)$.

Proof. From Proposition 4.4, we know that $\mathbf{C}_\omega^k$ is isomorphic to a subalgebra of $\mathbf{H}$. Then, $\mathcal{V}(\mathbf{C}_\omega^k) \subseteq \mathcal{V}(\mathbf{H})$.

On the other hand, we know that $\forall\mathbf{H}$ is a totally ordered cancellative Wajsberg hoop and then $\mathcal{V}_{\mathcal{WH}}(\mathbf{C}_\omega) = \mathcal{V}_{\mathcal{WH}}(\forall\mathbf{H})$ (see [2]). Then, by Lemma 4.5, we have that $\mathcal{V}(\mathbf{C}_{\omega}^{k}) = \mathcal{V}((\forall\mathbf{H})^{k})$. So, from Theorem 4.2, $\mathbf{H} \in \mathcal{V}(\mathbf{C}_{\omega}^{k})$. In consequence, $\mathcal{V}(\mathbf{H}) \subseteq \mathcal{V}(\mathbf{C}_{\omega}^{k})$. $\square$

Let $\mathbf{H}$ be a *bounded* monadic Wajsberg hoop. We define the *boolean width of* $\mathbf{H}$, denoted by $\mathrm{bwidth}(\mathbf{H})$, as the boolean width of the MV-reduct of $\mathbf{H}$.

Let $\mathbf{C}_{n,\omega}^{k}$, $\mathbf{C}_{n,\omega}^{k,1}$, and $\mathbf{C}_{n,\omega}^{k,\mathbf{P}}$ be the monadic Wajsberg hoop-reducts of the MMV-algebras $\mathbf{S}_{n,\omega}^{k}$, $\mathbf{S}_{n,\omega}^{k,1}$, and $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$, respectively.

Theorem 4.7. *Let* $\mathbf{H}$ *be a subdirectly irreducible monadic Wajsberg hoop such that* $\mathrm{width}(\mathbf{H}) = k$*.*

(a) If $\mathbf{H}$ *is cancellative then* $\mathcal{V}(\mathbf{H}) = \mathcal{V}(\mathbf{C}_{\omega}^{k})$*.*
(b) If $\mathrm{ord}(\forall\mathbf{H}) = m$*, then* $\mathcal{V}(\mathbf{H}) = \mathcal{V}(\mathbf{C}_{m}^{k})$*.*
(c) If $\mathrm{rank}(\forall\mathbf{H}) = n$*,* $\mathrm{ord}(\forall\mathbf{H}) = \omega$*,* $\mathrm{bwidth}(\mathbf{H}) = k$*, then* $\mathcal{V}(\mathbf{H}) = \mathcal{V}(\mathbf{C}_{n,\omega}^{k})$*.*
(d) If $\mathrm{rank}(\forall\mathbf{H}) = n$*,* $\mathrm{ord}(\forall\mathbf{H}) = \omega$*,* $\mathrm{bwidth}(\mathbf{H}) = 1$*, then* $\mathcal{V}(\mathbf{H}) = \mathcal{V}(\mathbf{C}_{n,\omega}^{k,1})$*.*
(e) If $\mathrm{rank}(\forall\mathbf{H}) = n$*,* $\mathrm{ord}(\forall\mathbf{H}) = \omega$*,* $\mathrm{bwidth}(\mathbf{H}) = s$ *with* $1 < s < k$*, then there exists a partition* $\mathbf{P} = \{P_1, \ldots, P_s\}$ *of the set* $\{1, \ldots, k\}$ *such that the equation*

$$\left[\left(\left(\bigwedge_{i=1}^{s} 2x_i^{n+1}\right) \to \forall\left(\bigwedge_{i=1}^{s} 2x_i^{n+1}\right)\right) \wedge \left(\bigwedge_{1\leq i<j\leq s} (2x_i^{n+1} \vee 2x_j^{n+1})\right) \wedge \right.$$
$$\left.\left(\bigwedge_{i=1}^{s} (\forall 2x_i^{n+1} \to \forall(\bigwedge_{i=1}^{s} 2x_i^{n+1}))\right) \wedge \left(\bigwedge_{i=1}^{s} \exists 2x_i^{n+1}\right)\right] \qquad (\gamma_n^{k,\mathbf{P}})$$
$$\to \left[\bigvee_{\sigma\in\mathbb{P}(\{1,\ldots,s\})} \left(\bigwedge_{i=1}^{s} \alpha^{p_{\sigma(i)}}_{\forall_{2x^{n+1}_{\sigma(i)}}}(z_1^{\sigma(i)}, \ldots, z_{p_{\sigma(i)}+1}^{\sigma(i)})\right)\right] \approx 1,$$

is satisfied in $\mathbf{H}$*, where* p_i *denotes the cardinal of each subset* P_i*,* $\mathbb{P}(\{1, \ldots, s\})$ *is the set of all permutations of the set* $\{1, \ldots, s\}$*, and*

$$\alpha^{p_{\sigma(i)}}_{\forall_{2x^{n+1}_{\sigma(i)}}}(z_1^{\sigma(i)}, \ldots, z_{p_{\sigma(i)}+1}^{\sigma(i)})$$

is an abbreviation of

$$\left(\forall\left(\bigvee_{r=1}^{p_{\sigma(i)}+1} \bigwedge Z_r^{-}\right) \vee 2x_{\sigma(i)}^{n+1}\right) \to \bigvee_{j=1}^{p_{\sigma(i)}+1} \left(\forall z_j^{\sigma(i)} \vee 2x_{\sigma(i)}^{n+1}\right),$$

where $Z = \{z_1^{\sigma(i)}, \ldots, z_{p_{\sigma(i)}+1}^{\sigma(i)}\}$ *and* $Z_r^{-} = Z - \{z_r^{\sigma(i)}\}$*. Moreover,* $\mathcal{V}(\mathbf{H}) = \mathcal{V}(\mathbf{C}_{n,\omega}^{k,\mathbf{P}})$*.*

Proof. Let $\mathbf{H}$ be a subdirectly irreducible monadic Wajsberg hoop such that $\mathrm{width}(\mathbf{H}) = k$. From Corollary 3.12, we know that $\mathbf{H}$ is bounded or cancellative.

If $\mathbf{H}$ is cancellative, the result follows from Theorem 4.6. Let us consider $\mathbf{H}$ bounded. Let $\mathbf{A}$ be the MMV-algebra obtained by considering the first element 0 of H as a constant. From Corollary 3.20 we know that $\mathcal{V}(\mathbf{H}) = \mathcal{S}^{hm}(\mathcal{V}(\mathbf{A}))$.

In case (b), $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_{m}^{k})$ by [10, Lemma 5.2].

In case (c), $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_{n,\omega}^{k})$ by [10, Proposition 5.12].

In case (d), $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}^{k,1}_{n,\omega})$ by [10, Theorem 5.19].

Observe that equation $(\gamma^{k,\mathbf{P}}_n)$ is analogous to equation $(f^{k,\mathbf{P}}_n)$. More precisely, we exclude from equation $(f^{k,\mathbf{P}}_n)$ superfluous implications and replace both appearances of 0 with $\forall(\bigwedge_{i=1}^{s} 2x_i^{n+1})$. Note that if $\{a_1, \dots, a_s\} \subseteq \mathrm{Rad}(\mathbf{A})$, then $2a_i^{n+1} = 1$ for all i, and $(\gamma^{k,\mathbf{P}}_n)$ holds in $\mathbf{A}$. If not,

$$\left[\left((\bigwedge_{i=1}^{s} 2a_i^{n+1}) \to \forall(\bigwedge_{i=1}^{s} 2a_i^{n+1})\right) \wedge \left(\bigwedge_{1\le i<j\le s} (2a_i^{n+1} \vee 2a_j^{n+1})\right) \wedge \left(\bigwedge_{i=1}^{s} (\forall 2a_i^{n+1} \to \forall(\bigwedge_{i=1}^{s} 2a_i^{n+1}))\right) \wedge \left(\bigwedge_{i=1}^{s} \exists 2a_i^{n+1}\right)\right] = 1$$

if and only if the set $\{2a_1^{n+1}, \dots, 2a_s^{n+1}\}$ is exactly the set of coatoms of $\mathbf{B}(\mathbf{A})$. In addition, if the last set is not the set of coatoms then

$$\left[\left((\bigwedge_{i=1}^{s} 2a_i^{n+1}) \to \forall(\bigwedge_{i=1}^{s} 2a_i^{n+1})\right) \wedge \left(\bigwedge_{1\le i<j\le s} (2a_i^{n+1} \vee 2a_j^{n+1})\right) \wedge \left(\bigwedge_{i=1}^{s} (\forall 2a_i^{n+1} \to \forall(\bigwedge_{i=1}^{s} 2a_i^{n+1}))\right) \wedge \left(\bigwedge_{i=1}^{s} \exists 2a_i^{n+1}\right)\right] = 0.$$

Therefore, $(\gamma^{k,\mathbf{P}}_n)$ holds in $\mathbf{A}$ if and only if $(f^{k,\mathbf{P}}_n)$ holds in $\mathbf{A}$. Then, by [10, Theorem 5.26], we have that $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}^{k,\mathbf{P}}_{n,\omega})$.

Now, from Corollary 3.20, we have the desired result. □

5. Monadic MV-closures of monadic Wajsberg hoops

For a given Wajsberg hoop $\mathbf{H}$, there exists an MV-algebra $\mathbf{MV(H)}$, called the MV-closure of $\mathbf{H}$, such that the underlying set H of $\mathbf{H}$ is a maximal filter of $\mathbf{MV(H)}$ and the quotient $\mathbf{MV(H)}/H$ is the two-element chain (see [1]). In this section, we show that if the given Wajsberg hoop $\mathbf{H}$ is a monadic Wajsberg hoop, then we can enrich $\mathbf{MV(H)}$ with an existential operator and a universal operator making $\mathbf{MV(H)}$ a monadic MV-algebra. We begin this section recalling briefly the construction of the MV-closure of a Wajsberg hoop. For more details, we refer the reader to [1].

Let $\mathbf{H} = \langle H; \odot_H, \to_H, 1_H\rangle$ be a Wajsberg hoop. The MV-closure of $\mathbf{H}$ is the MV-algebra defined as follows. Let

$$\mathbf{MV(H)} = \langle H \times \{0,1\}; \oplus_{mv}, \neg_{mv}, 0_{mv}\rangle,$$

where $0_{mv} := (1_H, 0)$, $\neg_{mv}(a,i) := (a, 1-i)$,

$$(a,i) \oplus_{mv} (b,j) := \begin{cases} (a \oplus_H b, 1) & \text{if } i = j = 1, \\ (b \to_H a, 1) & \text{if } i = 1 \text{ and } j = 0, \\ (a \to_H b, 1) & \text{if } i = 0 \text{ and } j = 1, \\ (a \odot_H b, 0) & \text{if } i = j = 0. \end{cases}$$

We define $1_{mv} := \neg_{mv} 0_{mv}$, $(a,i) \odot_{mv} (b,j) := \neg_{mv}(\neg_{mv}(a,i) \odot_{mv} \neg_{mv}(b,j))$, and $(a,i) \to_{mv} (b,j) := \neg_{mv}(a,i) \odot_{mv} (b,j)$, as usual. It is easily checked that

$$1_{mv} = (1_H, 1),$$

$$(a,i) \odot_{mv} (b,j) = \begin{cases} (a \odot_H b, 1) & \text{if } i = j = 1, \\ (a \to_H b, 0) & \text{if } i = 1 \text{ and } j = 0, \\ (b \to_H a, 0) & \text{if } i = 0 \text{ and } j = 1, \\ (a \oplus_H b, 0) & \text{if } i = j = 0, \end{cases}$$

$$(a,i) \to_{mv} (b,j) = \begin{cases} (a \to_H b, 1) & \text{if } i = j = 1, \\ (a \odot_H b, 0) & \text{if } i = 1 \text{ and } j = 0, \\ (a \oplus_H b, 1) & \text{if } i = 0 \text{ and } j = 1, \\ (b \to_H a, 1) & \text{if } i = j = 0. \end{cases}$$

The map $a \mapsto (a,1)$ is a hoop-embedding from $\mathbf{H}$ to $\mathbf{MV}(\mathbf{H})$. If we identify a and $(a,1)$ for every $a \in H$, we can consider $\mathbf{H}$ a subalgebra of the hoop-reduct of $\mathbf{MV}(\mathbf{H})$. If $b = (a,0)$ for some $a \in H$, then $b = \neg_{mv}(a,1)$ and we write $b = \neg_{mv} a$. Then we can consider $\mathbf{MV}(\mathbf{H})$ as the disjoint union of H and $\neg_{mv} H$.

The order relation on $\mathbf{MV}(\mathbf{H})$ is given by: $(a,i) \leq (b,j)$ if and only if one of the following conditions holds:

- $a \leq_H b$ for $i = j = 1$,
- $a \oplus_H b = 1_H$ for $i = 0$ and $j = 1$,
- $b \leq_H a$ for $i = j = 0$.

Let us consider now a monadic Wajsberg hoop $\mathbf{H} = \langle H; \odot_H, \to_H, \forall_H, 1_H \rangle$, and let us define in $H \times \{0,1\}$ the following quantifiers:

$$\exists(a,i) = \begin{cases} (\exists_H a, 1) & \text{if } i = 1, \\ (\forall_H a, 0) & \text{if } i = 0, \end{cases}$$

and

$$\forall(a,i) = \begin{cases} (\forall_H a, 1) & \text{if } i = 1, \\ (\exists_H a, 0) & \text{if } i = 0, \end{cases}$$

where $\exists_H a = \forall_H (a \to_H \forall_H a) \to_H \forall_H a$ (see §3).

Theorem 5.1. *If* $\mathbf{H}$ *is a monadic Wajsberg hoop, then*

$$\mathbf{MMV}(\mathbf{H}) = \langle H \times \{0,1\}; \oplus_{mv}, \neg_{mv}, \exists, 0_{mv} \rangle$$

is an MMV-algebra. Moreover, H is a monadic maximal filter of $\mathbf{MMV}(\mathbf{H})$ *and*

$$\mathbf{MMV}(\mathbf{H})/H \cong \mathbf{S}_1^1.$$

Proof. We only need to check that $\mathbf{MMV}(\mathbf{H})$ satisfies the MMV-axioms. Let $(a,i) \in H \times \{0,1\}$.

Let us suppose that $i = 1$. Then $\exists(a,1) = (\exists_H a, 1) \geq (a,1)$, since $\exists_H a \geq a$. If $i = 0$ then $\exists(a,0) = (\forall_H a, 0) \geq (a,0)$, since $\forall_H a \leq a$. So (MMV1) is satisfied.

Let us see that (MMV3) holds. Let us assume first that $i = 1$. Then $\exists\neg_{mv}\exists(a, 1) = \exists\neg_{mv}(\exists_H a, 1) = \exists(\exists_H a, 0) = (\forall_H \exists_H a, 0) = (\exists_H a, 0)$, since $\forall_H\exists_H a = \exists_H a$. However, $(\exists_H a, 0) = \neg_{mv}(\exists_H a, 1) = \neg_{mv}\exists(a, 1)$. So $\exists\neg_{mv}\exists(a, 1) = \neg_{mv}\exists(a, 1)$. If $i = 0$, and from $\exists_H\forall_H a = \forall_H a$, we have that $\exists\neg_{mv}\exists(a, 0) = \exists\neg_{mv}(\forall_H a, 0) = \exists(\forall_H a, 1) = (\exists_H\forall_H a, 1) = (\forall_H a, 1) = \neg_{mv}(\forall_H a, 0) = \neg_{mv}\exists(a, 0)$. In consequence, (MMV3) is satisfied.

Since $\exists_H(a\odot_H a) = \exists_H a\odot_H \exists_H a$, we have $\exists((a, 1)\odot_{mv}(a, 1)) = \exists((a\odot_H a, 1)) = (\exists_H(a\odot_H a), 1) = (\exists_H a\odot_H \exists_H a, 1) = (\exists_H a, 1)\odot_{mv}(\exists_H a, 1) = \exists(a, 1)\odot_{mv}\exists(a, 1)$. If $i = 0$, and from $\forall_H(a\oplus_H a) = \forall_H a\oplus_H \forall_H a$, then $\exists((a, 0)\odot_{mv}(a, 0)) = \exists((a\oplus_H a, 0)) = (\forall_H(a\oplus_H a), 0) = (\forall_H a\oplus_H \forall_H a, 0) = (\forall_H a, 0)\odot_{mv}(\forall_H a, 0) = \exists(a, 0)\odot_{mv}\exists(a, 0)$. Thus, (MMV5) holds.

Since $\exists_H(a\oplus_H a) = \exists_H a\oplus_H \exists_H a$ then $\exists((a, 1)\oplus_{mv}(a, 1)) = \exists((a\oplus_H a, 1)) = (\exists_H(a\oplus_H a), 1) = (\exists_H a\oplus_H \exists_H a), 1) = (\exists_H a, 1)\oplus_{mv}(\exists_H a, 1) = \exists(a, 1)\oplus_{mv}\exists(a, 1)$. Let us consider now that $i = 0$. Since $\forall_H(a\odot_H a) = \forall_H a\odot_H \forall_H a$, we have $\exists((a, 0)\oplus_{mv}(a, 0)) = \exists((a\odot_H a, 0)) = (\forall_H(a\odot_H a), 0) = (\forall_H a\odot_H \forall_H a, 0) = (\forall_H a, 0)\oplus_{mv}(\forall_H a, 0) = \exists(a, 0)\oplus_{mv}\exists(a, 0)$. Then, (MMV6) holds.

Let us prove now (MMV2). Let $(b, j) \in H\times\{0, 1\}$. If $i = j = 1$ then $\exists((a, 1)\vee_{mv}(b, 1)) = \exists(a\vee_H b, 1) = (\exists_H(a\vee_H b), 1) = (\exists_H a\vee_H \exists_H b, 1) = (\exists_H a, 1)\vee_{mv}(\exists_H b, 1) = \exists(a, 1)\vee_{mv}\exists(b, 1)$.

If $i = j = 0$ then $\exists((a, 0)\vee_{mv}(b, 0)) = \exists(((a, 0)\to_{mv}(b, 0))\to_{mv}(b, 0)) = \exists((b\to_H a, 1)\to_{mv}(b, 0)) = \exists(b\odot_H(b\to_H a), 0) = \exists(b\wedge_H a, 0) = (\forall_H(b\wedge_H a), 0) = (\forall_H b\wedge_H \forall_H a, 0) = (\forall_H a, 0)\vee_{mv}(\forall_H b, 0) = \exists(a, 0)\vee_{mv}\exists(b, 0)$.

If $i = 1$ and $j = 0$, and taking (MMV21) into account, we have $\exists((a, 1)\vee_{mv}(b, 0)) = \exists(((a, 1)\to_{mv}(b, 0))\to_{mv}(b, 0)) = \exists((a\odot_H b, 0)\to_{mv}(b, 0)) = \exists(b\to_H (a\odot_H b), 1) = (\exists_H(b\to_H (a\odot_H b)), 1) = (\forall_H b\to_H \exists_H a\odot_H \forall_H b, 1) = (\exists_H a\odot_H \forall_H b, 0)\to_{mv}(\forall_H b, 0) = ((\exists_H a, 1)\to_{mv}(\forall_H b, 0))\to_{mv}(\forall_H b, 0) = (\exists_H a, 1)\vee_{mv}(\forall_H b, 0) = \exists(a, 1)\vee_{mv}\exists(b, 0)$. So, (MMV2) holds.

Let us show now that (MMV4) is satisfied. If $i = j = 1$, then $\exists(\exists(a, 1)\oplus_{mv}\exists(b, 1)) = \exists((\exists_H a, 1)\oplus_{mv}(\exists_H b, 1)) = \exists((\exists_H a\oplus_H \exists_H b, 1)) = (\exists_H(\exists_H a\oplus_H \exists_H b), 1) = (\exists_H a\oplus_H \exists_H b, 1) = (\exists_H a, 1)\oplus_{mv}(\exists_H b, 1) = \exists(a, 1)\oplus_{mv}\exists(b, 1)$.

If $i = 0$ and $j = 1$, then $\exists(\exists(a, 0)\oplus_{mv}\exists(b, 1)) = \exists((\forall_H a, 0)\oplus_{mv}(\exists_H b, 1)) = \exists(\forall_H a\to_H \exists_H b, 1) = (\exists_H(\forall_H a\to_H \exists_H b), 1) = (\forall_H a\to_H \exists_H b, 1) = \exists(a, 0)\oplus_{mv}\exists(b, 1)$.

Finally, if $i = j = 0$ then we have $\exists(\exists(a, 0)\oplus_{mv}\exists(b, 0)) = \exists((\forall_H a, 0)\oplus_{mv}(\forall_H b, 0)) = \exists(\forall_H a\odot_H \forall_H b, 0) = (\forall_H(\forall_H a\odot_H \forall_H b), 0) = (\forall_H a\odot_H \forall_H b, 0) = (\forall_H a, 0)\oplus_{mv}(\forall_H b, 0) = \exists(a, 0)\oplus_{mv}\exists(b, 0)$. So, (MMV4) holds.

We have proved that $\mathbf{MMV(H)}$ is an MMV-algebra. From this and [1, Theorem 3.5], we have the desired result. □

A subset S of an MV-algebra satisfies the *finite product property* (fpp for short), provided 0 cannot be obtained as a finite product of elements of S, that is, the filter generated by S is proper.

Theorem 5.2. *Let* $\mathbf{H}$ *be a monadic Wajsberg hoop and* $\mathbf{A}$ *an MMV-algebra. If* $h\colon \mathbf{H}\to\mathbf{A}$ *is a monadic hoop-homomorphism, then there is a unique MMV-homomorphism* $\hat{h}\colon \mathbf{MMV(H)}\to\mathbf{A}$ *such that* $\hat{h}\restriction_H = h$.

$$\begin{tikzcd} \mathbf{H} \arrow[r, hook] \arrow[d, "h"'] & \mathbf{MMV}(\mathbf{H}) \arrow[dl, "\hat{h}"] \\ \mathbf{A} & \end{tikzcd}$$

In addition, if h is injective and $h[H]$ has the fpp in $\mathbf{A}$, then $\hat{h}$ is also injective.

Proof. For each $a \in H$ we define $\hat{h}(a,1) = h(a)$ and $\hat{h}(a,0) = \neg_A h(a)$. In view of [1, Theorem 3.6], we only need to verify that $\hat{h}$ is an MMV-homomorphism. Let $(a,1) \in MMV(H)$. Then

$$\hat{h}\exists(a,1) = \hat{h}(\exists_H a, 1) = h(\exists_H a) = \exists_A h(a) = \exists_A \hat{h}(a,1).$$

Now, let $(a,0) \in MMV(H)$. Then

$$\hat{h}\exists(a,0) = \hat{h}(\forall_H a, 0) = \neg_A h(\forall_H a) = \neg_A \forall_A h(a) = \exists_A \neg_A h(a) = \exists_A \hat{h}(a,0). \qquad \square$$

Corollary 5.3. *Let $h\colon \mathbf{H}_1 \to \mathbf{H}_2$ be a homomorphism of monadic Wajsberg hoops; then there is a unique MMV-homomorphism $\hat{h}\colon \mathbf{MMV}(\mathbf{H}_1) \to \mathbf{MMV}(\mathbf{H}_2)$ such that $\hat{h}\restriction_{H_1} = h$. Moreover, if h is injective, surjective, or bijective, so is $\hat{h}$.*

Now, we can define a functor $\mathbb{M}$ from the category $\mathbb{MWH}$ of monadic Wajsberg hoops into the category $\mathbb{MMV}$ of MMV-algebras as follows. Given a monadic Wajsberg hoop $\mathbf{H}$, let $\mathbb{M}(\mathbf{H}) = \mathbf{MMV}(\mathbf{H})$, and given a homomorphism $h\colon \mathbf{H}_1 \to \mathbf{H}_2$ of monadic Wajsberg hoops, let $\mathbb{M}(h) = \hat{h}$ defined in Corollary 5.3. As in [1], we can see that $\mathbb{M}$ is left adjoint to the forgetful functor $\mathbb{U}\colon \mathbb{MMV} \to \mathbb{MWH}$, and we call $\mathbf{MMV}(\mathbf{H})$ the *monadic MV-closure of* $\mathbf{H}$.

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